LUDWIG-MAXIMILIANS-UNIVERSITÄT MÜNCHEN

MATHEMATISCHES INSTITUT

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Functional Analysis

Tutorial 3

Problem 1 (Open and closed balls). Let (X, d) be a metric space. Let $x \in X$, $\varepsilon > 0$ and let $B_\varepsilon(x)$ denote the ball of radius ε centered at x, i.e. $B_\varepsilon(x) := \{y \in X \,|\, d(x, y) < \varepsilon\}$.

(*i*) Prove that $B_\varepsilon(x)$ is open.

(*ii*) Prove that $K_\varepsilon(x) := \{y \in X \,|\, d(x, y) \leqslant \varepsilon\}$ is closed.

(*iii*) Prove that $\overline{B_\varepsilon(x)} \subseteq K_\varepsilon(x)$.

(*iv*) Give an example of a metric space, where generally $\overline{B_\varepsilon(x)} \neq K_\varepsilon(x)$.

Problem 2 (Product topology). Let $(X, \mathcal{T}_Y)$ and $(Y, \mathcal{T}_Y)$ be topological spaces, and let $\mathcal{T}_{X\times Y}$ denote the product topology on $X \times Y$. Prove:

(*i*) Convergence in $(X \times Y, \mathcal{T}_{X\times Y})$ is coordinatewise, i.e. a sequence $((x_n, y_n))_{n\in\mathbb{N}}$ in $X \times Y$ converges to some $(x, y) \in X \times Y$, iff $x_n \to x$ in $(X, \mathcal{T}_Y)$ and $y_n \to y$ in $(Y, \mathcal{T}_Y)$ as $n \to \infty$.

(*ii*) If $\mathcal{B}_X$ is a base for $\mathcal{T}_X$ and $\mathcal{B}_Y$ is a base for $\mathcal{T}_Y$, then

$$\mathcal{B}_X \times \mathcal{B}_Y \;=\; \{U \times Y \,|\, U \in \mathcal{B}_X, V \in \mathcal{B}_Y\}$$

is a base for $\mathcal{T}_{X\times Y}$.

(*iii*) If $(X, \mathcal{T}_Y)$ and $(Y, \mathcal{T}_Y)$ are separable, then $(X \times Y, \mathcal{T}_{X\times Y})$ is separable.

(*iv*) If $(X, \mathcal{T}_Y)$ and $(Y, \mathcal{T}_Y)$ are first/second countable, then $(X \times Y, \mathcal{T}_{X\times Y})$ is first/second countable.