LUDWIG-MAXIMILIANS-UNIVERSITÄT MÜNCHEN

MATHEMATISCHES INSTITUT

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Functional Analysis

Tutorial 7

Problem 1 (Lemma 2.10). Let X, Y, and Z be normed spaces, and let $T : X \to Y$, and $S : Y \to Z$ be continuous linear maps. Prove:

(*i*) $\|T\|_{B(X,Y)} = \sup\big\{\|Tx\|_Y \,\big|\, x \in X, \|x\|_X = 1\big\}$.

(*ii*) $\|Tx\|_Y \leqslant \|T\|_{B(X,Y)}\, \|x\|_X$ for all $x \in X$.

(*iii*) $\|ST\|_{B(X,Z)} \leqslant \|S\|_{B(Y,Z)}\|T\|_{B(X,Y)}$.

Problem 2. Let $(X, \|\cdot\|)$ be a Banach space, $x_0 \in X$, $m \in \mathbb{N}$, and let $T : X \to X$ be a bounded linear map with $\|T^m\|_{B(X)} < 1$. Prove that the equation

$$x - Tx = x_0$$

has a unique solution $x \in X$. [*Hint:* Consider the m-th power of the map $x \mapsto x_0+Tx$.]

Problem 3 (Finite-dimensional norms are equivalent). Prove that, for each $N \in \mathbb{N}$ there exists $c_N > 0$ such that

$$p\left(\tfrac{1}{5}\right) \leqslant c_N \int_0^1 |p(x)|\, dx$$

for all polynomials $p : \mathbb{R} \to \mathbb{R}$ of degree at most N.

Problem 4 (Distance from a point to a set in a normed space). For $n \in \mathbb{N}$, consider the linear space P_n of real polynomials of degree at most n as a subset of $(C([-1, 1], \mathbb{R}), \|\cdot\|_\infty)$. We will calculate the distance from the monomial $m_n : [-1, 1] \to \mathbb{R}, x \mapsto x^n$ to P_{n-1}.

(*i*) For $n \in \mathbb{N}$ and $x \in [-1, 1]$, define $p_n(x) := \cos(n \arccos x)$. Prove that, for all $n \in \mathbb{N}$,

- (*a*) $p_n(x) + p_{n-2}(x) = 2x\, p_{n-1}(x)$ for all $x \in [-1, 1]$,
  [*Hint:* Recall that $\cos(a) + \cos(b) = 2\cos(\frac{a+b}{2})\cos(\frac{a-b}{2})$.]
- (*b*) $p_n - 2^{n-1}\, m_n \in P_{n-1}$.
- (*c*) If p is a polynomial with $\|p\|_\infty < \|p_n\|_\infty$, then $\deg(p-p_n) \geqslant n$.
- (*d*) If $p \in P_n$ is such that $p - 2^{n-1} m_n \in P_{n-1}$, then $\|p\|_\infty \geqslant \|p_n\|_\infty$.

(*ii*) Prove that, for all $n \in \mathbb{N}$, $\operatorname{dist}(m_n, P_{n-1}) = 2^{-n+1}$.