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Functional Analysis

Tutorial 5

Problem 1. Let X, Y, and Z be topological spaces, and let $X \times Y$ be equipped with the product topology. Prove that, if $f : X \times Y \to Z$ is continuous, then it is separately continuous in each variable. Is the converse true?

[*Hint:* Compare with Problem 3 c) on Exercise Sheet 3.]

Problem 2. Let $C([0,1])$ be equipped with the metric d_∞ (see Tutorial 4, Problem 1).

(i) Find a bounded sequence in $C([0,1])$ that does not admit a convergent subsequence.

(ii) Prove that the space of polynomials on $[0,1]$ is not open in $C([0,1])$.

Problem 3. For a topological space $(X, \mathcal{T})$ and a subset $A \subset X$, let $\mathcal{T}_A$ denote the relative topology on A. Moreover, for a metric space (X, d), let $\mathcal{T}_d$ denote the topology induced by d, and if $A \subset X$ is non-empty, let $d_A = d|_{A \times A}$ denote the induced metric on A. Prove:

(i) If (X, d) is a metric space, and $A \subset X$ is non-empty, then $\mathcal{T}_{d_A} = \mathcal{T}_A$.

(ii) If $(X, \mathcal{T})$ is a topological space, then a subset $A \subset X$ is compact in $(X, \mathcal{T})$ if and only if A is compact in $(A, \mathcal{T}_A)$.

Problem 4. Let (X, d) be a metric space. Prove that X is compact if and only if X is complete and pre-compact[1].

[*Hint:* Compare with the lecture, Theorem 1.49.]

[1] Note that, in the literature, the meaning of *pre-compactness* is not consistent. Sets that we call *pre-compact* are sometimes referred to as *totally bounded* sets, while *pre-compact* is then used interchangeably with *relatively compact.*