Motivic decomposition of a compactification of a Merkurjev-Suslin variety

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Abstract

We provide a motivic decomposition of a twisted form of a smooth hyperplane section of $\mathrm{Gr}(3,6)$. This twisted form is a norm variety corresponding to a symbol in the Milnor K-theory $\mathrm{K}_3^M/3$. As an application we construct a torsion element in the Chow group of this variety.

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1 Introduction

In the present paper we study certain twisted forms of a smooth hyperplane section of $\mathrm{Gr}(3,6)$. These twisted forms are smooth $\mathrm{SL}_1(A)$-equivariant compactifications of a Merkurjev-Suslin variety corresponding to a central simple algebra A of degree 3. On the other hand, these twisted forms are norm (generic splitting) varieties corresponding to symbols in the Milnor K-theory $\mathrm{K}_3^M/3$ given by the Serre-Rost invariant g_3 of Albert algebras. In the present paper we provide a complete decomposition of the Chow motives of these varieties.

The history of this problem goes back to Rost and Voevodsky. Namely, Rost obtained the celebrated decomposition of a norm quadric (see [23]) and later Voevodsky found some direct summand, called a generalized Rost motive, in the motive of any norm variety (see [25]). These varieties as well as Rost motives appearing in their motivic decompositions play an essential

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role in the proof of Milnor's conjecture and the drafts of the proof of the Bloch-Kato conjecture. The latter says that for all natural numbers n, prime numbers p, and fields k of characteristic different from p the Milnor K-theory $\mathrm{K}_n^M(k)/p$ is isomorphic to the étale cohomology $\mathrm{H}^n(k, \mu_p^{\otimes n})$ of the field k with coefficients in the twists of μ_p. We refer the reader to the introduction of paper [25] for an overview and general ideas of the proof of the Bloch-Kato conjecture.

For a field k and a central simple k-algebra A of degree 3 consider a variety D obtained by Galois descent from the variety $V/\operatorname{GL}_1(A_s)$, where $A_s = A \otimes_k k_s$, k_s denotes a separable closure of k and V is the variety of elements (α, β) in $A_s \times A_s$ with rank of $\alpha \oplus \beta$ equal 3 and the reduced norm $\operatorname{Nrd} \alpha = c \operatorname{Nrd} \beta$ for some fixed $c \in k^*$. Note that the F_4-varieties from paper [19] can be considered as a mod-3 analogue of a Pfister quadric (more precisely, of a maximal Pfister neighbour). In turn, our variety, being a norm variety of a symbol in $\mathrm{K}_3^M(k)/3$, can be considered as a mod-3 analogue of a norm quadric (see [24] § 1). Thus, there is a close connection between our variety and F_4-varieties.

The main result of the present paper (Theorem 3.7) asserts that the Chow motive of D is isomorphic to a direct sum of a non-pure motive R and some shifted copies of the motives of Severi-Brauer varieties $\mathcal{M}(\mathrm{SB}(A))$. The motive R has the property that over k_s it becomes isomorphic to a direct sum of shifted Lefschetz motives $\mathbb{Z} \oplus \mathbb{Z}(4) \oplus \mathbb{Z}(8)$ (cf. [25] formula between (5.4) and (5.5)).

As an application of this motivic decomposition we show that if D is anisotropic then the Chow group $\mathrm{CH}_2(D)$ has 3-torsion (Theorem 4.1).

The paper is organized as follows. In section 2 we provide background information on the category of Chow motives, Schubert calculus, and Steenrod operations. In section 3 we define a smooth compactification of a Merkurjev-Suslin variety $\mathrm{MS}(A, c)$ with A a central simple algebra of degree 3, describe its geometrical properties, and decompose its Chow motive. Section 4 is devoted to an application of the obtained motivic decomposition. Namely, using the ideas of Karpenko and Merkurjev we construct a 3-torsion element in the Chow group of our variety.

The main ingredients of our proofs are results of Białynicki-Birula [2], the Lefschetz hyperplane theorem, and the Segre embedding.

2 Notation

2.1. Let k denote a field. We use Galois descent language, i.e., identify a (quasi-projective) variety X over a field k with the variety $X_s = X \times_{\operatorname{Spec} k} \operatorname{Spec} k_s$ over a separable closure k_s equipped with an action of the absolute Galois group $\Gamma = \operatorname{Gal}(k_s/k)$. The set of k-rational points of X is precisely the set of k_s-rational points of X_s stable under the action of Γ.

We consider the Chow group $\mathrm{CH}^i(X)$ (resp. $\mathrm{CH}_i(X)$) of classes of algebraic cycles of codimension i (resp. of dimension i) on an irreducible algebraic variety X modulo rational equivalence (see [10]).

The Poincaré polynomial or generating function for a variety X is, by definition, the polynomial $\sum a_i t^i \in \mathbb{Z}[t]$ with $a_i = \operatorname{rk} \mathrm{CH}^i(X)$.

2.2 (Schubert calculus). The structure of the Chow ring of a Grassmann variety is of particular interest for us. We do a lot of computations using formulae from Schubert calculus (see [10] 14.7) which we briefly describe below.

Consider the Grassmann variety $\mathrm{Gr}(d+1, n+1)$ of $(d+1)$-dimensional subspaces in the affine space of dimension $n+1$. For each partition $\lambda = (\lambda_0, \ldots, \lambda_d)$ with $n-d \geq \lambda_0 \geq \ldots \geq \lambda_d \geq 0$ there exists a canonical generator

$$\mathit{\Delta}_\lambda \in \mathrm{CH}^{|\lambda|}(\mathrm{Gr}(d+1, n+1)) \tag{1}$$

with $|\lambda| = \sum_{i=0}^d \lambda_i$, called *a Schubert class*. The Schubert classes form a free $\mathbb{Z}$-basis of $\mathrm{CH}^*(\mathrm{Gr}(d+1, n+1))$.

The multiplication is determined by Pieri's formula:

$$\mathit{\Delta}_\lambda \cdot \sigma_m = \sum \mathit{\Delta}_\mu, \tag{2}$$

where the sum runs over μ with $n-d \geq \mu_0 \geq \lambda_0 \geq \ldots \geq \mu_d \geq \lambda_d$, $|\mu| = |\lambda| + m$ and $\sigma_m = \mathit{\Delta}_{(m,0,\ldots,0)}$ is *a special Schubert class.*

If $|\lambda| + |\mu| = (d+1)(n-d) = \dim \mathrm{Gr}(d+1, n+1)$, one has the Poincaré duality theorem:

$$\mathit{\Delta}_\lambda \cdot \mathit{\Delta}_\mu = \Big(\prod_{i=0}^{d} \delta_{\lambda_i + \mu_{d-i}, n-d}\Big)\, \mathit{\Delta}_{(n-d,\ldots,n-d)}\,. \tag{3}$$

2.3. Let A be a central simple k-algebra of degree n. A generalized Severi-Brauer variety $\mathrm{SB}_d(A)$ is the variety of right ideals of reduced dimension d in A (see [15] Definition (1.16)). This variety is a twisted form of $\mathrm{Gr}(d,n)$ (see [15] Theorem (1.18)). For $d=1$ we write $\mathrm{SB}(A)=\mathrm{SB}_1(A)$ for the usual Severi-Brauer variety. For two central simple algebras A and B there exists a Segre morphism

$$\mathrm{Seg}\colon\ \mathrm{SB}_d(A)\times\mathrm{SB}_{d'}(B)\to\mathrm{SB}_{dd'}(A\otimes_k B) \tag{4}$$

given by the tensor product of ideals.

We also assume that the reader is familiar with Chern classes and the tautological vector bundle on Grassmannians (see [10]).

2.4 (Lefschetz's hyperplane theorem). Assume that $\mathrm{char}\, k=0$ and the field k is algebraically closed. Let $X\subset\mathbb{P}^n$ be a smooth projective cellular variety over k, i.e., a variety which admits a filtration

$$\emptyset=X_{-1}\subset X_0\subset\ldots\subset X_m=X \tag{5}$$

by closed subvarieties such that $X_i\setminus X_{i-1}$ is a disjoint union of affine spaces for all $i=0,\ldots,m$.

Let $Y\overset{\imath}{\hookrightarrow}X$ be a smooth hyperplane section of X. Assume that Y is a cellular variety. The Lefschetz hyperplane theorem (see [11]) asserts that the pull-back $\imath^*\colon\ \mathrm{CH}^i(X)\to\mathrm{CH}^i(Y)$ is an isomorphism for $0\le i<[\frac{\dim Y}{2}]$, is injective for $i=[\frac{\dim Y}{2}]$, and the push-forward map with $\mathbb{Q}$-coefficients $\imath_*\otimes\mathbb{Q}\colon\ \mathrm{CH}_i(Y)\otimes\mathbb{Q}\to\mathrm{CH}_i(X)\otimes\mathbb{Q}$ is an isomorphism for $0\le i<[\frac{\dim Y}{2}]$, and is surjective for $i=[\frac{\dim Y}{2}]$.

2.5 ($\mathbb{G}_m$-varieties). We shall need the following result of Białynicki-Birula (see [2]). Let k be an algebraically closed field of arbitrary characteristic. Let X be a smooth projective variety over k equipped with an action of the multiplicative group $\mathbb{G}_m$ such that the fixed point locus $X^{\mathbb{G}_m}$ consists of a finite number r of isolated points. Then the variety X is a disjoint union of r affine cells. In particular, X is cellular.

There are further generalizations of this statement to the case of an arbitrary field k (see [5] Theorem 3.2). We don't use them in the present paper.

Next we introduce the category of Chow motives over a field k following [16] and [8]. We recall the notion of a rational cycle and state the Rost Nilpotence Theorem following [7].

2.6. Let k be an arbitrary field and $\mathcal{V}ar_k$ be the category of smooth projective varieties over k. For any variety X we set $\mathrm{Ch}(X) := \mathrm{CH}(X) \otimes_{\mathbb{Z}} \mathbb{Z}/3$. First, we define the category of *correspondences with* $\mathbb{Z}/3$*-coefficients* (over k) denoted by $\mathcal{C}or_k$. Its objects are smooth projective varieties over k. For morphisms, called correspondences, we set $\mathrm{Mor}(X, Y) := \coprod_{i=1}^{n} \mathrm{Ch}_{d_i}(X_i \times Y)$, where $X_1, \ldots, X_n$ are the irreducible components of X of dimensions $d_1, \ldots, d_n$. For any two correspondences $\alpha \in \mathrm{Ch}(X \times Y)$ and $\beta \in \mathrm{Ch}(Y \times Z)$ we define their composition $\beta \circ \alpha \in \mathrm{Ch}(X \times Z)$ as

$$\beta \circ \alpha = \mathrm{pr}_{13*}(\mathrm{pr}_{12}^*(\alpha) \cdot \mathrm{pr}_{23}^*(\beta)), \tag{6}$$

where pr_{ij} denotes the projection on the i-th and j-th factors of $X \times Y \times Z$ respectively and pr_{ij_*}, pr_{ij}^* denote the induced push-forwards and pull-backs for Chow groups.

The pseudo-abelian completion of $\mathcal{C}or_k$ is called the category of *Chow motives with* $\mathbb{Z}/3$*-coefficients* and is denoted by $\mathcal{M}_k$. The objects of $\mathcal{M}_k$ are pairs (X, p), where X is a smooth projective variety and $p \in \mathrm{Mor}(X, X)$ is an idempotent, that is, $p \circ p = p$. The morphisms between two objects (X, p) and (Y, q) are the compositions $q \circ \mathrm{Mor}(X, Y) \circ p$.

2.7. By construction, $\mathcal{M}_k$ is a tensor additive category, where the tensor product is given by the usual product $(X, p) \otimes (Y, q) = (X \times Y, p \times q)$. For any cycle $\alpha \in \mathrm{Ch}(X \times Y)$ we denote by α^t the corresponding transposed cycle.

2.8. Observe that the composition product $\circ$ induces the ring structure on the abelian group $\mathrm{Mor}(X, X)$. The unit element of this ring is the class of the diagonal map Δ_X, which is defined by $\Delta_X \circ \alpha = \alpha \circ \Delta_X = \alpha$ for all $\alpha \in \mathrm{Mor}(X, X)$. The motive (X, Δ_X) will be denoted by $\mathcal{M}(X)$ and called the (Chow) motive of X.

2.9. Consider the morphism $(e, \mathrm{id})\colon \{pt\} \times \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1$. The image by means of the induced push-forward $(e, \mathrm{id})_*(1)$ does not depend on the choice of the point $e\colon \{pt\} \to \mathbb{P}^1$ and defines a projector in $\mathrm{CH}_1(\mathbb{P}^1 \times \mathbb{P}^1)$ denoted by p_1. The motive $\mathbb{Z}(1) = (\mathbb{P}^1, p_1)$ is called *Lefschetz motive.* For a motive M and a nonnegative integer i we denote its twist by $M(i) = M \otimes \mathbb{Z}(1)^{\otimes i}$.

From now on we assume that all varieties under consideration are irreducible.

2.10. An isomorphism between twisted motives $(X,p)(m)$ and $(Y,q)(l)$ is given by correspondences $j_1 \in q \circ \mathrm{Ch}_{\dim X+m-l}(X \times Y) \circ p$ and $j_2 \in p \circ \mathrm{Ch}_{\dim Y+l-m}(Y \times X) \circ q$ such that $j_1 \circ j_2 = q$ and $j_2 \circ j_1 = p$.

2.11. Let X be a smooth projective cellular variety. The abelian group structure of $\mathrm{CH}(X)$ is well-known. Namely, X has a cellular filtration and the generators of Chow groups of the bases of this filtration correspond to the free additive generators of $\mathrm{CH}(X)$. Note that the product of two cellular varieties $X \times Y$ has a cellular filtration as well, and $\mathrm{CH}^*(X \times Y) \simeq \mathrm{CH}^*(X) \otimes \mathrm{CH}^*(Y)$ as graded rings. The correspondence product of two homogeneous cycles $\alpha = f_\alpha \times g_\alpha \in \mathrm{Ch}(X \times Y)$ and $\beta = f_\beta \times g_\beta \in \mathrm{Ch}(Y \times X)$ is given by (see [3] Lemma 5)

$$(f_\beta \times g_\beta) \circ (f_\alpha \times g_\alpha) = \deg(g_\alpha \cdot f_\beta)(f_\alpha \times g_\beta), \tag{7}$$

where $\deg\colon \mathrm{Ch}(Y) \to \mathrm{Ch}(pt) = \mathbb{Z}/3$ is the degree map. This formula is an obvious consequence of intersection theory.

2.12. Let X be a projective variety of dimension n over a field k. Let k_s be a separable closure of k and $X_s = X \times_{\mathrm{Spec}\, k} \mathrm{Spec}\, k_s$. We say a cycle $J \in \mathrm{Ch}(X_s)$ is *rational* if it lies in the image of the natural homomorphism $\mathrm{Ch}(X) \to \mathrm{Ch}(X_s)$. For instance, there is an obvious rational cycle Δ_{X_s} in $\mathrm{Ch}_n(X_s \times X_s)$ that is given by the diagonal class. Clearly, all linear combinations, intersections and correspondence products of rational cycles are rational.

2.13 (Rost Nilpotence). Finally, we shall also use the following fact (see [7] Theorem 8.2) called Rost Nilpotence theorem. Let X be a projective homogeneous variety over k. Then for any field extension l/k the kernel of the natural ring homomorphism $\mathrm{End}(\mathcal{M}(X)) \to \mathrm{End}(\mathcal{M}(X_l))$ consists of nilpotent elements.

In the last section of the present paper we use Steenrod operations and the notion of connected class of fields.

2.14 (Steenrod operations). We briefly recall the basic properties of Steenrod operations modulo 3 following Brosnan [4]. Note that these operations were defined before by Voevodsky in the context of motivic cohomology.

Let X be a smooth projective variety over a field k with $\mathrm{char}\, k \neq 3$. For every $i \geq 0$ there exist certain homomorphisms $S^i\colon \mathrm{Ch}^*(X) \to \mathrm{Ch}^{*+2i}(X)$

called Steenrod operations. The total Steenrod operation is the sum $S = S_X = S^0 + S^1 + \ldots \colon \mathrm{Ch}(X) \to \mathrm{Ch}(X)$. This map is a ring homomorphism. The restriction $S^i|_{\mathrm{Ch}^n(X)}$ is 0 for $i > n$ and is the map $\alpha \mapsto \alpha^3$ for $n = i$. The map S^0 is the identity. Moreover, the total Steenrod operation commutes with pull-backs and, in particular, preserves rationality of cycles.

The following Riemann-Roch formula holds for the total Steenrod operation:

$$f_*(S_Y(\alpha) \cdot c(-T_Y)) = S_X(f_*(\alpha)) \cdot c(-T_X), \tag{8}$$

where $f\colon Y \to X$ is a morphism of smooth projective k-varieties, f_* denotes the induced push-forward, c is the total Chern class, T_X is the tangent bundle of X and $c(-T_X) = c(T_X)^{-1}$.

2.15 (Connected class of fields and R-equivalence). Following [9] let X be a scheme over k and $\mathcal{A}(X)$ be the class of all field extensions K/k such that $X(K) \neq \emptyset$. In [9] Section 6 Chernousov and Merkurjev introduce the notion of X-equivalence on $\mathcal{A}(X)$ and a connectedness of $\mathcal{A}(X)$ with respect to this equivalence relation and prove the following statement ([9] Theorem 6.5): Let X be a proper scheme of finite type over k with $\overline{\mathrm{CH}}_0(X_L) := \mathrm{Ker}(\deg\colon \mathrm{CH}_0(X_L) \to \mathbb{Z}) = 0$ for any $L \in \mathcal{A}(X)$. Assume that the class $\mathcal{A}(X)$ is connected. Then $\overline{\mathrm{CH}}_0(X) = 0$.

Moreover, they show ([9] Theorem 11.3) that the class of all splitting fields of an Albert algebra arising from the first Tits construction over 3-special fields is connected (see also [15] Chapter IX).

Next we briefly recall the notion of R-equivalence. We say that two points $p, q \in X(k)$ are elementarily linked if there exists a rational morphism $\varphi\colon \mathbb{P}^1_k \dashrightarrow X$ such that $p, q \in \mathrm{Im}(\varphi(k))$. The R-equivalence relation is the equivalence relation generated by this relation.

We use this notion as well as results of Chernousov-Merkurjev once in the last section in order to show that $\overline{\mathrm{CH}}_0(D) = 0$ for our compactification D of a Merkurjev-Suslin variety.

3 Motivic decomposition

From now on we assume the characteristic of the base field k is 0. We need this assumption for validity of the Lefschetz hyperplane theorem.

It is well-known (see [11] Ch. 1, § 5, p. 193) that the Grassmann variety $\mathrm{Gr}(l, n)$ can be represented as the variety of $l \times n$ matrices of rank l modulo an obvious action of the group GL_l. Having this in mind we give the following definition.

3.1 Definition. Let A be a central simple algebra of degree 3 over a field k, $c \in k^*$. Fix an isomorphism $A_s \simeq \mathrm{M}_3(k_s)$. Consider the variety $D = D(A, c)$ obtained by Galois descent from the variety

(9)

$$\{\alpha \oplus \beta \in (A \oplus A)_s \simeq \mathrm{M}_{3,6}(k_s) \mid \mathrm{rk}(\alpha \oplus \beta) = 3,\ \mathrm{Nrd}(\alpha) = c\,\mathrm{Nrd}(\beta)\}/\,\mathrm{GL}_1(A_s),$$

where $\mathrm{GL}_1(A_s)$ acts on $A_s \oplus A_s$ by the left multiplication and Nrd stands for the reduced norm (see [15] § 1).

This variety was first considered by M. Rost.

Consider the Plücker embedding of $\mathrm{Gr}(3, 6)$ into a projective space (see [11] Ch. 1, § 5, p. 209). It is obvious that under this embedding for all $c \in k^*$ the variety $D(\mathrm{M}_3(k), c)$ is a hyperplane section of $\mathrm{Gr}(3, 6)$.

Moreover, it is easy to see from the definition of D that the closed embedding $D_s \to \mathrm{Gr}(3, 6)$ over k_s gives rise to a closed embedding $\imath \colon D \to \mathrm{SB}_3(\mathrm{M}_2(A))$ over k.

3.2 Lemma. *The variety D is smooth.*

Proof. (M. Florence) We can assume k is separably closed. Consider first the variety

(10)

$$V = \{\alpha \oplus \beta \in \mathrm{M}_3(k) \oplus \mathrm{M}_3(k) = \mathrm{M}_{3,6}(k) \mid \mathrm{rk}(\alpha \oplus \beta) = 3,\ \det(\alpha) = c \det(\beta)\}.$$

An easy computation of differentials shows that V is smooth. The variety V is a GL_3-torsor over D and, since GL_3 is smooth, the torsor V is locally trivial for étale topology. Therefore to prove the smoothness of D we can assume that the torsor V is split.

Since $D \times_k \mathrm{GL}_3$ is smooth, $D \times_k \mathrm{M}_3$ is also smooth. Therefore it suffices to prove that if $D \times_k \mathbb{A}^1$ is smooth, then D is smooth. But this is true for any variety. Indeed, for any point x on D we have $T_{(x,0)}(D \times_k \mathbb{A}^1) = T_x D \oplus T_0 \mathbb{A}^1 = T_x D \oplus k$ and $\dim T_x D = \dim T_{(x,0)}(D \times_k \mathbb{A}^1) - 1 = \dim(D \times_k \mathbb{A}^1) - 1 = \dim D$. □

3.3 Remark. One can associate to the variety D a Serre-Rost invariant $g_3(D) = (A) \cup (c) \in \mathrm{H}^3(k, \mathbb{Z}/3)$ (see [15] § 40). This invariant is trivial if and only if D is isotropic.

It is easy to see that $D^0 := \mathrm{MS}(A,c) := \{\alpha \in A \mid \mathrm{Nrd}(\alpha) = c\}$ is an open orbit under the natural right $\mathrm{SL}_1(A)$- or $\mathrm{SL}_1(A) \times \mathrm{SL}_1(A)$-action on D. Namely, the open orbit consists of all $\alpha \oplus \beta$ with $\mathrm{rk}(\alpha) = 3$. D^0 is called a Merkurjev-Suslin variety. In other words, the variety $D(A,c)$ is a smooth $\mathrm{SL}_1(A)$-equivariant compactification of the Merkurjev-Suslin variety $\mathrm{MS}(A,c)$.

3.4 Lemma. *For the variety D_s the following properties hold.*

1. There exists a $\mathbb{G}_m$-action on D_s with 18 fixed points. In particular, D_s is a cellular variety.

2. The generating function for $\mathrm{CH}(D_s)$ is equal to $g = t^8 + t^7 + 2t^6 + 3t^5 + 4t^4 + 3t^3 + 2t^2 + t + 1$.

3. The natural homomorphism $\mathrm{Pic}(D) \to \mathrm{Pic}(D_s)$ is an isomorphism. In particular, the Picard group $\mathrm{Pic}(D_s)$ is rational.

Proof. 1. We can assume $c = 1$. The right action of $\mathbb{G}_m$ on D_s is induced by the following action:

$$\begin{aligned}(\mathrm{M}_3(k_s) \oplus \mathrm{M}_3(k_s)) \times \mathbb{G}_m(k_s) &\to \mathrm{M}_3(k_s) \oplus \mathrm{M}_3(k_s)\\ (\alpha \oplus \beta, \lambda) &\mapsto \alpha\,\mathrm{diag}(\lambda, \lambda^5, \lambda^6) \oplus \beta\,\mathrm{diag}(\lambda^2, \lambda^3, \lambda^7)\end{aligned}$$

The exponents $1, 5, 6, 2, 3, 7$ of λ's are choosen in such a way that all of them are different and the sum of the first three equals the sum of the last three. Then the relation $\det\alpha = \det\beta$ is preserved.

The 18 fixed points of D_s are the $\binom{6}{3} = 20$ 3-dimensional standard subspaces in $\mathrm{Gr}(3,6)$ minus 2 subspaces, generated by the first and by the last 3 standard basis vectors. By the result of Białynicki-Birula (see 2.5) the variety D_s is cellular.

2. By the Lefschetz hyperplane theorem (see 2.4) the pull-back $\imath_s^*$ is an isomorphism in codimensions $i < \frac{\dim(\mathrm{Gr}(3,6))-1}{2}$. Therefore $\mathrm{rk}\,\mathrm{CH}^i(D_s) = \mathrm{rk}\,\mathrm{CH}^i(\mathrm{Gr}(3,6))$ for such i's. Since Poincaré duality holds, we have $\mathrm{rk}\,\mathrm{CH}_i(D_s) = \mathrm{rk}\,\mathrm{CH}_i(\mathrm{Gr}(3,6))$ for $i < \frac{\dim(\mathrm{Gr}(3,6))-1}{2} = 4$.

It remains to determine only the rank in the middle codimension. To do this observe that $\mathrm{rk}\,\mathrm{CH}^*(D_s) = 18$ (see 2.5). Therefore $\mathrm{rk}\,\mathrm{CH}^4(D_s) = 2\mathrm{rk}\,\mathrm{CH}^4(\mathrm{Gr}(3,6)) - 2 = 4$.

3. Consider the following commutative diagram:

(11)
$$\begin{array}{ccc} \mathrm{Pic}(\mathrm{SB}_3(\mathrm{M}_2(A))) & \xrightarrow{\imath^*} & \mathrm{Pic}(D) \\ \downarrow & & \downarrow \mathrm{res}^* \\ \mathrm{Pic}(\mathrm{Gr}(3,6)) & \xrightarrow{\imath_s^*} & \mathrm{Pic}(D_s) \end{array}$$

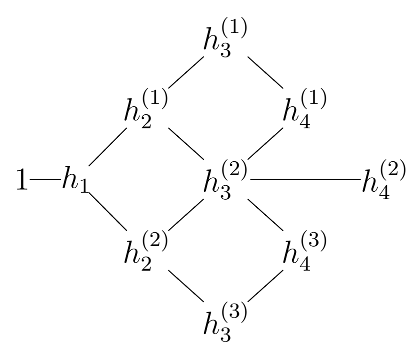
where the vertical arrows are the morphisms of scalar extension. By the Lefschetz hyperplane theorem the map $\imath_s^*$ restricted to $\mathrm{Pic}(\mathrm{Gr}(3,6))$ is an isomorphism. Since $\mathrm{Pic}(\mathrm{SB}_3(\mathrm{M}_2(A)))$ is rational (see [18] and [19] Lemma 4.3), i.e., the left vertical arrow is an isomorphism, the restriction map res^* is surjective. On the other hand, it immediatelly follows from the Hochschild-Serre spectral sequence (see [1] § 2) that $\mathrm{Pic}(D)$ can be identified with a subgroup of $\mathrm{Pic}(D_s)$. Since $\mathrm{Pic}(D_s) \simeq \mathbb{Z}$, we are done. □

3.5 Remark. It immediately follows from this Lemma that the variety D is not a twisted flag variety. Indeed, the generating functions of all twisted flag varieties over a separabely closed field are well-known and all of them are different from the generating function of D_s.

3.6. We must determine partially the multiplicative structure of $\mathrm{CH}(D_s)$. By the Lefschetz hyperplane theorem the generators in codimensions 0, 1, 2, and 3 are pull-backs of the canonical generators $\Delta_{(0,0,0)}$, $\Delta_{(1,0,0)}$, $\Delta_{(1,1,0)}$, $\Delta_{(2,0,0)}$, $\Delta_{(1,1,1)}$, $\Delta_{(2,1,0)}$, $\Delta_{(3,0,0)}$ of $\mathrm{Gr}(3,6)$ (see 2.2 and [10] 14.7). We denote these pull-backs as 1, h_1, $h_2^{(1)}$, $h_2^{(2)}$, $h_3^{(1)}$, $h_3^{(2)}$ and $h_3^{(3)}$, respectively. In codimension 4 the pull-back is injective and the pull-backs $h_4^{(1)} := \imath_s^*(\Delta_{(2,1,1)})$, $h_4^{(2)} := \imath_s^*(\Delta_{(2,2,0)})$, $h_4^{(3)} := \imath_s^*(\Delta_{(3,1,0)})$, where $\imath$ is as above, form a subbasis of $\mathrm{CH}^4(D_s)$.

Consider the following diagram, which visualizes the multiplicative structure of $\mathrm{CH}(D_s)$:

(12)



Since pull-backs are ring homomorphisms, it immediately follows from the Pieri formula (2) that

$$h_1 \cdot u = \sum_{u \to v} v, \tag{13}$$

where u is a vertex in the diagram which corresponds to a generator of codimension less than 4, and the sum runs through all the edges going from u one step to the right.

Next we compute some products in the middle codimension using in the main Poincaré duality (3).

Since $\varDelta_{(3,1,0)}\,\varDelta_{(2,1,1)} = \varDelta^2_{(2,2,0)} = 0$ and $\varDelta^2_{(2,1,1)} = \varDelta^2_{(3,1,0)} = \varDelta_{(2,2,0)}\,\varDelta_{(2,1,1)} = \varDelta_{(2,2,0)}\,\varDelta_{(3,1,0)} = \varDelta_{(3,3,2)}$ (see 2.2 and [10] 14.7), we have $h_4^{(1)}h_4^{(3)} = (h_4^{(2)})^2 = 0$ and $(h_4^{(1)})^2 = (h_4^{(3)})^2 = h_4^{(2)}h_4^{(3)} = h_4^{(1)}h_4^{(2)} = \imath_s^*(\varDelta_{(3,3,2)}) = pt$, where pt denotes the class of a rational point on D_s.

The next theorem shows that the Chow motive of D with $\mathbb{Z}/3$-coefficients is decomposable. Note that for any cycle h in $\mathrm{CH}(D_s)$ or in $\mathrm{CH}(D_s \times D_s)$ the cycle $3h$ is rational.

3.7 Theorem. *Let A denote a central simple algebra of degree 3 over a field k, $c \in k^*$, and $D = D(A, c)$. Then*

$$\mathcal{M}(D) \simeq R \oplus (\oplus_{i=1}^5 R'(i)), \tag{14}$$

where R is a motive such that over a separably closed field it becomes isomorphic to $\mathbb{Z} \oplus \mathbb{Z}(4) \oplus \mathbb{Z}(8)$ and $R' \simeq \mathcal{M}(\mathrm{SB}(A))$.

Proof. Consider the following commutative diagram (see [6] 5.5):

$$\begin{array}{ccccc}
D_s \times \mathbb{P}^2 & \xrightarrow{\imath_s \times \mathrm{id}_s} & \mathrm{Gr}(3,6) \times \mathbb{P}^2 & \xrightarrow{\mathrm{Seg}_s} & \mathrm{Gr}(3,18) \\
\downarrow & & \downarrow & & \downarrow \\
D \times \mathrm{SB}(A^{\mathrm{op}}) & \xrightarrow{\imath \times \mathrm{id}} & \mathrm{SB}_3(\mathrm{M}_2(A)) \times \mathrm{SB}(A^{\mathrm{op}}) & \xrightarrow{\mathrm{Seg}} & \mathrm{SB}_3(\mathrm{M}_2(A) \otimes_k A^{\mathrm{op}})
\end{array} \tag{15}$$

where the right horizontal arrows are Segre embeddings given by the tensor product of ideals (resp. linear subspaces) and the vertical arrows are canonical maps induced by the scalar extension k_s/k.

This diagram induces the commutative diagram of rings

(16)

$$\begin{array}{ccccc}
\mathrm{Ch}^*(D_s \times \mathbb{P}^2) & \xleftarrow[(\imath_s \times \mathrm{id}_s)^*]{} & \mathrm{Ch}^*(\mathrm{Gr}(3,6) \times \mathbb{P}^2) & \xleftarrow[\mathrm{Seg}_s^*]{} & \mathrm{Ch}^*(\mathrm{Gr}(3,18)) \\
\uparrow & & \uparrow & & \uparrow \simeq \\
\mathrm{Ch}^*(D \times \mathrm{SB}(A^{\mathrm{op}})) & \xleftarrow[(\imath \times \mathrm{id})^*]{} & \mathrm{Ch}^*(\mathrm{SB}_3(\mathrm{M}_2(A)) \times \mathrm{SB}(A^{\mathrm{op}})) & \xleftarrow[\mathrm{Seg}^*]{} & \mathrm{Ch}^*(\mathrm{SB}_3(\mathrm{M}_2(A) \otimes_k A^{\mathrm{op}}))
\end{array}$$

Observe that the right vertical arrow is an isomorphism since $\mathrm{M}_2(A) \otimes_k A^{\mathrm{op}}$ splits.

Let τ_3 and τ_1 be tautological vector bundles on $\mathrm{Gr}(3,6)$ and $\mathbb{P}^2$ respectively and let e denote the Euler class (the top Chern class). By [6] Lemma 5.7 the cycle $(\imath_s \times \mathrm{id}_s)^*(e(\mathrm{pr}_1^*\tau_3 \otimes \mathrm{pr}_2^*\tau_1)) \in \mathrm{Ch}(D_s \times \mathbb{P}^2)$ is rational.

We have $c(\tau_3) = \dfrac{1}{1+\sigma_1+\sigma_2+\sigma_3} = 1 - \sigma_1 - \sigma_2 - \sigma_3 + \sigma_1^2 + 2\sigma_1\sigma_2 - \sigma_1^3 = 1 - \sigma_1 + \mathit{\Delta}_{(1,1,0)} - \mathit{\Delta}_{(1,1,1)} \in \mathrm{Ch}(\mathrm{Gr}(3,6))$, where σ_i's denote special Schubert classes (see 2.2 and [6] 5.9). Therefore by the formula for Chern classes of a tensor product (see [10] 3.2) we have $r := -(\imath_s \times \mathrm{id}_s)^*(e(\mathrm{pr}_1^*\tau_3 \otimes \mathrm{pr}_2^*\tau_1)) = h_3^{(1)} \times 1 + h_2^{(1)} \times H + h_1 \times H^2 \in \mathrm{Ch}^3(D_s \times \mathbb{P}^2)$ (cf. [6] 5.10 and 5.11), where $H = -c_1(\tau_1)$ is the class of a smooth hyperplane section of $\mathbb{P}^2$.

Define the following rational cycles $\rho_i = r(h_1^i \times 1) \in \mathrm{Ch}^{3+i}(D_s \times \mathbb{P}^2)$ for $i = 1, \ldots, 4$, $\rho_0 = r + h_1^3 \times 1 \in \mathrm{Ch}^3(D_s \times \mathbb{P}^2)$ and $\rho_1' = r(h_1 \times 1) + h_1^4 \times 1$. A straightforward computation using the multiplication rules in 3.6 shows that $(-\rho_1') \circ \rho_3^t$ as well as $(-\rho_{4-i}) \circ \rho_i^t \in \mathrm{Ch}_2(\mathbb{P}^2 \times \mathbb{P}^2)$ is the diagonal $\Delta_{\mathbb{P}^2}$. Moreover, the opposite compositions $(-\rho_0)^t \circ \rho_4$, $(-\rho_1)^t \circ \rho_3$, $(-\rho_2)^t \circ \rho_2$, $(-\rho_3)^t \circ \rho_1'$, and $(-\rho_4)^t \circ \rho_0$ give rational pairwise orthogonal idempotents in $\mathrm{Ch}_8(D_s \times D_s)$.

To finish the proof of the theorem it remains by 2.10 to lift all these rational cycles ρ_i, ρ_j^t to $\mathrm{Ch}(D \times \mathrm{SB}(A^{\mathrm{op}}))$ and to $\mathrm{Ch}(\mathrm{SB}(A^{\mathrm{op}}) \times D)$ respectively in such a way that the corresponding compositions of their preimages give the diagonal $\Delta_{\mathrm{SB}(A^{\mathrm{op}})}$.

Fix an $i = 0, \ldots, 4$. Consider first any preimage $\alpha \in \mathrm{Ch}(D \times \mathrm{SB}(A^{\mathrm{op}}))$ of $-\rho_{4-i}$ and any preimage $\beta \in \mathrm{Ch}(\mathrm{SB}(A^{\mathrm{op}}) \times D)$ of ρ_i^t. These preimages exist, since the cycles ρ_i, ρ_{4-i} are rational. The image of the composition $\alpha \circ \beta$ under the restriction map is the diagonal $\Delta_{\mathbb{P}^2}$. Therefore by the Rost Nilpotence theorem for Severi-Brauer varieties (see 2.13) $\alpha \circ \beta = \Delta_{\mathrm{SB}(A^{\mathrm{op}})} + n$, where n is a nilpotent element in $\mathrm{End}(\mathcal{M}(\mathrm{SB}(A^{\mathrm{op}})))$. Since n is nilpotent $\alpha \circ \beta$ is invertible and $((\Delta_{\mathrm{SB}(A^{\mathrm{op}})} + n)^{-1} \circ \alpha) \circ \beta = \Delta_{\mathrm{SB}(A^{\mathrm{op}})}$. In other words, we can take $(\Delta_{\mathrm{SB}(A^{\mathrm{op}})} + n)^{-1} \circ \alpha$ as a preimage of $-\rho_{4-i}$ and β as a preimage of ρ_i^t (note that n is always a torsion element and since by [13] Criterion 7.1 $\mathrm{End}(\mathcal{M}(\mathrm{SB}(A^{\mathrm{op}}))) \simeq \mathrm{Mor}(\mathcal{M}(\mathrm{SB}(A)), \mathcal{M}(\mathrm{SB}(A^{\mathrm{op}})))$ and by [20] Theorem 2.3.7 $\mathrm{CH}(\mathrm{SB}(A))$ has no torsion, the projective bundle theorem (see [12] Proposition 4.3) implies that in fact $n = 0$).

Denote as R the remaining direct summand of the motive of D. Comparing the left and the right hand sides of the decomposition over k_s we see that the Poincaré polynomial of R_s over k_s is the difference $P(D_s) - \sum_{i=1}^{5} P(\mathbb{P}^2)t^i = 1+t^4+t^8 \in \mathbb{Z}[t]$, where $P(-)$ denotes the respective Poincaré polynomials. This implies that $R_s \simeq \mathbb{Z} \oplus \mathbb{Z}(4) \oplus \mathbb{Z}(8)$. □

3.8 Remark. Using messier computations one can show that the same proof works for the motive of D with integral coefficients. On the other hand, it follows from the general result [22] Theorem 2.16 that one can lift a modulo p motivic decomposition to a decomposition with $\mathbb{Z}$-coefficients for any prime number p.

4 Torsion

In this section we use Steenrod operations modulo 3 (see [4], [14] § 3, [17], and 2.14). We denote the total Steenrod operation by $S^\bullet = S^0 + S^1 + \ldots$.

Let X be a smooth projective variety over k. For any cycle $p \in \mathrm{CH}(X \times X)$ we define its realization $p_\star \colon \mathrm{CH}(X) \to \mathrm{CH}(X)$ as $p_\star(\alpha) = \mathrm{pr}_{2*}(\mathrm{pr}_1^*(\alpha)p)$, $\alpha \in \mathrm{CH}(X)$, where $\mathrm{pr}_1, \mathrm{pr}_2 \colon X \times X \to X$ denote the first and the second projections. As $\deg \colon \mathrm{CH}_0(X) \to \mathbb{Z}$ we denote the usual degree map.

The goal of the present section is to prove the following theorem.

4.1 Theorem. *Assume that the variety D is anisotropic. Then $\mathrm{CH}_2(D)$ contains 3-torsion.*

4.2. The proof of this Theorem consists of several parts. First we define an important element d as follows. The kernel of the push-forward map

$$(\imath_s)_*|_{\mathrm{CH}_4(D_s)} \colon \mathrm{CH}_4(D_s) \to \mathrm{CH}_4(\mathrm{Gr}(3,6)) \tag{17}$$

has rank 1, since by the Lefschetz hyperplane theorem (see 2.4) the push-forward $((\imath_s)_*|_{\mathrm{CH}_4(D_s)}) \otimes \mathbb{Q}$ is surjective. Denote as $d \in \mathrm{CH}_4(D_s)$ a generator of this kernel. The projection formula immediately implies that $(\imath_s)_*(\alpha d) = 0$ for any $\alpha \in \mathrm{Im}\, \imath_s^*$ and therefore by the Lefschetz hyperplane theorem $\alpha d = 0$.

4.3 Lemma. *We have*

1. $d^2 \neq 0 \mod 3$,

2. *the total Chern class of the tangent bundle*

$$c(-T_{D_s}) = 1 + h_1 + h_1^2 - h_1^3 - h_1^4 - h_1^5 \in \mathrm{Ch}(D_s), \tag{18}$$

and

3. $S^\bullet(d) = d$.

Proof. The first equality is just a routine computation using Poincaré duality on $\mathrm{CH}(D_s)$.

Next we compute the total Chern class of the tangent bundle T_{D_s}. Since D_s is a hyperplane section of $\mathrm{Gr}(3,6)$ we have the following exact sequence:

$$0 \to T_{D_s} \to \imath_s^*(T_{\mathrm{Gr}(3,6)}) \to \imath_s^*(\mathcal{O}_{\mathrm{Gr}(3,6)}(1)) \to 0 \tag{19}$$

Therefore $c(T_{D_s})\imath_s^*(c(\mathcal{O}_{\mathrm{Gr}(3,6)}(1))) = \imath_s^*(c(T_{\mathrm{Gr}(3,6)}))$. Since $\imath_s^*(c(\mathcal{O}_{\mathrm{Gr}(3,6)}(1))) = 1+h_1$ and $\imath_s^*(c(T_{\mathrm{Gr}(3,6)})) = 1-h_1^2-h_1^3+h_1^5$ (this formula follows immediatelly from Schubert calculus), we have $c(T_{D_s}) = (1-h_1^2-h_1^3+h_1^5)(1+h_1)^{-1} = 1-h_1-h_1^3+h_1^4$ and $c(-T_{D_s}) = c(T_{D_s})^{-1} = 1+h_1+h_1^2-h_1^3-h_1^4-h_1^5$.

To prove the last assertion note that $\Delta_{D_s} = \Delta' \pm d \times d$, where Δ' is the part of the diagonal Δ_{D_s} which does not involve d, i.e., which comes from $\mathrm{Gr}(3,6)$. Let $\delta\colon D_s \to D_s \times D_s$ denote the diagonal morphism.

Now

$$S^\bullet(\pm d \times d) = S^\bullet(\Delta_{D_s} - \Delta') = S^\bullet(\delta_*(1) - \Delta') = S^\bullet(\delta_*(1)) - S^\bullet(\Delta'). \tag{20}$$

To prove that $S^\bullet(d) = d$ we must show that the right hand side does not contain summands of the form $d \times \alpha$, $\alpha \in \mathrm{Ch}(D_s)$, different from $\pm d \times d$. Therefore the summand $S^\bullet(\Delta')$ is not interesting for us.

We have

$$\begin{aligned} S^\bullet(\delta_*(1)) = c(T_{D_s\times D_s})\delta_*(S^\bullet_{D_s}(1)c(-T_{D_s})) &= c(T_{D_s\times D_s})(c(-T_{D_s}) \times 1)\delta_*(1) \\ &= c(T_{D_s\times D_s})(c(-T_{D_s}) \times 1)\Delta_{D_s}, \end{aligned}$$

where the first equality follows from the Riemann-Roch formula (8) and the second from the projection formula. But by item 2. the Chern classes $c_i(T_{D_s})$ don't involve d, i.e., lie in the image of $\imath_s^*$. Since $\alpha d = 0$ for all $\alpha \in \mathrm{Im}\,\imath^*$ (see above), the lemma is proved. □

In the notation of Theorem 3.7 denote as $p \in \mathrm{Ch}_8(D \times D)$ the projector corresponding to the motive R, i.e., $R = (D, p)$. From the proof of Theorem 3.7 it is easy to see that

$$p_s = 1 \times pt \pm d \times d + pt \times 1. \tag{21}$$

Since the natural map $\mathrm{Pic}(D) \to \mathrm{Pic}(D_s)$ is an isomorphism (see Lemma 3.4(3)), we denote as h_1 the canonical generator of $\mathrm{Pic}(D_s)$ as well as the correpronding generator of $\mathrm{Pic}(D)$.

4.4 Lemma. *The following properties of D hold:*

1. *The natural group homomorphism $\mathrm{CH}_0(D) \to \mathrm{CH}_0(D_s)$ is injective. Its image is generated by zero cycles of degree divisible by* 3.
2. $S^1(p_\star(h_1^6)) = h_1^8$.

Proof. 1. To prove the statement we use the notion of connectedness of classes of fields introduced by Chernousov and Merkurjev (see 2.15).

By [9] Theorem 6.5 it suffices to show that the class $\mathcal{A}(D)$ of all field extensions E/k such that $D(E) \neq \emptyset$ is connected and for any $L \in \mathcal{A}(D)$ the group $\mathrm{CH}_0(D_L) = \mathbb{Z}$. By [15] Corollary (40.11) the Jordan algebra $J(A, c)$ obtained by the first Tits construction splits if and only if $c \in \mathrm{Nrd}_A(A^*)$. Therefore the connectedness of $\mathcal{A}(D)$ follows from [9] Theorem 11.3, since the variety $D = D(A, c)$ has an E-point for a field extension E/k if and only if $c \in \mathrm{Nrd}_{A_E}(A_E^*)$.

To prove that $\mathrm{CH}_0(D_L) = \mathbb{Z}$ for any $L \in \mathcal{A}(D)$ it suffices to check that for any field extension E/L any two rational points of D_E are rationally equivalent (see [9] Lemma 5.2). If the algebra A_E is not split, then all rational points of D_E are contained in $\mathrm{MS}(A_E, 1) \simeq \mathrm{SL}_1(A_E)$. Since $\mathrm{SL}_1(A)$ is a rational and homogeneous variety, this implies that D_E is R-trivial (see 2.15), and, hence, $\mathrm{CH}_0(D_E) = \mathbb{Z}$. If the algebra A_E splits, then obviously $\mathrm{CH}_0(D_E) = \mathbb{Z}$.

Since D is anisotropic, it follows from [21] Cor. on p. 205 that the image of the degree map $\mathrm{CH}_0(D) \to \mathrm{CH}_0(D_s) = \mathbb{Z}$ is divisible by 3.

2. The proof of this item is similar to the proof of Corollary 4.9 [14]. By [14] Lemma 3.1

$$S^\bullet(p_\star(h_1^6)) = S^\bullet(p)_\star(h_1^6(1 + h_1^2)^6 c(-T_D)). \tag{22}$$

Therefore $S^1(p_\star(h_1^6))$ equals the 0-dimensional component of the right hand side. Assume that

$$S^1(p)_\star(h_1^6) = 0. \tag{23}$$

Then by Lemma 4.3 $S^\bullet(p)_\star(h_1^6(1+h_1^2)^6 c(-T_D)) = S^\bullet(p)_\star(h_1^6(1+h_1+h_1^2-h_1^3-h_1^4-h_1^5+e))$, where $e \in \mathrm{Ch}^{\geq 2}(D)$ is a torsion element. Therefore the 0-dimensional component equals $S^0(p)_\star(h_1^8)+S^1(p)_\star(h_1^6) = p_\star(h_1^8) = h_1^8$ (cf. [14]).

To prove (23) it suffices to show that $\deg S^1(p)_\star(h_1^6)$ is divisible by 9 (cf. [14] Proof of Corollary 4.5). Without loss of generality we can compute this degree over k_s. It follows from the proof of Corollary 4.5 [14] that $\deg S^1(p)_\star(h_1^6) = \deg h_1^6 \mathrm{pr}_{1*}(S^1(p_s))$. But $\mathrm{pr}_{1*}(S^1(p_s))$ is divisible by 3 (see Lemma 4.3(3)) and for any $\alpha \in \mathrm{Ch}^2(D_s)$ the product $h_1^6\alpha$ is divisible by 3. We are done. □

Now we are ready to prove Theorem 4.1. Consider the cycle $S^1(p_\star(h_1^6))$. Since $\deg h_1^8 = 42 = 3 \cdot 14$ and D is anisotropic, by lemma 4.4 this cycle is non-zero. Therefore $p_\star(h_1^6) \in \mathrm{Ch}_2(D)$ is non-zero. On the other hand, $(p_s)_\star(h_1^6) = 0$. In other words, $p_\star(h_1^6)$ is a non-trivial torsion element in $\mathrm{Ch}_2(D)$.

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