Higher Tits indices of linear algebraic groups

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Abstract

Let G be a semisimple algebraic group over a field k. We introduce the higher Tits indices of G as the set of all Tits indices of G over all field extensions K/k. In the context of quadratic forms this notion coincides with the notion of the higher Witt indices introduced by M. Knebusch and classified by N. Karpenko and A. Vishik. We classify the higher Tits indices for exceptional algebraic groups. Our main tools include the Chow groups and the Chow motives of projective homogeneous varieties, Steenrod operations, and the notion of the J-invariant introduced in [PSZ07].

1 Introduction

Let G denote a semisimple algebraic group of inner type defined over a field k. In his famous paper [Ti66] J. Tits defined the *Tits index* of G as the data consisting of the Dynkin diagram of G with some vertices being circled. Let K be an arbitrary field extension of k. In the present paper we investigate the following problem: What are the possible values of the Tits index of the group G_K?

In the theory of central simple algebras (i.e., when G is a group of type A_n) this problem is equivalent to the index reduction formula of A. Blanchet, A. Schofield, and M. Van den Bergh (see [SVB92]). Later their result was generalized by A. Merkurjev, I. Panin, and A. Wadsworth (see [MPW96] and [MPW98]).

In the theory of quadratic forms (i.e, when G is an orthogonal group) the above problem is equivalent to the study of the *higher Witt indices* of

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quadratic forms. The higher Witt indices were introduced by M. Knebusch [Kn76]. They provide a nice discrete invariant of quadratic forms over any field k. Numerous results have been obtained so far. One of the main achievements here is the celebrated result of N. Karpenko [Ka03], where he proves Hoffmann's conjecture about all possible values of the first higher Witt index. One should also mention the interesting papers of N. Karpenko, A. Merkurjev and A. Vishik [Ka04], [KM03], [Vi04], [Vi06] concerning closely related problems in the theory of quadratic forms.

The main result of M. Knebusch in his paper [Kn76] asserts that one can always consider only a certain finite number of field extensions K_i/k, $i = 0, \ldots, h$ such that the Tits index of G over a field K/k equals one of the Tits indices of G_{K_i}. The fields K_i appearing in the Knebusch theory are the fields of rational functions on certain projective G-homogeneous varieties (see Section 2 below).

Moreover, the result of Karpenko [Ka04, Theorem 2.6] asserts that information on the higher Tits indices is hidden in a subring of the Chow ring of certain projective G-homogeneous varieties[1]. In the present paper we exploit this connection further. Our main tools include the Steenrod operations in the Chow theory, Tits' classification, Chow motives, and motivic invariants, like the *J-invariant* of algebraic groups introduced in [PSZ07].

It turns out, that in the most cases the splitting behaviour of an algebaic group G depends not on the base field k, but on triviality or non-triviality of a certain discrete invariant of G called the *J-invariant*. By definition this invariant measures the "size" of the subring of rational cycles in the Chow ring of the G-variety of complete flags (see Section 4). But usually the (non)-triviality of this invariant can be expressed in terms of the (non)-triviality of the Tits algebras of G and/or of certain cohomological invariants of G, like the Rost invariant.

In Section 6 we study the existence of the anisotropic kernels of type D_6 in the groups of type E_7 over field extensions K/k of the base field ("index reduction formula" for groups of type E_7). We prove that this problem is equivalent to the problem of existence of zero-cycles of degree 1 on certain anisotropic projective homogeneous varieties. The latter problem has a long history starting with the paper [Serre] and obviously has a positive answer (in the sense that an anisotropic projective variety does not have a zero-

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[1]His result concerns only quadrics, but can be straightforwardly generalized to arbitrary projective homogeneous varieties.

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cycle of degree 1) over fields k whose absolute Galois group $\mathrm{Gal}(k_s/k)$ a pro p-group. We refer the reader to papers [Fl04], [Par05], [To04] that discuss this problem. In the situation of E_7 this problem was solved in article [GS08] that was partially motivated by the results of the present paper.

The main results of our paper are Theorems 5.3 and 6.1 that allow to compute all possible higher Tits indices for groups of type F_4, $^1E_6$, and E_7 with trivial Tits algebras, and to classify all generically cellular varieties of exceptional types.

One suprising corollary of our results is a very short proof of the triviality of the kernel of the Rost invariant for groups of type E_7 (see Corollary 6.11). The arguments of that proof belong to S. Garibaldi.

The present paper contains an Appendix due to M. Florence who shows in a uniform way that for all Dynkin types there exist a group whose set of higher Tits indices is maximal possible (see Theorem 6.12).

The main goal of the present paper is to study general restrictions on the splitting behaviour of algebraic groups, i.e., those restrictions which don't depend on the base field.

Acknowledgements

2   Tits' classification and Knebusch theory

We recall the definition and basic properties of the Tits indices of semisimple algebraic groups following J. Tits [Ti66] and [Ti71].

Let k be a field, k_s be a separable closure of k, G a semisimple algebraic group defined over k, S a maximal split torus of G defined over k, T a maximal torus containing S and defined over k, $\Delta = \Delta(G)$ the system of simple roots of G with respect to T and $\Delta_0 = \Delta_0(G)$ the subsystem of those roots which vanish on S.

In the present paper we consider only groups of *inner type*, i.e., groups that are twisted forms of a split group by means of a 1-cocycle in $H^1_{\mathrm{et}}(k, G_0)$, where G_0 denotes the split adjoint group over k of the same type as G. Equivalently, this means that the $*$-action (see [Ti66]) of the absolute Galois group on Δ is trivial. Therefore we don't define the $*$-action and don't include it in the definition of the Tits index of G.

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The *index* of G is a pair (Δ, Δ_0). We represent the index as the Dynkin diagram of G with the vertices that don't belong to Δ_0 being circled.

There exists a certain subgroup of G called the *semisimple anisotropic kernel*. We refer the reader to [Ti66] for its definition. Note that the index of the semisimple anisotropic kernel of G can be easiely reduced from the index of G by removing the vertices of the Dynkin diagram which are circled.

To any semisimple group over k one can functorially associate certain central simple algebras, called the *Tits algebras*. We refer the reader to [Ti71] for a definition and description.

2.1 Examples. 1. Let A be a central simple k-algebra of degree $n + 1$ and $G = \mathrm{SL}_1(A)$ the respective group of type A_n. Then the index of A equals $\dfrac{n+1}{r+1}$, where r is the number of circled vertices on the Tits diagram of G.

The Tits algebras of G are the λ-powers $\lambda^i A$, $i = 1, \ldots, n$.

2. Let (V, q) be a regular odd-dimensional quadratic space over k and $G = \mathrm{Spin}(V, q)$ be the respective group of type B_n. Then the number of circled vertices on its Tits diagram equals the Witt index of q.

The Tits algebra of G is the even Clifford algebra $C_0(V, q)$.

One can give similar descriptions for all semisimple algebraic groups over k.

Next we recall the construction of the generic splitting tower of Knebusch for semisimple algebraic groups (see [Kn76], [Kn77]).

Consider the set

$$\{\mathrm{ind}(G_K)_{\mathrm{an}} \mid K/k \text{ is a field extension}\}, \tag{1}$$

where $\mathrm{ind}(G_K)_{\mathrm{an}}$ stands for the Tits index of the semisimple anisotropic kernel of G_K.

2.2 Definition. Set (1) is called the set of the Higher Tits Indices of G.

This set can be obtained using the *generic splitting tower*, i.e., it suffices to consider not all field extensions K/k, but just a finite number of *generic* ones. The latter are defined inductively as follows.

First we set $K^0 = k$, $G^1 = G_{\mathrm{an}}$ and consider the function fields $K_i^1 = K^0(X_i)$, $i \in \Delta(G^1) \subset \Delta(G)$, of the projective varieties of the maximal parabolic subgroups of G^1 of type i. Note that there are precisely $\mathrm{rk}G_{\mathrm{an}} = |\Delta(G_{\mathrm{an}})|$ such varieties.

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Next for each $j \in \Delta(G^1)$ we consider the group $G_j^2 = (G_{K_j^1}^1)_{\mathrm{an}}$ over K_j^1 and apply the same procedure, i.e., consider the fields $K_i^2 = K_j^1(X_i)$, $i \in \Delta(G_j^2) \subset \Delta(G)$, where X_i stands for the projective variety over K_j^1 of the maximal parabolic subgroups of type i of the group G_j^2. Proceeding further we obtain a set (a tower) of fields K_*^*. Its main property is that

$$\{\mathrm{ind}(G_K)_{\mathrm{an}} \mid K/k \text{ is a field extension}\} \tag{2}$$

$$= \{\mathrm{ind}(G_{K_i^j})_{\mathrm{an}} \mid K_i^j/k \text{ is an element in the generic splitting tower of } G\}.$$

The maximal value of the upper index of K_*^*'s is called the *height* of G.

Conversely, given the set of the higher Tits index of G, the Tits index of $G_{k(X_i)}$ can be restored as the minimal higher Tits index containing i. An overview of this ideas can be found in [KR94].

It is sometimes convenient to represent the right hand side of identity (2) as an oriented labelled graph as following. First we define a graph whose vertices are the anisotropic groups G_j^s appearing in the construction above and which we represent by their Dynkin diagrams. There is an edge from G_j^s to $G_{j'}^{s'}$ with the label i if and only if $s' = s + 1$, $i = j'$ and $G_i^{s+1} = ((G_j^s)_{K_j^{s-1}(X_i)})_{\mathrm{an}}$. Next we identify the vertices of this graph which correspond to the same Dynkin types. Thus, the vertices of this graph represent all possible Tits indices of G_K without repetitions for all K/k. The height of G is the maximal length of the paths on the graph.

2.3 Examples. 1. Let G be an anisotropic group of type B_2. Then its splitting graph is: B_2 (enumeration of simple roots follows Bourbaki).
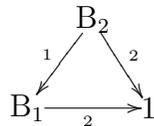
2. Let G be an anisotropic group of type B_3 and q the respective quadratic form of discriminant 1. Then its splitting graph is: B_3 if q has trivial
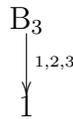
Clifford invariant and otherwise.
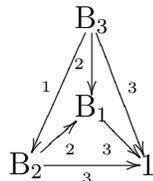
There exist generalizations for quadratic forms of bigger dimensions due to N. Karpenko and A. Vishik.

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The pictures in the examples resemble an automaton. Therefore one can call the graphs defined above the *Tits automata.*

2.4 Remark. Note that the first step, i.e., the groups G_*^2 are the most important ones. Indeed, the next groups $G_*^{\geq 3}$ involved in the construction are anisotropic kernels of the groups G_*^2. Since the rank of G_*^2 is smaller than the rank of G^1, we are deduced to the same situation but for groups of a smaller rank.

3   Cycles on projective homogeneous varieties and Chow motives

In this section we briefly describe the main properties of projective homogeneous varieties and their Chow rings (see [De74], [Hi82]).

Let G be a split semisimple algebraic group of rank n defined over a field k. We fix a split maximal torus T in G and a Borel subgroup B of G containing T and defined over k. We denote by Φ the root system of G, by $\Pi = \{\alpha_1, \ldots, \alpha_n\}$ the set of simple roots of Φ with respect to B, by W the Weyl group, and by $S = \{s_1, \ldots, s_n\}$ the corresponding set of fundamental reflections.

Let $P = P_\Theta$ be the (standard) parabolic subgroup corresponding to a subset $\Theta \subset \Pi$, i.e., $P = BW_\Theta B$, where $W_\Theta = \langle s_\theta, \theta \in \Theta \rangle$. Denote

$$W^\Theta = \{w \in W \mid \forall s \in \Theta \quad l(ws) = l(w) + 1\},$$

where l is the length function. It is easy to see that W^Θ consists of all representatives in the left cosets W/W_Θ which have minimal length.

As P_i we denote the maximal parabolic subgroup $P_{\Pi \setminus \{\alpha_i\}}$ *of type i* and as w_0 the longest element of W. Enumeration of simple roots follows Bourbaki.

Any projective G-homogeneous variety X is isomorphic to G/P_Θ for some subset Θ of the simple roots.

Now consider the Chow ring of the variety $X = G/P_\Theta$. It is known that $\mathrm{CH}^*(G/P_\Theta)$ is a free abelian group with a basis given by varieties $[X_w]$ that correspond to the elements $w \in W^\Theta$. The degree (codimension) of the basis element $[X_w]$ equals $l(w_\theta) - l(w)$, where w_θ is the longest element of W_Θ.

Moreover, there exists a natural injective pull-back homomorphism

$$\mathrm{CH}^*(G/P) \to \mathrm{CH}^*(G/B)$$

$$[X_w] \mapsto [X_{ww_\theta}]$$

The following results provide tools to perform computations in the Chow ring $\mathrm{CH}(G/P_\Theta)$.

In order to multiply two basis elements $h = [X_w]$ and $g = [X_{w'}]$ of $\mathrm{CH}^*(G/P_\Theta)$ such that $\deg h + \deg g = \dim G/P_\Theta$ we use the following formula (Poincaré duality):

$$[X_w] \cdot [X_{w'}] = \delta_{w, w_0 w' w_\theta} \cdot [X_1]. \tag{3}$$

In view of Poincaré duality we denote as $[Z_w]$ the cycle dual to $[X_w]$ with respect to the canonical basis. In other words, $[Z_w] = [X_{w_0 w w_\theta}]$.

In order to multiply two basis elements of $\mathrm{CH}^*(G/B)$ one of which is of codimension 1 we use the following formula (Pieri formula):

$$[X_{w_0 s_\alpha}][X_w] = \sum_{\beta \in \Phi^+, \, l(ws_\beta) = l(w) - 1} \langle \beta^\vee, \bar\omega_\alpha \rangle [X_{ws_\beta}], \tag{4}$$

where α is a simple root and the sum runs through the set of positive roots $\beta \in \Phi^+$, s_β denotes the reflection corresponding to β and $\bar\omega_\alpha$ is the fundamental weight corresponding to α. Here $[X_{w_0 s_\alpha}]$ is the element of codimension 1.

The *Poincaré polynomial* of a free abelian $\mathbb{Z}$-graded finitely generated group A^* is, by definition, the polynomial $g(A^*, t) = \sum_{i=-\infty}^{+\infty} a_i t^i \in \mathbb{Z}[t, t^{-1}]$ with $a_i = \mathrm{rk} A^i(X)$. The following formula (the Solomon theorem) allows to compute the Poincaré polynomial of $\mathrm{CH}^*(X)$:

$$g(\mathrm{CH}^*(X), t) = \frac{r(\Pi)}{r(\Theta)}, \quad r(-) = \prod_{i=1}^{l} \frac{t^{d_i(-)} - 1}{t - 1}, \tag{5}$$

where $d_i(\Theta)$ (resp. $d_i(\Pi)$) denote the degrees of the fundamental polynomial invariants of the root subsystem of Φ generated by Θ (resp. Π) and l its rank (see [Ca72]). The dimension of X equals $\deg g(\mathrm{CH}^*(X), t)$. There exists a Maple package [St] of J. Stembridge that provides tools to compute the Poincaré polynomials of projective G-homogeneous varieties.

Let $\mathrm{P} = \mathrm{P}(\Phi)$ denote the weight space. We denote as $\bar\omega_1, \dots \bar\omega_n$ the basis of P consisting of the fundamental weights. The symmetric algebra $S^*(\mathrm{P})$ is isomorphic to $\mathbb{Z}[\bar\omega_1, \dots \bar\omega_n]$. The Weyl group W acts on P, hence, on $S^*(\mathrm{P})$. Namely, for a simple root α_i

$$s_i(\bar\omega_j) = \begin{cases} \bar\omega_i - \alpha_i, & i = j; \\ \bar\omega_j, & \text{otherwise.} \end{cases}$$

We define a linear map $c\colon S^*(\mathrm{P})^{W_\Theta} \to \mathrm{CH}^*(G/P_\Theta)$ as follows. For a homogeneous W_Θ-invariant $u \in \mathbb{Z}[\bar\omega_1,\dots,\bar\omega_n]$

$$c(u) = \sum_{w \in W^\Theta,\, l(w)=\deg(u)} \Delta_w(u)[X_{w_0 w w_\theta}],$$

where for $w = s_{i_1}\dots s_{i_k}$ we denote by Δ_w the composition of derivations $\Delta_{s_{i_1}} \circ \dots \circ \Delta_{s_{i_k}}$ and the derivation $\Delta_{s_i}\colon S^*(\mathrm{P}) \to S^{*-1}(\mathrm{P})$ is defined by $\Delta_{s_i}(u) = \dfrac{u - s_i(u)}{\alpha_i}$.

Let $U = \Sigma_u(P_\Theta)$ denote the set of the (positive) roots lying in the unipotent radical of the parabolic subgroup P_Θ. Then the elementary symmetric polynomials $\sum_{u \in U} \sigma_i(u)$ are W_{P_Θ}-invariant and, in fact, coincide with the Chern classes of the tangent bundle T_X:

$$c(T_X) = c(\prod_{\gamma \in U}(1 + \gamma)). \tag{6}$$

The Maple package [map] provides efficient tools to compute the Chern classes of the tangent bundles.

To multiply two cycles $[X_{w_1}]$ and $[X_{w_2}]$ in $\mathrm{CH}^*(X)$ we proceed as follows. First, we find preimages of $[X_{w_1}]$ and $[X_{w_2}]$ in $S^*(\mathrm{P})\otimes\mathbb{Q} = \mathbb{Q}[\bar\omega_1,\dots,\bar\omega_n]$ (the preimages always exist; see below), then we either expand the product in the polynomial ring $\mathbb{Q}[\bar\omega_1,\dots,\bar\omega_n]$ and apply the function c, or apply c directly using the Pieri formula (4), the Leibniz rule [Hi82, Ch. IV, Lemma 1.1(e)] or/and Poincaré duality (3).

To find a preimage of some $[X_w]$ we do the following. It is well known that the map $c \otimes \mathbb{Q}\colon \mathbb{Q}[\bar\omega_1,\dots\bar\omega_n]^{W_\Theta} \to \mathrm{CH}^*(G/P_\Theta) \otimes \mathbb{Q}$ defined above is a ring epimorphism, and the ring $\mathbb{Q}[\bar\omega_1,\dots\bar\omega_n]^{W_\Theta}$ is generated by $\bar\omega_i$, $i \notin \Theta$, and by the W_Θ-invariant fundamental polynomials for the semisimple part of the Levi part of P_Θ, i.e., for the split group of type $\langle\Theta\rangle \subset \Phi$. The latter polynomials (as well as their degrees called degrees of fundamental polynomial invariants) are known. Explicit formulas for them are provided in [Meh88]. Now, since we know a generating set of $\mathbb{Q}[\bar\omega_1,\dots\bar\omega_n]^{W_\Theta}$, we can compute its image in $\mathrm{CH}^*(G/P_\Theta) \otimes \mathbb{Q}$ and, thus, find a set of generators of $\mathrm{CH}^*(G/P_\Theta) \otimes \mathbb{Q}$ together with their preimages in $\mathbb{Q}[\bar\omega_1,\dots\bar\omega_n]^{W_\Theta}$. Therefore we can compute a preimage of any element in $\mathrm{CH}^*(G/P_\Theta) \otimes \mathbb{Q}$. Observe that we don't loose any information extending scalars to $\mathbb{Q}$, since the group $\mathrm{CH}^*(G/P_\Theta)$ is free abelian.

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The effective procedures to multiply cycles in the Chow rings of projective homogeneous varieties are implemented in the Maple package [map][2].

Next we briefly describe Steenrod operations and motivic decompositions of projective G-homogeneous varieties with isotropic group G following [CGM05] and [Br05]. We refer the reader to the book [EKM] of R. Elman, N. Karpenko, and A. Merkurjev or to the original paper [Ma68] of Yu. Manin for the definition and properties of the Chow motives.

The main result of papers [CGM05] and [Br05] asserts that the Chow motive a projective G-homogenous variety X with isotropic group G decomposes into a direct sum of (twisted) motives of anisotropic projective G_{an}-homogeneous varieties Y_i. Moreover, one has an explicit algorithm to compute these motivic decompositions.

In the present section we give a combinatorial interpretation of these decompositions in terms of the Hasse diagrams of the weak Bruhat order. Namely, consider an oriented labelled graph, called *Hasse diagram* of X, whose vertices are elements of W^Θ, where Θ denotes the type of the variety X, i.e., a graph whose vertices correspond to the free additive generators of $CH^*(\overline{X})$, where $\overline{X} = X \times_{\operatorname{Spec} k} \operatorname{Spec} k_s$ and k_s stands for a separable closure of k. There is an edge from a vertex w to a vertex w' labelled with i if and only if $l(w) < l(w')$ and $w' = s_i w$.

Consider now the Chow motive of X and erase from the Hasse diagram of X all edges with labels not in Δ_0 (see Section 2). The Hasse diagram splits then into several non-connected components which correspond to the varieties Y_i. To illustrate this construction we give the following example.

3.1 Example. Let G be an isotropic group of type E_7 such that G_{an} has type D_4. This means that the vertices 1, 6, and 7 on the Tits diagram of G are circled. Consider the projective G-homogeneous variety X of parabolic subgroups of type 7. Its Hasse diagram is provided in [PlSeVa, Figure 21]. Cutting the Hasse diagram along the edges with labels 1, 6, and 7 we see that the diagram splits into 14 components: 8 alone standing vertices which correspond to the elements of W^Θ of length 0, 1, 9, 10, 17, 18, 26, and 27, and therefore to the (twisted) Lefschetz motives $\mathbb{Z}$, $\mathbb{Z}(1)$, $\mathbb{Z}(9)$, $\mathbb{Z}(10)$, $\mathbb{Z}(17)$, $\mathbb{Z}(18)$, $\mathbb{Z}(26)$, $\mathbb{Z}(27)$, and 6 diagrams that correspond to different varieties of type D_4. It is well known and easy to see that G_{an} corresponds to a(n anisotropic) 3-fold Pfister form φ and therefore by the celebrated result of M. Rost [Ro98] the Chow motives of a projective G_{an}-homogeneous

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[2]Created in collaboration with S. Nikolenko and K. Zainoulline.

variety splits into a direct sum of (twisted) Rost motives R which depend only on φ. The Rost motive R is indecomposable and over k_s (where φ splits) $R_s \simeq \mathbb{Z} \oplus \mathbb{Z}(3)$. Thus,

$$\mathcal{M}(X) \simeq (\oplus_{i=0,1,9,10,17,18,26,27} \mathbb{Z}(i)) \oplus (\oplus_{i=2}^{22} R(i)) \oplus R(11) \oplus R(12) \oplus R(13).$$

One should note that in the category of the Chow motives with finite coefficients of projective homogeneous varieties the Krull-Schmidt theorem holds (see [CM06, Theorem 9.6]). Therefore the motivic decompositions are unique.

Now we briefly recall the basic properties of Steenrod operations constructed by V. Voevodsky. We follow P. Brosnan [Br03]

Let X be a smooth projective variety over a field k with $\operatorname{char} k \neq 2$ and $p = 2$. For every $i \geq 0$ there exist certain homomorphisms $S^i = Sq^{2i} \colon \operatorname{Ch}^*(X) \to \operatorname{Ch}^{*+i}(X)$ called Steenrod operations. The total Steenrod operation is the sum $S = S_X = S^0 + S^1 + \ldots \colon \operatorname{Ch}^*(X) \to \operatorname{Ch}^*(X)$. This map is a ring homomorphism. The restriction $S^i|_{\operatorname{Ch}^n(X)}$ is 0 for $i > n$ and is the map $\alpha \mapsto \alpha^2$ for $n = i$. The map S^0 is the identity. Moreover, the total Steenrod operation commutes with pull-backs and, in particular, preserves rationality of cycles.

To compute the Steenrod operations on a projective G-homogeneous variety with a split group G we use an algorithm described in details in [DuZ07]. This algorithm is implemented in the Maple package [map].

4 J-invariant

In this section we recall the definition and the main properties of a motivic invariant of a semisimple algebraic group introduced in [PSZ07] and called the J-invariant. It was shown in [PSZ07] that this invariant determines the motivic behaviour of generically split projective homogeneous varieties (see the definition below).

Let G_0 be a split semisimple algebraic group over k with a split maximal torus T and a Borel subgroup B containing T. Let $G = {}_\gamma G_0$ be the twisted form of G_0 given by a 1-cocycle $\gamma \in H^1(k, G_0)$.

Let X be a projective G-homogeneous variety and p a prime integer. To simplify the notation we denote $\operatorname{Ch}^*(X) = \operatorname{CH}^*(X) \otimes \mathbb{Z}/p$ and $\overline{X} = X \times_{\operatorname{Spec} k} \operatorname{Spec} k_s$, where k_s stands for a separable closure of k. We say that

a cycle $J \in \mathrm{CH}^*(\overline{X})$ (resp. $J \in \mathrm{Ch}^*(\overline{X})$) is *rational* if it lies in the image of the natural restriction map res: $\mathrm{CH}^*(X) \to \mathrm{CH}^*(\overline{X})$ (resp. res: $\mathrm{Ch}^*(X) \to \mathrm{Ch}^*(\overline{X})$). We denote as $\overline{\mathrm{CH}}^*(X)$ (resp. as $\overline{\mathrm{Ch}}^*(X)$) the image of this map.

From now on and till the end of this section we consider the variety $X = {}_\gamma(G_0/B)$ of complete flags. Let $\widehat{T}$ denote the group of characters of T and $S(\widehat{T}) \subset S^*(\mathrm{P})$ be the symmetric algebra (see Section 3). By R^* we denote the image of the characteristic map $c \colon S(\widehat{T}) \to \mathrm{Ch}^*(X)$ defined above. According to [KM05, Theorem 6.4] $R^* \subseteq \overline{\mathrm{Ch}}^*(X)$.

Let $\mathrm{Ch}^*(\overline{G})$ denote the Chow ring with $\mathbb{Z}/p$-coefficients of the group $(G_0)_{k_s}$. An explicit presentation of $\mathrm{Ch}^*(\overline{G})$ in terms of generators and relations is known for all groups and all primes p. Namely, by [Kc85, Theorem 3]

$$\mathrm{Ch}^*(\overline{G}) = (\mathbb{Z}/p)[x_1, \ldots, x_r]/(x_1^{p^{k_1}}, \ldots, x_r^{p^{k_r}}) \qquad (7)$$

for certain numbers k_i, $i = 1, \ldots, r$, and $\deg x_i = d_i$ for certain numbers $1 \le d_1 \le \ldots \le d_r$ coprime to p. A complete list of numbers $\{d_i p^{k_i}\}_{i=1,\ldots,r}$, called *p-exceptional degrees* of G_0, is provided in [Kc85, Table II]. Taking the p-primary and p-coprimary parts of each p-exceptional degree one immediately restores the respective k_i's and d_i's.

Now we introduce an order on the set of additive generators of $\mathrm{Ch}^*(\overline{G})$, i.e., on the monomials $x_1^{m_1} \ldots x_r^{m_r}$. To simplify the notation, we denote the monomial $x_1^{m_1} \ldots x_r^{m_r}$ by x^M, where M is an r-tuple of integers $(m_1, \ldots, m_r)$. The codimension (in the Chow ring) of x^M is denoted by $|M|$. Observe that $|M| = \sum_{i=1}^{r} d_i m_i$.

Given two r-tuples $M = (m_1, \ldots, m_r)$ and $N = (n_1, \ldots, n_r)$ we say $x^M \le x^N$ (or equivalently $M \le N$) if either $|M| < |N|$, or $|M| = |N|$ and $m_i \le n_i$ for the greatest i such that $m_i \ne n_i$. This gives a well-ordering on the set of all monomials (r-tuples) known also as *DegLex order*.

Consider the pull-back induced by the quotient map

$$\pi \colon \mathrm{Ch}^*(\overline{X}) \to \mathrm{Ch}^*(\overline{G})$$

According to [Gr58, Rem. 2°] π is surjective with the kernel generated by the subgroup of the non-constant elements of R^*.

Now we are ready to define the J-invariant of a group G.

4.1 Definition. Let $X = {}_\gamma(G_0/B)$ be the twisted form of the variety of complete flags by means of a 1-cocycle $\gamma \in H^1(k, G_0)$. Denote as $\overline{\mathrm{Ch}}^*(G)$ the

image of the composite map

$$\mathrm{Ch}^*(X) \xrightarrow{\mathrm{res}} \mathrm{Ch}^*(\overline{X}) \xrightarrow{\pi} \mathrm{Ch}^*(\overline{G}).$$

Since both maps are ring homomorphisms, $\overline{\mathrm{Ch}}^*(G)$ is a subring of $\mathrm{Ch}^*(\overline{G})$.

For each $1 \leq i \leq r$ set j_i to be the smallest non-negative integer such that the subring $\overline{\mathrm{Ch}}^*(G)$ contains an element a with the greatest monomial $x_i^{p^{j_i}}$ with respect to the DegLex order on $\mathrm{Ch}^*(\overline{G})$, i.e., of the form

$$a = x_i^{p^{j_i}} + \sum_{x^M \lneq x_i^{p^{j_i}}} c_M x^M, \quad c_M \in \mathbb{Z}/p.$$

The r-tuple of integers $(j_1, \ldots, j_r)$ is called the *J-invariant of G modulo p* and is denoted by $J_p(G)$. Note that $j_i \leq k_i$ for all i.

In case p is not a torsion prime of G we have $\mathrm{Ch}^*(\overline{G}) = \mathbb{Z}/p$. Therefore the J-invariant is interesting only for torsion primes (see [Gr58, Definition 3] for a definition of torsion primes). A table of possible values of the J-invariants is given in [PSZ07, Section 6].

To illustrate Definition 4.1 of the J-invariant we give the following example. For a prime integer p we denote as v_p the p-adic valuation.

4.2 Example. Let p be a prime integer and A and B be central simple k-algebras that generate the same subgroup in the Brauer group $\mathrm{Br}(k)$. Set $G = \mathrm{PGL}_1(A) \times \mathrm{PGL}_1(B)$.

Then $J_p(G) = (v_p(\mathrm{ind} A), 0)$. Indeed, the Chow ring

$$\mathrm{Ch}^*(\overline{G}) = (\mathbb{Z}/p)[x_1, x_2]/(x_1^{p^{k_1}}, x_2^{p^{k_2}})$$

with $k_1 = v_p(\deg A)$, $k_2 = v_p(\deg B)$. Therefore r in the definition of the J-invariant equals 2. Denote $J_p(G) = (j_1, j_2)$ and consider the map

$$\mathrm{res} \colon \mathrm{Pic}(X_A \times X_B) \to \mathrm{Pic}(\overline{X}_A \times \overline{X}_B),$$

where X_A (resp. X_B) denote the $\mathrm{PGL}_1(A)$- (resp. $\mathrm{PGL}_1(B)$-) variety of complete flags and Pic stands for the Picard group modulo p. Denote by h_A (resp. h_B) the image of $\bar{\omega}_1 \in S(\mathrm{P})$ in $\mathrm{Pic}(\overline{X}_A)$ (resp. $\mathrm{Pic}(\overline{X}_B)$) by means of the map c defined in Section 3.

Since A and B generate the same subgroup in the Brauer group, the cycle $1 \times h_B + \alpha h_A \times 1 \in \mathrm{Pic}(\overline{X}_A \times \overline{X}_B)$ is rational for some $\alpha \in (\mathbb{Z}/p)^\times$ (see [MT95]

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for the description of the Picard groups of projective homogeneous varieties). The image of this cycle in $\mathrm{Ch}^*(\overline{G})$ by means of π equals $x_2 + \alpha x_1$ (at least we can choose the generators x_1 and x_2 in such a way). Therefore, since $x_1 < x_2$ in the DegLex order, $j_2 = 0$. The proof that $j_1 = v_p(\mathrm{ind}A)$ is the same as in [PSZ07, Section 7, case A_n] and we omit it.

Next we describe some useful properties of the J-invariant.

4.3 Proposition. *Let G be a semisimple algebraic group of inner type over k, p a prime integer and $J_p(G) = (j_1, \ldots, j_r)$. Then*

1. *Let K/k be a field extension. Denote $J_p(G_K) = (j_1', \ldots, j_r')$. Then $j_i' \leq j_i$, $i = 1, \ldots, r$.*

2. *Fix an $i = 1, \ldots, r$. Assume that in the presentation (7) for the semisimple anisotropic kernel G_{an} of G none of x_j has degree d_i. Then $j_i = 0$.*

3. *Assume $d_i = 1$ for some $i = 1, \ldots, r$.*

   Then $j_i \leq \max_A v_p(\mathrm{ind}A)$ where A runs through all Tits algebras of G. Conversely, if $j_i > 0$, then there exists a Tits algebra A of G with $v_p(\mathrm{ind}A) > 0$.

4. *Assume that the group G does not have simple components of type E_8 and for all primes p the J-invariant $J_p(G)$ is trivial. Then G is split.*

Proof. 1. This is an obvious consequences of the definition of the J-invariant.

2. Let $\mathrm{Ch}^*(\overline{G}) = (\mathbb{Z}/p)[x_1, \ldots, x_r]/(x_1^{p^{k_1}}, \ldots, x_r^{p^{k_r}})$ with $\deg x_i = d_i$, $\mathrm{Ch}^*(\overline{G_{\mathrm{an}}}) = (\mathbb{Z}/p)[x_1', \ldots, x_{r'}']/(x_1'^{p^{k_1'}}, \ldots, x_{r'}'^{p^{k_{r'}'}})$ with $\deg x_i' = d_i'$, and $J_p(G_{\mathrm{an}}) = (j_1', \ldots, j_{r'}')$. It follows from the [Kc85, Table II] that $r' \leq r$ and $\{d_i', i = 1, \ldots, r'\} \subset \{d_i, i = 1, \ldots, r\}$. On the other hand, by [PSZ07, Corollary 5.4], the polynomials $\prod_{i=1}^r \dfrac{1 - x^{d_i p^{j_i}}}{1 - x^{d_i}}$ and $\prod_{i=1}^{r'} \dfrac{1 - x^{d_i' p^{j_i'}}}{1 - x^{d_i'}}$ are equal. This implies the claim.

3. Assume that $j_i > 0$ and all Tits algebras of G are trivial. Then by [MT95] the group $\mathrm{Pic}(\overline{X})$, where X denotes the G-variety of complete flags, is rational. Therefore by the very definition of the J-invariant $j_i = 0$. A contradiction.

Let A be a Tits algebra of G corresponding to a vertex t of the Dynkin diagram such that $\pi(h_t) = x_i \in \mathrm{Pic}(\overline{G})$, where $h_t \in \mathrm{Pic}(\overline{X})$ is the image of $\bar{\omega}_t \in S(\mathrm{P})$ by means of the map c constructed above. We show now that $j_i \le v_p(\mathrm{ind}A) =: s$, where A is the Tits algebra corresponding to the vertex t.

Consider the projective homogeneous variety $X \times \mathrm{SB}(A)$, where $\mathrm{SB}(A)$ denotes the Severi-Brauer variety of right ideals of A of reduced dimension 1. Denote by $h_A \in \mathrm{Pic}(\overline{\mathrm{SB}(A)}) = \mathrm{Pic}(\mathbb{P}^{\deg A - 1})$ the canonical generator as in Section 3.

By the results of A. Merkurjev and J.-P. Tignol [MT95] the cycle $\alpha = h_t \times 1 - 1 \times h_A \in \mathrm{Pic}(\overline{X} \times \overline{\mathrm{SB}(A)})$ is rational. Since the cycles $\alpha^{p^s} = h_t^{p^s} \times 1 - 1 \times h_A^{p^s} \in \mathrm{Ch}^*(\overline{X} \times \overline{\mathrm{SB}(A)})$ and $h_A^{p^s} \in \mathrm{Ch}^*(\overline{\mathrm{SB}(A)})$ are rational, the cycle $h_t^{p^s} \times 1 \in \mathrm{Ch}^*(\overline{X} \times \overline{\mathrm{SB}(A)})$ is rational as well.

The projection morphism $\mathrm{pr} \colon X \times \mathrm{SB}(A) \to X$ is a projective bundle by [PSZ07, Corollary 3.4]. In particular, $\mathrm{CH}^*(X \times \mathrm{SB}(A)) = \bigoplus_{j=0}^{\deg A - 1} \mathrm{CH}^{*-j}(X)$. Therefore the pull-back pr^* has a section δ. By the construction of this section it is compatible with a base change. Passing to the splitting field k_s we obtain that the cycle $\bar{\delta}(h_t^{p^s} \times 1) = h_t^{p^s} \in \mathrm{Ch}^*(\overline{X})$ is rational and the image $\pi(h_t^{p^s}) = x_i^{p^s}$. By the definition of the J-invariant, $j_i \le s$.

4. The statement follows from [PSZ07, Corollary 6.10] and [Gi97, Theorem C]. $\qquad\square$

4.4 Remark. The fact that j_i (with $d_i = 1$) provides an upper bound for $v_p(\mathrm{ind}A)$, where A runs through the Tits algebras of G, is not true. A counter-example is e.g. a group of type E_7 with a Tits algebra of index more than 2.

5 Generically split varieties

In this section we begin to study the higher Tits indices of semisimple algebraic groups over k. First, we would like to understand under what conditions our group G splits over the field of rational functions of a projective G-homogeneous variety X.

5.1 Definition. Let G be a semisimple algebraic group over k and X a projective G-homogeneous variety. We say that X is *generically split*, if the group G splits (i.e., contains a split maximal torus) over $k(X)$.

5.2 Remark. If X is generically split, then the Chow motive of X splits over $k(X)$ as a direct sum of Lefschetz motives. This explains the terminology "generically split". One can also call such varieties *generically cellular*, since over $k(X)$ they are cellular via the Bruhat decomposition.

5.3 Theorem. *Let G_0 be a split semisimple algebraic group over k, $G = {}_\gamma G_0$ the twisted form of G_0 given by a 1-cocycle $\gamma \in H^1(k, G_0)$, and X a projective G-homogeneous variety. If X is generically split, then for all primes p the following identity on the Poincaré polynomials holds:*

$$\frac{g(\mathrm{Ch}^*(\overline{X}), t)}{g(\overline{\mathrm{Ch}}^*(X), t)} = \prod_{i=1}^{r} \frac{t^{d_i p^{j_i}} - 1}{t^{d_i} - 1}, \tag{8}$$

where $J_p(G) = (j_1, \ldots, j_r)$ and d_i's are the p-coprimary parts of the p-exceptional degrees of G_0.

Proof. In the proof of this theorem we use results established in our paper [PSZ07].

Let p be a prime integer. We fix preimages e_i of $x_i \in \mathrm{Ch}^*(\overline{G})$. For an r-tuple $M = (m_1, \ldots, m_r)$ we set $e^M = \prod_{i=1}^{r} e_i^{m_i}$.

First we recall the definition of filtrations on $\mathrm{Ch}^*(\overline{X})$ and $\overline{\mathrm{Ch}}^*(X)$ (see [PSZ07, Definition 4.13]). Given two pairs (L, l) and (M, m), where L and M are r-tuples and l and m are integers, we say that $(L, l) \leq (M, m)$ if either $L < M$, or $L = M$ and $l \leq m$.

The (M, m)-th term of the filtration on $\mathrm{Ch}^*(\overline{X})$ is the subgroup of $\mathrm{Ch}^*(\overline{X})$ generated by the elements $e^I \alpha$, $I \leq M$, $\alpha \in R^{\leq m}$. We denote as $A^{*,*}$ the graded ring associated to this filtration. As $A_{\mathrm{rat}}^{*,*}$ we denote the graded subring of $A^{*,*}$ associated to the subring $\overline{\mathrm{Ch}}^*(X) \subset \mathrm{Ch}^*(\overline{X})$ of rational cycles with the induced filtration.

Consider the Poincaré polynomial of A_{rat} with respect to the grading induced by the usual grading of $\mathrm{Ch}^*(\overline{X})$. Proposition 4.18 of [PSZ07] which explicitely describes a $\mathbb{Z}/p$-basis of $A_{\mathrm{rat}}^{*,*}$ implies that the Poincaré polynomial $g(A_{\mathrm{rat}}, t) =: \sum_{i=0}^{\dim X} a_i t^i$ $(a_i \in \mathbb{Z})$ of A_{rat} equals the right hand side of formula (8).

On the other hand, $\dim \overline{\mathrm{Ch}}^*(X) = \dim A_{\mathrm{rat}}$ and the coefficients b_i of the Poincaré polynomial $g(\overline{\mathrm{Ch}}^*(X), t) =: \sum_{i=0}^{\dim X} b_i t^i$ are obviously bigger than or equal to a_i for all i. Therefore $g(\overline{\mathrm{Ch}}^*(X), t) = g(A_{\mathrm{rat}}, t)$. This finishes the proof of the theorem. $\square$

The right hand side of formula (8) depends only on the value of the J-invariant of G. In turn, the left hand side depends on the rationality of cycles on X. Available information on cycles that are rational as sure as fate, allows to establish the following result.

5.4 Theorem. *Let G be a group of type $\Phi = F_4$, E_6, E_7 or E_8 given by a 1-cocycle from $H^1(k, G_0)$, where G_0 stands for the split adjoint group of the same type as G, and let X be the variety of the parabolic subgroups of G of type i. The variety X is not generically split if and only if*

1	$\Phi = F_4$, $i = 4$, $J_2(G) = (1)$
2	$\Phi = E_6$, $i = 1, 6$, $J_2(G) = (1)$
3	$\Phi = E_6$, $i = 2, 4$, $J_3(G) = (j_1, *)$, $j_1 \neq 0$
4	$\Phi = E_7$, $i = 1, 3, 4, 6$, $J_2(G) = (j_1, *, *, *)$, $j_1 \neq 0$
5	$\Phi = E_7$, $i = 1, 6, 7$, $J_2(G) = (*, j_2, *, *)$, $j_2 \neq 0$
6	$\Phi = E_7$, $i = 7$, $J_3(G) = (1)$
7	$\Phi = E_8$, $i = 1, 6, 7, 8$, $J_2(G) = (j_1, *, *, *)$, $j_1 \neq 0$
8	$\Phi = E_8$, $i = 7, 8$, $J_3(G) = (1, *)$

("$*$" means any value).

Proof. First we prove using Theorem 5.3 that the cases listed in the table are not generically split. Indeed, assume ther contrary. Then in cases 3) and 4) the right hand side of formula (8) does not have a term of degree 1. On the other hand, the Picard group $\mathrm{Pic}(\overline{X})$ is rational.

In cases 1), 5), 7) the right hand side of formula (8) does not have a term of degree 3. On the other hand, the group $\mathrm{Ch}^3(\overline{X})$ is rational: in all these cases it is contained in the subring generated by (rational) $\mathrm{Ch}^1(\overline{X})$, $\mathrm{Ch}^2(\overline{X})$, and by the Chern classes of the tangent bundle $T_{\overline{X}}$ (which are rational).

In case 8) one comes to the same contradiction considering $\mathrm{Ch}^4(\overline{X})$. In case 2) one easiely comes to a contradiction, since in this case the right hand side of formula (8) does not divide the Poincaré polynomial $g(\mathrm{Ch}^*(\overline{X}), t)$.

Next we show that all other varieties not listed in the table are generically split. Let G and X be an exceptional group and a G-variety not listed in the table. Consider $G_{k(X)}$. Using Proposition 4.3(2) one immediatelly sees case-by-case that for all primes p the J-invariant of the anisotropic kernel of $G_{k(X)}$ is trivial. Therefore this anisotropic kernel is trivial by Proposion 4.3(4) and the group G splits over $k(X)$. $\square$

5.5 Remark. The cases 3) and 4) in the table above also follow from the index reduction formula for exceptional groups [MPW98].

5.6 Remark. In view of the results obtained in [PSZ07] and in the present paper the following holds:

1. $\Phi = \mathrm{F}_4, \mathrm{E}_6$, $J_2(G) = (1)$ if and only if G has a non-trivial cohomological invariant f_3.

2. $\Phi = \mathrm{E}_6$, $J_3(G) = (j_1, *)$ or $\Phi = \mathrm{E}_7$, $J_2(G) = (j_1, *, *, *)$, $j_1 \neq 0$, if and only if G has a non-trivial Tits algebra.

3. $\Phi = \mathrm{E}_6, \mathrm{E}_7$, $J_3(G) = (1)$ if and only if G has a non-trivial cohomological invariant g_3.

6 Index of groups of type E_7

In this section we prove an index reduction formula for groups of type E_7. Our result can be considered as a generalization of the usual index reduction formula for central simple algebras.

Other variations on this theme are the Main Tool Lemma of A. Vishik [Vi07, Theorem 3.1] and an application of the Rost degree formula [Me03, Theorem 7.2].

To prove the main result of this section we use Chow motives and Steenrod operation and a relation between rational cycles on projective homogeneous varieties and their splitting properties.

To simplify the notation we will denote the Lefschetz motives in the category of Chow motives with $\mathbb{Z}/p$-coefficient not as $(\mathbb{Z}/p)(i)$, but still as $\mathbb{Z}(i)$. The restriction on the characteristic in the following theorem comes from Steenrod operations that we use in the proof, since so far they are not constructed in characteristic 2.

6.1 Theorem. *Let G be an anisotropic group of type E_7 and X (resp. Y) be the projective G-homogeneous variety of parabolic subgroups ot type 1 (resp. 7) over a field k with $\operatorname{char} k \neq 2$. Then Y has a $k(X)$-rational point if and only if Y has a zero-cycle of degree 1.*

In particular, Y has no $k(X)$-rational points if G has a non-trivial Tits algebra, or if the absolute Galois group $\operatorname{Gal}(k_s/k)$ is a pro p-group, or if the Rost invariant of G has order divisible by 4, or if k is a perfect field with $\operatorname{char} k \neq 2, 3$.

Proof. Before proving this theorem we discuss briefly the plan of the proof. First, assuming that Y has a $k(X)$-rational point we find a rational cycle, say α, in codimension 17 on the product $X \times X$. Using Steenrod operations we produce certain projector-like cycle, say β, on $X \times X$. Applying duality arguments to β we find a sub-cycle, say γ, in the cycle α. On the other hand, multiplying α by a certain cycle known to be always rational, we obtain again a projector-like cycle, but of another shape because of the existence of γ inside of α. Applying duality arguments again we come to a contradiction.

From now on an till the end of the proof of this theorem we set $p = 2$. To simplify the notation we denote as $\mathrm{pt} = [X_1]$ the class of a rational point on $\overline{X}$ (or $\overline{Y}$).

Assume first that the variety Y does not have a zero-cycle of degree 1, but Y has a $k(X)$-rational point. All claims below are proved under this assumption.

6.2 Claim. *The varieties X and Y are not generically split.*

Assume that $J_2(G)$ is trivial. Then G splits over an odd degree field extension by [PSZ07, Corollary 6.10]. On the other hand, the variety Y becomes isotropic over a quadratic extension of k by [Fe72, Corollary 3.4]. Therefore Y has a zero-cycle of degree 1. Contradiction.

Thus, $J_2(G)$ is not trivial. Now the variety X is not generically split by Corollary 5.4. On the other hand, the variety Y is not generically split, since it has a rational point over $k(X)$ and G is not split over $k(X)$.

In the following claims we use the Sweedler notation for Hopf algebras to denote the cycles in the Chow rings of projective homogeneous varieties, i.e., we do not write the sums and the indices.

6.3 Claim. *Some power of any element in a finite monoid is an idempotent. In particular, for $q = 1 \times \mathrm{pt} + x_{(1)} \times x_{(2)} \in \mathrm{Ch}^{\dim X}(\overline{X} \times \overline{X})$ with $x_{(2)} \in \mathrm{Ch}^{<\dim X}(\overline{X})$ there exists $n \in \mathbb{N}$ such that $q^{\circ n}$ is a non-trivial projector.*

The first statement of the claim is well known. For completeness we give its proof. Let q be an element of our monoid. Since the monoid is finite, we can find in the sequence $\{q^i\}_{i \in \mathbb{N}}$ two equal cycles q^{n_1} and q^{n_2} with $n_2 \geq 2n_1$. Define $n = (n_2 - n_1)n_1$. Then $q^{2n} = q^n$, i.e., q^n is an idempotent.

6.4 Lemma. *Let X and Y be arbitrary projective homogeneous varieties such that $X_{k(Y)}$ and $Y_{k(X)}$ have zero-cycles of degree prime to p. Then the Chow*

motives $\mathcal{M}(X)$ and $\mathcal{M}(Y)$ with $\mathbb{Z}/p$-coefficients have a common non-trivial direct summand R such that $R_{k_s} \simeq \mathbb{Z} \oplus M$ for some motive M.

Proof. Since X has a $k(Y)$-rational cycle of degree prime to p and Y has a $k(X)$-rational cycle of degree prime to p, we can apply [PSZ07, Lemma 1.6] (generic point argument) and get two cycles $\alpha \in \mathrm{Ch}^{\dim Y}(X \times Y)$ and $\beta \in \mathrm{Ch}^{\dim X}(Y \times X)$ such that $\overline{\alpha} = 1 \times \mathrm{pt} + x_{(1)} \times x_{(2)}$ and $\overline{\beta} = 1 \times \mathrm{pt} + x'_{(1)} \times x'_{(2)}$. The compositions $\alpha \circ \beta$ and $\beta \circ \alpha$ give cycles on $Y \times Y$ and $X \times X$ as in the previous claim. Therefore some powers $(\alpha \circ \beta)^{\circ n}$ and $(\beta \circ \alpha)^{\circ n}$ define projectors over k_s. The mutually inverse isomorphisms between the motives corresponding to these projectors are given by the rational maps α and $\beta \circ (\alpha \circ \beta)^{\circ n-1}$. Applying the Rost nilpotence theorem [CGM05, Section 8] we finish the proof of the lemma. $\qquad\square$

Now it easily follows from the classification of Tits indices that our varieties X and Y of the Theorem have a common motivic summand as above. We denote this summand as R.

6.5 Claim. $R_{k(X)} \simeq \mathbb{Z} \oplus \mathbb{Z}(17) \oplus R'$ *for some motive R'.*

Consider the motive $R_{k(X)}$. We claim first that $R_{k(X)} \simeq \mathbb{Z} \oplus \mathbb{Z}(l) \oplus R'$ for some motive R' and some $l \in \mathbb{Z}$. Indeed, let $q \in \mathrm{Ch}_{\dim X}(X \times X)$ be a projector corresponding to R. Consider $\overline{q}$ over k_s. The cycle $\overline{q} \cdot \overline{q}^t$, where $\overline{q}^t$ denotes the transposed cycle, equals $n\mathrm{pt} \times \mathrm{pt}$, where n is the dimension of the realization $\mathrm{Ch}^*(X, \overline{q})$. The number n is, of course, even, since otherwise we get a rational zero-cycle on X of odd degree, and, since by [Fe72, Corollary 3.4] the group $2\,\mathrm{CH}_0(\overline{Y})$ is rational, we get a zero-cycle of degree 1 on Y which contradicts the assumptions of the Theorem. On the other hand, the Krull-Schmidt theorem [CM06, Theorem 9.6] in the category of Chow motives implies that $R_{k(X)}$ is a direct sum of the Lefschetz motives and the indecomposable Rost motives. (The Rost motives are indecomposable, since $(G_{k(X)})_{\mathrm{an}}$ has type D_4 by our assumptions and by Claim 6.2). Therefore $R_{k(X)}$ must contain as a direct summand some Lefschetz motive $\mathbb{Z}(l)$ matching with $\mathbb{Z}$.

Next we compute l. By assumptions the motive R is a common motive of X and Y. Therefore the number l has the property that $\mathbb{Z}(l)$ is a direct summand of the motives $\mathcal{M}(X)$ and $\mathcal{M}(Y)$ over $k(X)$ (or over $k(Y)$). Using the Hasse diagrams of X and Y (see [PlSeVa, Figures 21 and 23]) one can easily see as in Example 3.1 that there is only one such common dimension, namely $l = 17$.

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In the following claim we put all computations that we need to prove our theorem. The computations were done using algorithms described in Section 3. As $[i_1, \ldots, i_l]$ we denote the product $s_{i_1} \ldots s_{i_l}$ in the Weyl group.

6.6 Claim. *a) The 16-th Steenrod operation (modulo 2) of*

$$f := Z_{[7,6,5,4,3,2,4,5,6,1,3,4,5,2,4,3,1]} \in \mathrm{Ch}^{17}(\overline{X})$$

equals $S^{16}(f) = X_{[1]} = \mathrm{pt}.$
   b) The 16-th Chern class of the tangent bundle $T_{\overline{X}}$ *of* $\overline{X}$ *equals*

$$c := Z_{[6,5,4,2,3,1,4,3,5,4,2,6,5,4,3,1]} + Z_{[4,2,3,1,4,3,6,5,4,2,7,6,5,4,3,1]}$$
$$+ Z_{[4,3,1,5,4,3,6,5,4,2,7,6,5,4,3,1]} \in \mathrm{Ch}^{16}(\overline{X}).$$

   c) For any $g \in \mathrm{Ch}^9(\overline{X})$ *and* $h \in \mathrm{Ch}^8(\overline{X})$ $S^8(g)S^8(h) = cgh.$
   d) $c \cdot \mathrm{Ch}^6(\overline{X}) = c \cdot \mathrm{Ch}^{12}(\overline{X}) = 0.$
   e) $S^8(\mathrm{Ch}^8(\overline{X})) = (\mathbb{Z}/2)c.$

   Proof of Theorem 6.1: Consider the cycle $f = Z_{[7,6,5,4,3,2,4,5,6,1,3,4,5,2,4,3,1]} \in \mathrm{Ch}^{17}(\overline{X})$. This cycle is rational over $k(X)$ by [CGM05, Proposition 6.1]. Indeed, the Hasse diagram for $\overline{X}$ is represented in [PlSeVa, Figure 23] (That figure contains the left half of the Hasse diagram which is too big to be represented in whole. One should symmetrically reflect that diagram to get a complete picture). Since by assumption the group $(G_{k(X)})_{\mathrm{an}}$ has type D_4, one should erase from the Hasse diagram of $\overline{X}$ all edges with labels 1, 6, and 7. One immediately sees that the vertex corresponding to f splits away from the diagram. Thus, f is defined over $k(X)$.

   By [PSZ07, Lemma 1.6] the cycle $a := 1 \times f + x_{(1)} \times x_{(2)} \in \mathrm{Ch}^{17}(\overline{X} \times \overline{X})$, $x_{(2)} \in \mathrm{Ch}^{<17}(\overline{X})$, is rational (i.e., defined over k). By Claim 6.6(a) the cycle $S^{16}(a) = 1 \times \mathrm{pt} + x'_{(1)} \times x'_{(2)} \in \mathrm{Ch}^{33}(\overline{X} \times \overline{X})$, $x'_{(2)} \in \mathrm{Ch}^{<33}(\overline{X})$ ($\dim \overline{X} = 33$). Therefore using Claim 6.3 one obtains a (rational) projector on $\overline{X} \times \overline{X}$. We denote this projector as q.

   We claim that this projector contains a summand of the form $r \times s$ with $r \in \mathrm{Ch}^{17}(\overline{X})$, $s \in \mathrm{Ch}^{16}(\overline{X})$ and $rs = \mathrm{pt}$. Indeed, by Claim 6.5 the motive $\mathcal{M}(X)$ has an indecomposable direct summand R such the projector corresponding to R_{k_s} contains the sum $1 \times \mathrm{pt} + r' \times s'$ with $r' \in \mathrm{Ch}^{17}(\overline{X})$, $s' \in \mathrm{Ch}^{16}(\overline{X})$ and $r's' = \mathrm{pt}$. Our projector q contains the summand $1 \times \mathrm{pt}$. Therefore the Krull-Schmidt theorem for Chow motives with finite coefficients [CM06, Theorem 9.6] implies that q must also contain a summand $r \times s$ with $r \in \mathrm{Ch}^{17}(\overline{X})$, $s \in \mathrm{Ch}^{16}(\overline{X})$ and $rs = \mathrm{pt}$.

Since q comes from the cycle a by means of S^{16}, the cycle a contains a summand $g \times h$ with $g \in \mathrm{Ch}^9(\overline{X})$ and $h \in \mathrm{Ch}^8(\overline{X})$. Indeed, otherwise we can't get the cycle $r \times s$ in q, since $S^i(\alpha) = 0$ for all $\alpha \in \mathrm{Ch}^{<i}(\overline{X})$. Moreover, a contains a summand $g \times h$ ($g \in \mathrm{Ch}^9(\overline{X})$, $h \in \mathrm{Ch}^8(\overline{X})$) with $S^8(g)S^8(h) = \mathrm{pt}$, since $rs = \mathrm{pt}$.

Consider now the (rational) product $(1 \times c) \cdot a \in \mathrm{Ch}^{33}(\overline{X} \times \overline{X})$. This product contains the sum $1 \times cf + g \times ch = 1 \times \mathrm{pt} + g \times ch$. As in Claim 6.3 we may assume that $(1 \times c) \cdot a$ is a projector which contains the summands $1 \times \mathrm{pt}$ and $g \times ch$ by Claim 6.6(c). The motive corresponding to this sum has the Poincaré polynomial $1 + t^9$. Consider now the motive of X over $k(X)$. As in Example 3.1 one can see that $\mathcal{M}(X)$ splits as a direct sum of (twisted) Lefschetz motives and (twisted) Rost motives corresponding to the anisotropic kernel of $G_{k(X)}$ which has strongly inner type D_4 by our assumptions and by Claim 6.2. The Poincaré polynomial of the Rost motive appearing in the motivic decomposition is $1 + t^3$. Moreover, the Lefschetz motive $\mathbb{Z}(9)$ does not appear in the motivic decomposition. Therefore by the Krull-Schmidt theorem [CM06, Theorem 9.6] the summand $g \times ch$ of the product $(1 \times c) \cdot a$ is a part of a twisted Rost motive. Therefore the product $(1 \times c) \cdot a$ must contain a matching summand for $g \times ch$. Since the Poincaré polynomial of our Rost motive is $1 + t^3$ the matching to $g \times ch$ summand has the form $\widetilde{g} \times \widetilde{ch}$ with $\widetilde{g} \in \mathrm{Ch}^{9\pm3}(\overline{X})$. This leads to a contradiction with Claim 6.6(d).

Assume now that Y does not have a $k(X)$-rational point. We will show that the variety Y does not have a zero-cycle of degree 1. We may assume that the Tits algebras of G are trivial, since over each field which makes Y isotropic the Tits algebras of G are split and, hence, Y does not have a zero-cycle of degree 1.

Consider $Y_{k(X)}$. Using the method of Chernousov-Gille-Merkurjev and Brosnan (see Section 3) one can decompose its motive as follows: $\mathcal{M}(Y_{k(X)}) \simeq \mathcal{M}(Q) \oplus \mathcal{M}(Z)(6) \oplus \mathcal{M}(Q)(17)$, where Q is the quadric corresponding to $G_{k(X)}$ and Z is its maximal orthogonal Grassmannian. In particular, the image of the degree map $\deg\colon \mathrm{CH}_0(Y_{k(X)}) \to \mathbb{Z}$ coincides with the image of $\deg\colon \mathrm{CH}_0(Q) \to \mathbb{Z}$ which is known to be $2\mathbb{Z}$ by Springer's theorem. Thus, $Y_{k(X)}$ does not have a zero-cycle of degree 1 and therefore Y also doesn't. The theorem is proved.

The variety Y does not have a zero-cycle of degree 1 in the cases listed in the Theorem either by obvious reasons, or by [GS08]. $\qquad\square$

6.7 Remark. The index reduction formula for central simple algebras [MPW98, Section 8, Type E_7] implies that Theorem 6.1 holds if the group G has a non-trivial Tits algebra. Our proof does not depend on the (non)-triviality of the Tits algebras.

6.8 Remark. The following papers discuss the problem of the existence of zero-cycles of degree 1 on anisotropic varieties: [Fl04], [Par05], [Serre], [To04].

Now we summarize the results of the present paper and provide a complete list of the the higher Tits indices for anisotropic groups of type F_4 and $^1E_6$.

F_4	$J_2 = (0)$	$\{1, F_4\}$
F_4	$J_2 = (1)$	$\{1, B_3, F_4\}$
E_6	$J_2 = (0),\ J_3 = (0, 1)$	$\{1, E_6\}$
E_6	$J_2 = (1),\ J_3 = (0, *)$	$\{1, D_4, E_6\}$
E_6	$J_2 = (0),\ J_3 = (j_1, *),\ j_1 \neq 0$	$\{1, 2A_2, E_6\}$
E_6	$J_2 = (1),\ J_3 = (j_1, *),\ j_1 \neq 0$	$\{1, 2A_2, D_4, E_6\}$

Let now G be an anisotropic group of type E_7 with trivial Tits algebras over a field k. Denote as $\Omega(G)$ the higher Tits index of G. Theorems 5.4 and 6.1 imply that

$D_4 \in \Omega(G)$ iff $J_2(G)$ is non-trivial,

$E_6 \in \Omega(G)$ iff $J_3(G)$ is non-trivial, and

if char $k \neq 2$, then $D_6 \in \Omega(G)$ iff the variety of parabolic subgroups of G of type 7 does not have a zero-cycle of degree 1.

In partucular, using [GS08, Corollary 2.2] we have:

6.9 Proposition. *Let k be a perfect field and* char $k \neq 2, 3$. *Then $\Omega(G) \supset \{1, D_4, D_6, E_7\}$. In particular, there exists a field extension K/k such that G_K has semisimple anisotropic kernel of type D_6.*

The following corollaries of this statement and their proofs are due to S. Garibaldi.

6.10 Corollary. *Let G be a group of type E_7 with trivial Tits algebras over a perfect field with* char $k \neq 2, 3$. *Then we can speak about the Rost invariant $r(G)$ of G (see [Inv, § 31.B]). Assume that $r(G)$ lies in $H^3(k, \mathbb{Z}/4)$ and is a symbol. Then G is isotropic.*

Proof. For sake of contradiction, suppose G is anisotropic. Take K as in Proposition 6.9. By assumptions the group $G \otimes_k K$ has semisimple anisotropic kernel $\mathrm{Spin}(q)$ for a 12-dimensional quadratic form q with the Arason invariant $e_3(q)$ a symbol. By [Ga07, Lemma 12.5] the form q is isotropic. $\square$

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6.11 Corollary (Kernel of the Rost invariant). *Let G_0 denote a split simply connected group of type E_7 over an arbitrary field k. Suppose $\eta \in H^1(k, G_0)$ and the Rost invariant of η is trivial. Then $\eta = 0$.*

Proof. By [Gi00] we may assume that char $k = 0$. By Corollary 6.10 the twisted group $_\eta(G_0)$ is isotropic. So η is equivalent to a cocycle with values in E_6^{sc} (a split simply connected group of type E_6) or $\mathrm{Spin}_{12}(q)$ (q split). But the triviality of the kernel of the Rost invariant for these two groups is well known (see [Ga01b, Theorem 0.5]). $\qquad\square$

Thus, we provided a shortened proof of the triviality of the kernel of the Rost invariant for groups of type E_7; cf. [Ga01b] and [C03].

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Appendix
by Mathieu Florence

We prove here the following result:

6.12 Theorem. *Let Φ be a root system. There exists a field K and a group G over K of type Φ such that the higher Tits index of G is the maximal possible, i.e.,*

$$\{\mathrm{ind}(G_L)_{\mathrm{an}} \mid L/K \ a \ field \ extension\}$$

$$= \{\mathrm{ind}(H_l)_{\mathrm{an}} \mid l \ a \ field, H \ a \ group \ over \ l \ of \ type \ \Phi\}.$$

Proof. Let k be a prime field and G_0 a split semisimple simply connected group of type Φ over k. Its automorphism group $H = \mathrm{Aut}(G_0)$ is an algebraic group. Consider a generically free finite-dimensional representation V of H over k. We can find an open subset $U \subset \mathbb{A}(V)$ of the affine space of V which is H-invariant such that the categorical quotient $U \to U/H$ exists and is an H-torsor. Let η be the generic point of U/H and $(G_0)_\eta$ (resp. H_η, P) denote the pullback $G_0 \times_{\mathrm{Spec}\,k} \eta$ (resp. $H \times_{\mathrm{Spec}\,k} \eta$, $U \times_{U/H} \eta$). We view $(G_0)_\eta$ and H_η as algebraic groups over η, and P as a H_η-torsor. Denote by G the twist of $(G_0)_\eta$ by P. For a standard parabolic subset $\Psi \subset \Phi$, let $X_\Psi \longrightarrow \eta$ be the homogenous space of parabolic subgroups of G of type Ψ. We claim that there exists a k-variety S, with the function field η, a faithfully flat group scheme $\mathfrak{G} \longrightarrow S$ and $\mathfrak{G}$-homogeneous spaces $\mathfrak{X}_\Psi \longrightarrow S$ which are models of G and of the X_Ψ's respectively, and which are generic in the following sense:

(*) For every field extension K/k with K infinite, and every semisimple simply connected group G' of type Φ over K, there exists a point $x \in S(K)$ such that the pullback of $\mathfrak{G} \longrightarrow S$ by x is isomorphic to G', and such that, for all standard parabolic subsets $\Psi \subset \Phi$, the homogeneous space of parabolic subroups of type Ψ of G' is isomorphic to the pullback of $\mathfrak{X}_\Psi \longrightarrow S$ by x.

Moreover, the same remains true if we replace S by a nonempty open suvariety $S' \subset S$. If we remove the assertion concerning standard parabolic subgroups from (*), then the result is well-known[3]. The proof of (*) is then a formal consequence of the following facts:

(i) For K, G' and Ψ as in (*), the homogeneous space of parabolic subgroups of type Ψ of G' is isomorphic to the twist of the homogeneous space of

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[3]G. Berhuy, G. Favi, Essential dimension: a functorial point of view, *Doc. Math.* **8** (2003), 279–330, Proposition 4.11.

parabolic subgroups of type Ψ of $G_0 \times_{\mathrm{Spec}\,k} \mathrm{Spec}\,K$ by the H-torsor (defined over K) corresponding to G';

(ii) Let Y denote any H-variety (over k), and P be an H-torsor over a base T/k which is integral and of finite-type. Let l denote the field of functions of T. Then the twist of $Y \times_{\mathrm{Spec}\,k} \mathrm{Spec}\,l$ by $P \times_T \mathrm{Spec}\,l$ can already be defined over a nonempty open subvariety S of T.

We omit the proofs of these assertions (for the second one, one has to use standard finite-typeness arguments).

Now let K/k be a field extension, with K infinite, and G' a semisimple simply connected group of type Φ, with anisotropic kernel of type $\Psi \subset \Phi$. Let η_Ψ denote the generic point of X_Ψ. We claim that $G_\Psi := G \times_\eta \eta_\Psi$ has anisotropic kernel of type Ψ. Indeed, assume the contrary. Then G_Ψ has a strictly smaller type $\Psi' \subset \Psi$. Thus there exists a rational map $X_\Psi \dashrightarrow X_{\Psi'}$ defined over η. Shrinking S, we may assume that there exists an S-morphism $\mathfrak{X}_\Psi \longrightarrow \mathfrak{X}_{\Psi'}$. Now let $x \in S(K)$ be a rational point as in (*). Specializing at x, we obtain that G' — which possesses parabolic subgroups of type Ψ by assumption — also possesses parabolic subgroups of type Ψ', a contradiction. $\qquad\square$

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