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Functional Analysis
Tutorial 8

Problem 1. Decide which of the following operators are bounded:

(*a*) $T_1 : (C^1([0,1)]), \|\cdot\|_\infty) \to (C([0,1]), \|\cdot\|_\infty)$, $f \mapsto f'$.

(*b*) $T_2 : (P_n([0,1]), \|\cdot\|_\infty) \to (C([0,1]), \|\cdot\|_\infty)$, $f \mapsto f'$, where $P_n([0,1])$ denotes the polynomials on $[0,1]$ of degree at most n.

(*c*) $T_3 : (C^1([0,1)]), \|\cdot\|_{1,\infty}) \to (C([0,1]), \|\cdot\|_\infty)$, $f \mapsto f'$, where $\|f\|_{1,\infty} := \|f\|_\infty + \|f'\|_\infty$.

Here, f' denotes the derivative of f.

Problem 2. Let c_0 be equipped with $\|\cdot\|_\infty$. Prove the following statements:

(*i*) If $\{e_n\}_{n\in\mathbb{N}}$ denotes the family of sequences given by $(e_n)_k := \delta_{nk}$ for $k \in \mathbb{N}$, then

$$\lim_{n\to\infty} \left\| x - \sum\nolimits_{k=1}^{n} x_k e_k \right\|_\infty = 0\,.$$

(*ii*) The map $I : \ell^1 \to (c_0)'$, $(Iy)(x) := \sum_n x_n y_n$ is a well-defined bounded linear map, satisfying $\|Iy\| = \|y\|_1$ for all $y \in \ell^1$ (i.e. I is an isometry).

(*iii*) $(c_0)' \cong \ell^1$, i.e. $(c_0)'$ and ℓ^1 are isometrically isomorphic.

Problem 3. Prove that the following maps are bounded linear functionals and compute their norms:

(*a*) $\phi_1 : (c_0, \|\cdot\|_\infty) \to \mathbb{C}$, $x \mapsto \sum_{n=1}^\infty 2^{-n+1}\, x_n$.

(*b*) $\phi_2 : (\ell^1, \|\cdot\|_1) \to \mathbb{C}$, $x \mapsto \sum_{n=1}^\infty (1-\frac{1}{n})\, x_n$.

Problem 4 (Hardy operator). For $f \in C([0,1])$ and $x \in [0,1]$, let

$$(Tf)(x) \;:=\; \begin{cases} \frac{1}{x}\int_0^x f(y)dy & \text{if } \; x \in (0,1]\,, \\ 0 & \text{if } \; x = 0\,. \end{cases}$$

Prove that this defines a bounded linear map $T : C([0,1]) \to C([0,1])$ and compute $\|T\|$.