LUDWIG-MAXIMILIANS-UNIVERSITÄT MÜNCHEN
MATHEMATISCHES INSTITUT

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Functional Analysis

Tutorial 2

Problem 1. Let X be a non-empty set and let $\mathcal{T}_0$ be the trivial/indiscrete topology on X, and $\mathcal{T}_d$ the discrete topology on X. Prove that in $(X, \mathcal{T}_0)$ every sequence converges to all points in X, and that in $(X, \mathcal{T}_d)$ all convergent sequences are eventually constant.

Problem 2 (Hausdorff spaces). Let $(X, \mathcal{T})$ be a Hausdorff space. Prove:

(i) The limit of a convergent sequence in X is unique.

(ii) For each $x \in X$ the singleton $\{x\}$ is closed.

Problem 3. Let $(X, \mathcal{T})$ be a topological space, and let $E, F \subset X$.

(i) Prove that if $E \subset F$ then $\mathring{E} \subset \mathring{F}$ and $\overline{E} \subset \overline{F}$.

(ii) Prove that $\overline{E \cap F} \subset \overline{E} \cap \overline{F}$ and give an example for strict inclusion.

(iii) Let $(x_k)_{k\in\mathbb{N}}$ be a sequence in E that converges to some $x \in X$. Prove that $x \in \overline{E}$.

Problem 4.

(a) Consider the following subsets of $\mathbb{R}$:

$$A := [0, 1), \quad B := \mathbb{Q}, \quad C := \left\{\tfrac{1}{n} \middle| \, n \in \mathbb{N}\right\} \cup (2, 3] \,.$$

Determine

(i) all interior points,

(ii) all adherent points,

(iii) all limit points,

(iv) and all boundary points

of the given sets A, B, and C. What are the closure and the interior of A, B, and C?

(b) Consider the following topological spaces $X \subset \mathbb{R}$ equipped with the relative topology $\mathcal{T}_X$ induced by the Euclidean topology on $\mathbb{R}$:

(v) $X = [0, 1] \cup [2, 3]$,

(vi) $X = \mathbb{Q}$.

Produce a non-trivial subset of X which is both, open and closed.