LUDWIG-MAXIMILIANS-UNIVERSITÄT MÜNCHEN

MATHEMATISCHES INSTITUT

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Summer term 2016
May 30, 2016

Functional Analysis

Excercise Sheet 7 – Solution to Problem 4

Problem 4 (Operator norm)**.** Let $X := \{x = (x_n)_{n\in\mathbb{N}} \mid \forall n \in \mathbb{N} : x_n \in \mathbb{R}\}$ denote the set of all sequences in $\mathbb{R}$. Consider the operator $H : X \to X$ defined by

$$(Hx)_n := \frac{1}{n}\sum_{j=1}^{n} x_j, \qquad n \in \mathbb{N}. \tag{1}$$

a) For $p > 1$, and $a = (a_n)_{n=1}^{\infty} \in X$ with $a_n \geq 0$ for all $n \in \mathbb{N}$, prove that

$$(Ha)_n^p - \frac{p}{p-1}(Ha)_n^{p-1}a_n \leq \frac{1}{p-1}((n-1)(Ha)_{n-1}^p - n(Ha)_n^p). \tag{2}$$

Hint: Young's inequality.

b) For $p > 1$, $N \in \mathbb{N}$, and $a = (a_n)_{n=1}^{\infty} \in X$ with $a_n \geq 0$ for all $n \in \mathbb{N}$, prove that

$$\left(\sum_{n=1}^{N}(Ha)_n^p\right)^{1/p} \leq \frac{p}{p-1}\left(\sum_{n=1}^{N} a_n^p\right)^{1/p}. \tag{3}$$

Hint: Hölder's inequality.

c) Prove that $H : \ell^p \to \ell^p$ is a well-defined, bounded linear operator.

d) Compute $\|H\|_{B(\ell^p)}$.

[2+2+3+3 Points]

Proof.

a) Observe that $a_n = n(Ha)_n - (n-1)(Ha)_{n-1}$. Then we obtain

$$\begin{aligned}(Ha)_n^p - \frac{p}{p-1}(Ha)_n^{p-1}a_n &= (Ha)_n^p - \frac{p}{p-1}(Ha)_n^{p-1}\Big(n(Ha)_n - (n-1)(Ha)_{n-1}\Big)\\ &= (Ha)_n^p\Big(1-\frac{np}{p-1}\Big) + \frac{n-1}{p-1}\,p\,(Ha)_n^{p-1}(Ha)_{n-1}\,.\end{aligned}$$

By Young's inequality, $ab \leq \frac{a^q}{q}+\frac{b^p}{p}$ (here $\frac{1}{p}+\frac{1}{q}=1$), with $a = (Ha)_n^{p-1} = (Ha)_n^{p/q} \geq 0$ and $b = (Ha)_{n-1} \geq 0$, it follows that

$$p\,(Ha)_n^{p-1}(Ha)_{n-1} \leq (p-1)(Ha)_n^p + (Ha)_{n-1}^p\,,$$

and therefore

$$(Ha)_n^p - \frac{p}{p{-}1}(Ha)_n^{p-1}a_n \le (Ha)_n^p\Big(1{-}\frac{np}{p{-}1}\Big) + (n{-}1)(Ha)_n^p + \frac{n{-}1}{p{-}1}(Ha)_{n-1}^p$$
$$= \frac{n{-}1}{p{-}1}(Ha)_{n-1}^p - \frac{n}{p{-}1}(Ha)_n^p\ ,$$

which proves inequality (2).

b) Let $\alpha_n := n(Ha)_n^p$. Summing in (2) over $n = 1, \dots, N$ leads to

$$\sum_{n=1}^{N}\left((Ha)_n^p - \frac{p}{p{-}1}(Ha)_n^{p-1}a_n\right) \le \frac{1}{p{-}1}\sum_{n=1}^{N}(\alpha_{n-1}{-}\alpha_n) = -\frac{\alpha_N}{p{-}1} \le 0\,,$$

and therefore by Hölder, $\sum_{n=1}^N |r_n s_n| \le \left(\sum_{n=1}^N |r_n|^q\right)^{1/q}\left(\sum_{n=1}^N |s_n|^p\right)^{1/p}$, where $\frac{1}{p}{+}\frac{1}{q} = 1$ and $r_n = (Ha)_n^{p-1}$, $s_n = a_n$,

$$\sum_{n=1}^{N}(Ha)_n^p \le \frac{p}{p{-}1}\sum_{n=1}^{N}(Ha)_n^{p-1}a_n \le \frac{p}{p{-}1}\left(\sum_{n=1}^{N}(Ha)_n^p\right)^{1-\frac{1}{p}}\left(\sum_{n=1}^{N}a_n^p\right)^{\frac{1}{p}}. \qquad (*)$$

If $\sum_{n=1}^N (Ha)_n^p = 0$, then (3) is trivially true, and if $\sum_{n=1}^N (Ha)_n^p > 0$, then dividing $(*)$ by $(\sum_{n=1}^N (Ha)_n^p)^{1-\frac{1}{p}}$ gives (3).

c) Linearity is obvious. By (3), for each $N \in \mathbb{N}$,

$$\sum_{n=1}^{N}|(Hx)_n|^p \leqslant \left(\frac{p}{p{-}1}\right)^p\sum_{n=1}^{N}|x_n|^p \leqslant \left(\frac{p}{p{-}1}\right)^p\sum_{n=1}^{\infty}|x_n|^p\,,$$

i.e. $\|Hx\|_p \le \frac{p}{p-1}\|x\|_p$ for all $x \in \ell^p$, and thus $H : \ell^p \to \ell^p$, H is bounded, and $\|H\| \le \frac{p}{p-1}$.

d) We claim that $\|H\| = \frac{p}{p-1}$. For each $N \in \mathbb{N}$, let

$$x_n^{(N)} := n^{-1/p}\ \forall n \le N\,, \quad x_n^{(N)} := 0\ \forall n > N\,.$$

Note that $\|x^{(N)}\|_p^p = \sum_{n=1}^N \frac{1}{n} \to \infty$ as $N \to \infty$. We have

$$\|Hx^{(N)}\|_p^p = \sum_{n=1}^{\infty}|(Hx^{(N)})_n|^p = \sum_{n\le N}\left|\frac{1+\cdots+n^{-1/p}}{n}\right|^p + \sum_{n>N}\left|\frac{1+\cdots+N^{-1/p}}{n}\right|^p$$
$$\ge \sum_{n\le N}\left|\frac{1+\cdots+n^{-1/p}}{n}\right|^p > \sum_{n\le N}\left|\frac{\int_1^n x^{-1/p}dx}{n}\right|^p$$
$$= \left(\frac{p}{p-1}\right)^p\sum_{n\le N}\left|\frac{(n^{1-1/p}-1)}{n}\right|^p = \left(\frac{p}{p-1}\right)^p\sum_{n\le N}\frac{1}{n}\left(1{-}\frac{1}{n^{1-1/p}}\right)^p$$
$$\ge \left(\frac{p}{p-1}\right)^p\sum_{n\le N}\frac{1}{n}\left(1{-}\frac{p}{n^{1-1/p}}\right) = \left(\frac{p}{p-1}\right)^p\sum_{n\le N}\left(\frac{1}{n}{-}\frac{p}{n^{2-1/p}}\right)$$
$$= \left(\frac{p}{p-1}\right)^p\left(\sum_{n\in\mathbb{N}}|x_n^{(N)}|^p - p\sum_{n\le N}\frac{1}{n^{2-1/p}}\right).$$

In particular, if we write $C_N := \sum_{n\leq N} \frac{1}{n^{2-1/p}}$, then

$$\frac{\|Hx^{(N)}\|_p}{\|x^{(N)}\|_p} \geq \frac{p}{p-1}\left(1 - p\frac{C_N}{\|x^{(N)}\|_p^p}\right)^{1/p} \longrightarrow \frac{p}{p-1}$$

as $N \to \infty$, since $(C_N)_{N\in\mathbb{N}}$ converges, and $\|x^{(N)}\|_p^p \to \infty$.

$\square$