ADVANCED ANALYSIS – WiSe 2019/20

Exercise sheet 9

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Exercise 1. [15 points]

Let $1 < p < \infty$. Assume that $\{u_j\}$ is a bounded sequence in $W^{1,p}(\Omega)$. Prove that there exists a subsequence $\{u_{i_k}\}$ and $u \in W^{1,p}(\Omega)$ such that $u_{i_k} \rightharpoonup u$ in L^p and $Du_{i_k} \rightharpoonup Du$ in $L^p(\Omega)$.

Exercise 2. [15 points]

Let $1 \leq p < \infty$. The Sobolev space with zero boundary values $W_0^{1,p}(\Omega)$ is the completion of $C_c^\infty(\Omega)$ with respect to the Sobolev norm ($u \in W_0^{1,p}(\Omega)$ iff there exists $\{u_i\} \in C_c^\infty(\Omega)$ such that $u_i \to u$ in $W^{1,p}(\Omega)$ as $i \to \infty$). The space $W_0^{1,p}(\Omega)$ is endowed with the norm of $W^{1,p}(\Omega)$. Prove that if $u \in W^{1,p}(\Omega)$ ans supp$u$ is a compact set of Ω, then $u \in W_0^{1,p}(\Omega)$.

Exercise 3. [10 points]

Let f_j be a sequence of functions, bounded in $H^1(\mathbb{R}^n)$. Prove that there exist a subsequence f_{j_k} and a function $f \in H^1(\mathbb{R}^n)$ such that $f_{j_k} \rightharpoonup f$ as $j \to \infty$. Prove the same for a sequence a functions bounded in $H^{1/2}(\mathbb{R}^n)$.