ADVANCED ANALYSIS – WiSe 2019/20

Exercise sheet 4

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Exercise 1. [15 points]

Under the same hypothesis of Hardy-Littlewood-Sobolev inequality (Theorem 4.3. in Analysis by Lieb and Loss), prove that

$$\int_0^\infty v(a)^{1-\lambda/n} \int_{a^{p/r}}^\infty \omega(b)\, db da \leq \frac{1}{pr}\left(\frac{\lambda/n}{1-1/r}\right)^{\lambda/n}, \tag{1}$$

where, recall that

$$\omega(b) = \int_{\mathbb{R}^n} \chi_{\{h>b\}}, \qquad v(a) = \int_{\mathbb{R}^n} \chi_{\{f>a\}}. \tag{2}$$

Exercise 2. [10 points]

Let f be a function in $L^1(\mathbb{R}^n)$ and denote by $\widehat{f}$ its Fourier transform. Prove that

1. the map $f \to \hat{f}$ is linear in f,
2. $\widehat{\tau_h f}(k) = e^{-2\pi i(k,h)}\widehat{f}(k)$, $h \in \mathbb{R}^n$
3. $\widehat{\delta_\lambda f}(k) = \lambda^n \widehat{f}(\lambda k)$ $\lambda > 0$,

where τ_h is the translation operator, i.e., $(\tau_h f)(x) = f(x-h)$, and δ_λ is the scaling operator such that $(\delta_\lambda f)(x) = f(x/\lambda)$.

Exercise 3. [15 points]

Let $f \in L^1(\mathbb{R}^n)$ and let $\widehat{f}$ be its Fourier transform. Prove that $\widehat{f}(k) \to 0$ as $|k| \to \infty$.