ADVANCED ANALYSIS – WiSe 2019/20

Exercise sheet 2

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Exercise 1. [10 points]

Let $1 < p < \infty$ and let f_k be a sequence in $L^p(\Omega)$. Consider the set $\widetilde{K} \subset L^p(\Omega)$,

$$\widetilde{K} := \left\{ \sum_{k=1}^{m} d_k f_k, \text{ with } m \text{ arbitrary and } \sum_{k=1}^{m} d_k = 1,\, d_k \geq 0 \right\}$$

and let $K \subset L^p(\Omega)$ be defined as

$$K := \widetilde{K} \cup \left\{ f \in L^p(\Omega) \,|\, f \text{ is a limit of Cauchy sequences of elements of } \widetilde{K} \right\}$$

Prove that K is a closed set.

Exercise 2. [10 points]

Let $\mathcal{H}$ be an Hilbert space. Prove that whenever $\{\ell_n\}$ is a collection of bounded linear functionals on $\mathcal{H}$ such that for every $x \in \mathcal{H}$, $\sup_n |\ell_n(x)| < \infty$. Prove that $\sup_n \|\ell_n\| < \infty$.

Exercise 3. [10 points]

Let $\mathcal{H}$ be an Hilbert space and let K be a convex set in $\mathcal{H}$ which is also a norm closed set. Let $x \in \mathcal{H}$ but not in K and define the distance as

$$D := \text{dist}(x, K) = \inf_{y \in K} \|x - y\|_{\mathcal{H}}$$

Prove that then there exists $\bar{y} \in K$ such that $D = \|x - \bar{y}\|_{\mathcal{H}}$.

Exercise 4. [10 points]

Prove that

1. If $f, g \in L^1(\mathbb{R}^n)$ then $f * g \in L^1(\mathbb{R}^n)$.
2. If $f \in L^p(\mathbb{R}^n)$ and $g \in L^q(\mathbb{R}^n)$ with $\frac{1}{p} + \frac{1}{q} = 1$, then $f * g \in L^\infty(\mathbb{R}^n)$ and $f * g$ tends to 0 at infinity.