The J-invariant, Tits algebras and triality

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Abstract

In the present paper we set up a connection between the indices of the Tits algebras of a semisimple linear algebraic group G and the degree one indices of its motivic J-invariant. Our main technical tools are the second Chern class map and Grothendieck's γ-filtration.

As an application we provide lower and upper bounds for the degree one indices of the J-invariant of an algebra A with orthogonal involution σ and describe all possible values of the J-invariant in the trialitarian case, i.e., when degree of A equals 8. Moreover, we establish several relations between the J-invariant of (A, σ) and the J-invariant of the corresponding quadratic form over the function field of the Severi-Brauer variety of A.

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Introduction

The notion of a *Tits algebra* was introduced by Jacques Tits in his celebrated paper on irreducible representations [Ti71]. This invariant of a linear algebraic group G plays a crucial role in the computation of the K-theory of twisted flag varieties by Panin [Pa94] and in the index reduction formulas by Merkurjev, Panin and Wadsworth [MPW96]. It has important applications to the classification of linear algebraic groups, and to the study of the associated homogeneous varieties.

Another invariant of a linear algebraic group, the *J-invariant*, has been recently defined in [PSZ08]. It extends the J-invariant of a quadratic form which was studied during the last decade, notably by Karpenko, Merkurjev, Rost and Vishik. The J-invariant is a discrete invariant which describes the motivic behavior of the variety of Borel subgroups of G. It plays an important role in the classification of generically split projective homogeneous varieties and in studying of cohomological invariants of G (see [GPS10], [PS10]). Apart from this, it plays a crucial role in the solution of a problem posed by Serre about compact Lie groups of type E_8 (see [Sem09]).

The main goal of the present paper is to set up a connection between the indices of the Tits algebras of a group G and the elements of its motivic J-invariant corresponding to degree 1 generators. The main results are Cor. 4.2

and Thm. 4.7, which consist of inequalities relating those integers. As a crucial ingredient, we use Panin's computation of $K_0(\mathfrak{X})$, where $\mathfrak{X}$ is the variety of Borel subgroups of G [Pa94]. The result is obtained using Grothendieck's γ-filtration on $K_0(\mathfrak{X})$, and relies on the properties of the Steinberg basis and on Lemma 4.10, which describes Chern classes of rational bundles of the first two layers of the γ-filtered group $K_0(\mathfrak{X})$.

Let (A, σ) be a central simple algebra endowed with an involution of orthogonal type and trivial discriminant. Its automorphism group is a group of type D_n, and the J-invariant in this setting provides a discrete motivic invariant of (A, σ). The most interesting case is the degree 8 case, treated in Thm. 6.3, where the proof is based on triality, and precisely on its consequences on Clifford algebras (see [KMRT, §42.A]). In this case using results of Section 4, Dejaiffe's direct sum of algebras with involutions, and Garibaldi's "orthogonal sum lemma" we give a list of all possible values of the J-invariant.

The J-invariant of (A, σ) is an r-tuple of integers $(j_1, j_2, \dots, j_r)$ with $0 \leq j_i \leq k_i$ for some explicit upper bounds k_i (see Section 3). Moreover, the Steenrod operations provide additional restrictions on values of the J-invariant (see [PSZ08, 4.12] and Appendix below for a precise table for algebras with orthogonal involutions). For quadratic forms, this was already noticed by Vishik [Vi05, §5], who also checked that these restrictions are the only ones for quadratic forms of small dimension (*loc. cit.* Question 5.13). As opposed to this, we prove in Cor. 6.1 that some values for the J-invariant, which are not excluded by the Steenrod operations, do not occur. This happens already in the trialitarian case, and follows from a classical result on algebras with involution [KMRT, (8.31)], due to Tits and Allen.

The paper is organized as follows. In Sections 1 and 3 we introduce notation and explain known results. Sections 2 and 4 are devoted to the inequalities relating Tits algebras and the J-invariant. In Sections 5 and 6 we give applications to algebras with orthogonal involutions. Finally, we study in Section 7 the properties of the quadratic form attached to an orthogonal involution σ after generic splitting of the underlying algebra A. In particular, when this form belongs to the s-th power of the fundamental ideal of the Witt ring, we get interesting consequences on the J-invariant of (A, σ) in Thm. 7.2.

Note that Junkins in [Ju11] has successfully applied and extended our results for prime $p = 3$ to characterize the behavior of Tits indices of exceptional groups of type E_6.

1 Preliminaries. Notation.

1.1 *Roots and weights.* We work over a base field F of characteristic different from 2. Let G_0 be a split semisimple linear algebraic group of rank n over F. We fix a split maximal torus $T_0 \subset G_0$, and a Borel subgroup $B_0 \supset T_0$, and we let $\hat{T}_0$ be the character group of T_0. We let $\Pi = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$ be a set of simple roots with respect to B_0, and $\{\omega_1, \omega_2, \dots, \omega_n\}$ the respective set of fundamental weights, so that $\alpha_i^\vee(\omega_j) = \delta_{ij}$. The roots and weights are always numbered as in Bourbaki [Bou].

Recall that $\Lambda_r \subset \hat{T}_0 \subset \Lambda_\omega$, where Λ_r and Λ_ω are the root and weight lattices, respectively. The lattice $\hat{T}_0$ coincides with Λ_r (respectively Λ_ω) if and only if G_0 is adjoint (resp. simply connected).

1.2 *The Picard group.* We let $\mathfrak{X}_0$ be the variety of Borel subgroups of G_0, or equivalently of its simply connected cover G_0^{sc}. The Picard group $\mathrm{Pic}(\mathfrak{X}_0)$ can be computed as follows. Since any character $\lambda \in \hat{T}_0$ extends uniquely to B_0, it defines a line bundle $\mathcal{L}(\lambda)$ over $\mathfrak{X}_0$. Hence, there is a natural map $\hat{T}_0 \to \mathrm{Pic}(\mathfrak{X}_0)$, which is an isomorphism if G_0 is simply connected. So, we can identify the Picard group $\mathrm{Pic}(\mathfrak{X}_0)$ with the weight lattice Λ_ω (this fact goes back to Chevalley, see also [MT95, Prop. 2.2]).

1.3 *Inner forms.* Throughout the paper, G denotes a twisted form of G_0, and $T \subset G$ is the corresponding maximal torus. We always assume that G is an inner twisted form of G_0, and even a little bit more, that is $G = {}_\xi G_0$ for some cocycle $\xi \in Z^1(F, G_0)$. Note that not every twisted form of G_0 can be obtained in this way. For instance, if G_0 is simply connected, then G is a strongly inner twisted form of G_0.

We denote by $\mathfrak{X} = {}_\xi \mathfrak{X}_0$ the corresponding twisted variety. Observe that $\mathfrak{X}$ is the variety of Borel subgroups of G and, hence, is a projective homogeneous G-variety (see e.g. [MPW96, §1]). The varieties $\mathfrak{X}$ and $\mathfrak{X}_0$ are defined over F, and they are isomorphic over a separable closure F_s of F.

1.4 *Tits algebras.* Consider the simply connected cover G_0^{sc} of G_0 and the corresponding twisted group $G^{\mathrm{sc}} = {}_\xi G_0^{\mathrm{sc}}$, $\xi \in Z^1(F, G_0)$. We denote by $\Lambda_\omega^+ \subset \Lambda_\omega$ the cone of dominant weights. Since G is an inner twisted form of G_0, for any $\omega \in \Lambda_\omega^+$ the corresponding irreducible representation $G_0^{\mathrm{sc}} \to \mathrm{GL}(V)$, viewed as a representation of $G^{\mathrm{sc}} \times F_s$, descends to a representation $G^{\mathrm{sc}} \to \mathrm{GL}_1(A_\omega)$, where A_ω is a central simple algebra over F, called a Tits algebra of ξ (cf. [Ti71, §3,4] or [KMRT, §27]). In particular, to any fundamental weight ω_ℓ corresponds a Tits algebra A_{ω_ℓ}.

Taking Brauer classes, the assignment $\Lambda_\omega^+ \ni \omega \mapsto A_\omega$ induces a homomorphism $\beta\colon \Lambda_\omega/\hat{T}_0 \to \mathrm{Br}(F)$ known as the *Tits map* [Ti71].

For any $\omega \in \Lambda_\omega$ we denote by $\bar{\omega}$ its class in $\Lambda_\omega/\hat{T}_0$, by $i(\omega)$ the index of the Brauer class $\beta(\bar{\omega})$, that is the degree of the underlying division algebra. For fundamental weights, $i(\omega_\ell)$ is the index of the Tits algebra A_{ω_ℓ}.

1.5 *Algebras with involution.* We refer to [KMRT] for definitions and classical facts on algebras with involution. Throughout the paper, (A, σ) always stands for a central simple algebra of even degree $2n$, endowed with an involution of orthogonal type with trivial discriminant. In particular, this implies that the Brauer class $[A]$ of the algebra A is an element of order 2 of the Brauer group $\mathrm{Br}(F)$. Because of the discriminant hypothesis, the Clifford algebra of (A, σ), endowed with its canonical involution, is a direct product $(\mathcal{C}(A,\sigma), \underline{\sigma}) = (\mathcal{C}_+, \sigma_+) \times (\mathcal{C}_-, \sigma_-)$ of two central simple algebras. If moreover n is even, the involutions σ_+ and σ_- are also of orthogonal type.

1.6 *Hyperbolic involutions.* We refer to [KMRT, §6] for the definition of isotropic and hyperbolic involutions. In particular, recall that A has a hyperbolic involution if and only if it decomposes as $A = M_2(A')$ for some central simple algebra A' over F. When this occurs, A has a unique hyperbolic involution σ_0 up to isomorphism. Moreover, σ_0 has trivial discriminant, and if additionally the degree of A is divisible by 4, then its Clifford algebra has a split component by [KMRT, (8.31)].

1.7 *The cocenter for* D_n. The connected component of the automorphism group of (A,σ) is denoted by $\mathrm{PGO}^+(A,\sigma)$. Since the involution has trivial discriminant, it is an inner twisted form of PGO^+_{2n} (see 1.3). Both groups are adjoint of type D_n. We recall from Bourbaki [Bou] the description of their cocenter Λ_ω/Λ_r in terms of the fundamental weights, for $n \geq 3$:

If $n = 2m$ is even, then $\Lambda_\omega/\Lambda_r \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, and the three non-trivial elements are the classes of ω_1, ω_{2m-1} and ω_{2m} if $m \geq 2$.

If $n = 2m+1$ is odd, then $\Lambda_\omega/\Lambda_r \simeq \mathbb{Z}/4\mathbb{Z}$, and the generators are the classes of ω_{2m} and ω_{2m+1}. Moreover, the element of order 2 is the class of ω_1.

1.8 *Fundamental relations.* The Tits algebras A_{ω_1}, $A_{\omega_{2m-1}}$ and $A_{\omega_{2m}}$ of the group $G = \mathrm{PGO}^+(A,\sigma)$ are respectively the algebra A and the two components $\mathcal{C}_+$ and $\mathcal{C}_-$ of the Clifford algebra of (A,σ) (see [KMRT, §27.B]). Applying the Tits map, and taking into account the description of $\Lambda_\omega/\hat{T}_0 = \Lambda_\omega/\Lambda_r$, we get the so-called fundamental relations [KMRT, (9.12)] relating their Brauer classes, namely:

If $n = 2m$ is even, that is $\deg(A) \equiv 0 \mod 4$, then $[\mathcal{C}_+]$ and $[\mathcal{C}_-]$ are of order at most 2, and $[A] + [\mathcal{C}_+] + [\mathcal{C}_-] = 0 \in \mathrm{Br}(F)$. In other words, any of those three algebras is Brauer equivalent to the tensor product of the other two.

If $n = 2m+1$ is odd, that is $\deg(A) \equiv 2 \mod 4$, then $[\mathcal{C}_+]$ and $[\mathcal{C}_-]$ are of order dividing 4, and $[A] = 2[\mathcal{C}_+] = 2[\mathcal{C}_-] \in \mathrm{Br}(F)$.

2 Characteristic maps and restriction maps

2.1 *Characteristic map for Chow groups.* Let $\mathrm{CH}^*(-)$ be the graded Chow ring of algebraic cycles modulo rational equivalence. Since $\mathfrak{X}_0$ is smooth projective, the first Chern class induces an isomorphism between the Picard group $\mathrm{Pic}(\mathfrak{X}_0)$ and $\mathrm{CH}^1(\mathfrak{X}_0)$ [Ha, Cor. II.6.16]. Combining with the isomorphism $\Lambda_\omega \simeq \mathrm{Pic}(\mathfrak{X}_0)$ of 1.2, we get an isomorphism, which is the simply connected degree 1 characteristic map:

$$\mathfrak{c}^{(1)}_{\mathrm{sc}} \colon \Lambda_\omega \xrightarrow{\sim} \mathrm{CH}^1(\mathfrak{X}_0).$$

Hence, the cycles $h_i := c_1(\mathcal{L}(\omega_i))$, $i = 1 \ldots n$, form a $\mathbb{Z}$-basis of the group $\mathrm{CH}^1(\mathfrak{X}_0)$.

In general, the degree 1 characteristic map is the restriction of this isomorphism to the character group of T_0,

$$\mathfrak{c}^{(1)} \colon \hat{T}_0 \hookrightarrow \Lambda_\omega \longrightarrow \mathrm{CH}^1(\mathfrak{X}_0).$$

Hence, it maps $\lambda = \sum_{i=1}^n a_i\omega_i \in \hat{T}_0$, where $a_i \in \mathbb{Z}$, to $c_1(\mathcal{L}(\lambda)) = \sum_{i=1}^n a_i h_i$. For instance, in the adjoint case, the image of $\mathfrak{c}^{(1)}$ is generated by linear combinations $\sum_j c_{ij}h_j$, where $c_{ij} = \alpha_i^\vee(\alpha_j)$ are the coefficients of the Cartan matrix.

The degree 1 characteristic map extends to a characteristic map

$$\mathfrak{c}\colon\ S^*(\hat{T}_0) \longrightarrow \mathrm{CH}^*(\mathfrak{X}_0),$$

where $S^*(\hat{T}_0)$ is the symmetric algebra of $\hat{T}_0$ (see [Gr58, §4], [De74, §1.5]). Its image $\mathrm{im}(\mathfrak{c})$ is generated by the elements of codimension one, that is by the image of $\mathfrak{c}^{(1)}$.

2.2 **Example.** We let $p = 2$ and consider the Chow group with coefficients in $\mathbb{F}_2$ $\mathrm{Ch}^1(\mathfrak{X}_0) = \mathrm{CH}^1(\mathfrak{X}_0) \otimes_{\mathbb{Z}} \mathbb{F}_2$. Assume G_0 is of type D_4. Using the simply connected characteristic map, we can identify the degree 1 Chow group modulo 2 with the $\mathbb{F}_2$-lattice $\mathrm{Ch}^1(\mathfrak{X}_0) = \mathbb{F}_2h_1 \oplus \mathbb{F}_2h_2 \oplus \mathbb{F}_2h_3 \oplus \mathbb{F}_2h_4$.

In the adjoint case the image of the characteristic map $\mathfrak{c}_{\mathrm{ad}}^{(1)}$ with $\mathbb{F}_2$-coefficients is the subgroup $\mathrm{im}(\mathfrak{c}_{\mathrm{ad}}^{(1)}) = \mathbb{F}_2h_2 \oplus \mathbb{F}_2(h_1 + h_3 + h_4) \subset \mathrm{Ch}^1(\mathfrak{X}_0)$.

In the half-spin case, that is when one of the two weights ω_3, ω_4 is in $\hat{T}_0$, say $\omega_3 \in \hat{T}_0$, we get $\mathrm{im}(\mathfrak{c}_{\mathrm{hs}}^{(1)}) = \mathbb{F}_2h_2 \oplus \mathbb{F}_2h_3 \oplus \mathbb{F}_2(h_1 + h_4) \subset \mathrm{Ch}^1(\mathfrak{X}_0)$.

2.3 *Restriction map for Chow groups.* Let G and $\xi \in Z^1(F, G_0)$ be as in 1.3, so that $G = {}_\xi G_0$. The cocycle ξ induces an identification $\mathfrak{X} \times_F F_s \simeq \mathfrak{X}_0 \times_F F_s$. Moreover, since $\mathfrak{X}_0$ is split, $\mathrm{CH}^*(\mathfrak{X}_0 \times_F F_s) = \mathrm{CH}^*(\mathfrak{X}_0)$. Hence the restriction map can be viewed as a map

$$\mathrm{res}_{\mathrm{CH}}\colon\ \mathrm{CH}^*(\mathfrak{X}) \longrightarrow \mathrm{CH}^*(\mathfrak{X} \times_F F_s) \simeq \mathrm{CH}^*(\mathfrak{X}_0).$$

2.4 **Definition.** A cycle of $\mathrm{CH}^*(\mathfrak{X}_0)$ is called *rational* if it belongs to the image of the restriction $\mathrm{res}_{\mathrm{CH}}$.

In [KM06, Thm. 6.4(1)], it is proven that, under the hypothesis (1.3), any cycle in the image of the characteristic map $\mathfrak{c}$ is rational, i.e. $\mathrm{im}(\mathfrak{c}) \subseteq \mathrm{im}(\mathrm{res}_{\mathrm{CH}})$ (See [KM06, §7] to compare their $\bar{\varphi}_G$ with our characteristic map.)

2.5 **Remark.** Note that the image of the restriction map does not depend on the choice of G in its isogeny class, while the image of the characteristic map does.

For a split group G_0, the restriction map is an isomorphism, and this inclusion is strict, except if $H^1(F, G_0)$ is trivial. On the other hand, generic torsors are defined as the torsors for which it is an equality:

2.6 **Definition.** A cocycle $\xi \in Z^1(F, G_0)$ defining the twisted group $G = {}_\xi G_0$ is said to be *generic* if any rational cycle is in $\mathrm{im}(\mathfrak{c})$, so that $\mathrm{im}(\mathfrak{c}) = \mathrm{im}(\mathrm{res}_{\mathrm{CH}})$.

Observe that a generic cocycle always exists over some field extension of F by [KM06, Thm. 6.4(2)].

2.7 *Characteristic map for K_0.* Using the identification between Λ_ω and $\mathrm{Pic}(\mathfrak{X}_0)$ of 1.2, one also gets a characteristic map for K_0 (see [De74, §2.8]),

$$\mathfrak{c}_K\colon \mathbb{Z}[\hat{T}_0] \to K_0(\mathfrak{X}_0),$$

where $\mathbb{Z}[\hat{T}_0] \subset \mathbb{Z}[\Lambda_\omega]$ denotes the integral group ring of the character group $\hat{T}_0$. Any generator e^λ, $\lambda \in \hat{T}_0$, maps to the class of the associated line bundle $[\mathcal{L}(\lambda)] \in K_0(\mathfrak{X}_0)$.

Combining a theorem of Pittie [Pi72] (see also [Pa94, §0]), and Chevalley's description of the representation rings of the simply connected cover G_0^{sc} of G_0 and its Borel subgroup B_0^{sc}, one can check that $K_0(\mathfrak{X}_0)$ is isomorphic to the tensor product $\mathbb{Z}[\Lambda_\omega] \otimes_{\mathbb{Z}[\Lambda_\omega]^W} \mathbb{Z}$. That is, the simply-connected characteristic map

$$\mathfrak{c}_{K,\mathrm{sc}} \colon \mathbb{Z}[\Lambda_\omega] \to K_0(\mathfrak{X}_0)$$

is surjective, and its kernel is generated by the elements of the augmentation ideal that are invariant under the action of the Weyl group W.

2.8 *The Steinberg basis.* Consider the weights ρ_w defined for every w in the Weyl group W by $\rho_w = \sum_{\{\alpha_k \in \Pi,\ w^{-1}(\alpha_k) \in \Phi^-\}} w^{-1}(\omega_k)$, where Φ^- denotes the set of negative roots with respect to Π. The elements

$$g_w := \mathfrak{c}_{K,\mathrm{sc}}(e^{\rho_w}) = [\mathcal{L}(\rho_w)], \quad w \in W,$$

form a $\mathbb{Z}$-basis of $K_0(\mathfrak{X}_0)$, called the *Steinberg basis* (see [St75, §2] and [Pa94, §12.5]). Note that if w is the reflection $w = s_i$, $1 \leq i \leq n$, associated to the root α_i, we get

$$\rho_{s_i} = \sum_{\{\alpha_k \in \Pi,\ s_i(\alpha_k) \in \Phi^-\}} s_i(\omega_k) = s_i(\omega_i) = \omega_i - \alpha_i.$$

2.9 **Definition.** The elements of the Steinberg basis $g_i = [\mathcal{L}(\rho_{s_i})]$, $i = 1 \ldots n$, will be called *special* elements.

2.10 *Restriction map for K_0 and the Tits algebras.* As we did for Chow groups, we use the identification $\mathfrak{X} \times_F F_s \simeq \mathfrak{X}_0 \times_F F_s$ to view the restriction map for K_0 as a morphism

$$\mathrm{res}_{\mathrm{K}_0} \colon K_0(\mathfrak{X}) \to K_0(\mathfrak{X}_0) = \bigoplus_{w \in W} \mathbb{Z} \cdot g_w.$$

By Panin's theorem [Pa94, Thm. 4.1], the image of the restriction map, whose elements are called rational bundles, is the sublattice with basis

$$\{i(\rho_w) \cdot g_w,\ w \in W\},$$

where $i(\rho_w)$ is the index of the Brauer class $\beta(\bar{\rho}_w)$, that is the index of any corresponding Tits algebra (see 1.4).

Note that since the Weyl group acts trivially on $\Lambda_\omega/\hat{T}_0$, we have

$$\bar{\rho}_w = \sum_{\{\alpha_k \in \Pi | w^{-1}(\alpha_k) \in \Phi^-\}} \bar{\omega}_k.$$

Therefore, the corresponding Brauer class is given by

$$\beta(\bar{\rho}_w) = \Sigma_{\{\alpha_k \in \Pi | w^{-1}(\alpha_k) \in \Phi^-\}} \beta(\bar{\omega}_k).$$

In particular, for special elements we get $\beta(\bar{\rho}_{s_i}) = \beta(\bar{\omega}_i)$, so that $i(\rho_{s_i})$ is the index of the Tits algebra A_{ω_i}.

2.11 *Rational cycles versus rational bundles.* Since the total Chern class of a rational bundle is a rational cycle, the graded-subring $\mathfrak{B}^*$ of $\mathrm{CH}^*(\mathfrak{X}_0)$ generated by Chern classes of rational bundles consists of rational cycles. We use Panin's description of rational bundles to compute $\mathfrak{B}^*$. The total Chern class of $i(\rho_w)\cdot g_w$ is given by

$$c(i(\rho_w)\cdot g_w) = \big(1 + c_1(\mathcal{L}(\rho_w))\big)^{i(\rho_w)} = \sum_{k=1}^{i(\rho_w)} \binom{i(\rho_w)}{k} c_1(\mathcal{L}(\rho_w))^k$$

Therefore, $\mathfrak{B}^*$ is generated as a subring by the homogeneous elements

$$\binom{i(\rho_w)}{k} c_1(\mathcal{L}(\rho_w))^k, \text{ for } w \in W,\ 1 \le k \le i(\rho_w).$$

Let p be a prime number, and denote by $\mathfrak{i}_w$ the p-adic valuation of $i(\rho_w)$, so that $i(\rho_w) = p^{\mathfrak{i}_w} q$ for some prime-to-p integer q. By Luca's theorem [Di, p. 271] the binomial coefficient $\binom{i(\rho_w)}{p^{\mathfrak{i}_w}}$ is congruent to q modulo p. Hence its image in $\mathbb{F}_p$ is invertible. Considering the image in the Chow group modulo p of the rational cycle $\binom{i(\rho_w)}{p^{\mathfrak{i}_w}} c_1(\mathcal{L}(\rho_w))^{p^{\mathfrak{i}_w}}$, we get:

2.12 **Lemma.** *Let p be a prime number. For any w in the Weyl group, the cycle*

$$c_1(\mathcal{L}(\rho_w))^{p^{\mathfrak{i}_w}} \in \mathrm{CH}(\mathfrak{X}_0) \otimes_{\mathbb{Z}} \mathbb{F}_p \text{ \textit{is rational.}}$$

3 The J-invariant

In this section, we briefly recall the definition and key properties of the J-invariant following [PSZ08].

3.1 *The Chow ring of G_0.* Let us denote by $\pi\colon \mathrm{CH}^*(\mathfrak{X}_0) \to \mathrm{CH}^*(G_0)$ the pullback induced by the natural projection $G_0 \to \mathfrak{X}_0$, where $\mathfrak{X}_0$ is the variety of Borel subgroups of G_0. By [Gr58, §4, Rem. 2], π is surjective and its kernel is the ideal $I(\mathfrak{c}) \subset \mathrm{CH}^*(\mathfrak{X}_0)$ generated by non-constant elements in the image of the characteristic map (see Section 2). Therefore, there is an isomorphism of graded rings

$$\mathrm{CH}^*(\mathfrak{X}_0)/I(\mathfrak{c}) \simeq \mathrm{CH}^*(G_0).$$

In particular, in degree 1, we get

$$\mathrm{CH}^1(G_0) \simeq \mathrm{CH}^1(\mathfrak{X}_0)/(\operatorname{im} \mathfrak{c}^{(1)}) \simeq \Lambda_\omega/\hat{T}_0. \tag{1}$$

By [Kc85, Thm. 3] the Chow ring of G_0 with $\mathbb{F}_p$-coefficients is isomorphic as an $\mathbb{F}_p$-algebra (and even as a Hopf algebra) to

$$\mathrm{Ch}^*(G_0) \simeq \mathbb{F}_p[x_1, \ldots, x_r]/(x_1^{p^{k_1}}, \ldots, x_r^{p^{k_r}}) \tag{2}$$

for some integers r, k_i and degrees d_i of x_i such that $d_1 \le \ldots \le d_r$. Below in 3.5 we explain, how we choose this isomorphism in D_{2m} case. The number of generators of degree 1 of $\mathrm{Ch}^*(G_0)$, denoted by s, is the dimension over $\mathbb{F}_p$ of the vector space $\Lambda_\omega/\hat{T}_0 \otimes_{\mathbb{Z}} \mathbb{F}_p$.

3.2 *Motivic decomposition.* Let G be an inner form of G_0 that is $G = {}_\xi G_0$ for some $\xi \in Z^1(F, G_0)$. Consider the variety $\mathfrak{X}$ of Borel subgroups of G. Recall that $\mathfrak{X} \simeq {}_\xi\mathfrak{X}_0$. The main result (Thm. 5.13) in [PSZ08] asserts that the Chow-motive $M(\mathfrak{X})$ splits as a direct sum of twisted copies of some indecomposable motive $R_p(G)$ and the Poincaré polynomial of $R_p(G)$ over a separable closure of F (see [PSZ08, §1.3]) is given by

$$P(R_p(G) \times_F F_s, t) = \prod_{i=1}^{r} \frac{1 - t^{d_i p^{j_i}}}{1 - t^{d_i}}, \text{ for some } \ 0 \leq j_i \leq k_i. \tag{3}$$

The parameters r, d_i and k_i for $i = 1, \dots, r$ are the same as in the Chow ring of G_0, but the integers j_i depend on ξ.

In this way we obtain a multiset of pairs

$$\{(d_1, j_1), \dots, (d_r, j_r)\} \tag{4}$$

with $d_1 \leq \ldots \leq d_r$ and $0 \leq j_i \leq k_i$ for each $i = 1 \ldots r$. Recall from [PSZ08, §4] that $j_i = k_i$ for each $i = 1 \ldots r$ if ξ is a generic cocycle, and $j_1 = \ldots = j_r = 0$ if and only if $\mathfrak{X}$ has a point of degree prime to p or, equivalently, if ξ splits over a p-primary closure of the base field F. Moreover, as explained in [PSZ08, 4.7] the integers j_i can only decrease after extension of the base field.

Observe that the multiset $\{(d_1, j_1), \dots, (d_r, j_r)\}$ depends only on the group G and can be viewed as an invariant of G.

Let now G_{an} denote the semisimple anisotropic kernel of G and let

$$\{(d'_1, j'_1), \dots, (d'_m, j'_m)\}$$

be its (unordered) J-invariant. It follows from [PSZ08, Cor. 5.19] and formula (3) that the the multisets $\{(d_l, j_l) \mid j_l \neq 0\}$ and $\{(d'_l, j'_l) \mid j'_l \neq 0\}$ are equal, i.e. the non-zero entries in the J-invariants of G and G_{an} are the same.

3.3 *The J-invariant.* Choosing a cocycle $\xi \in Z^1(F, G_0)$ and an isomorphism (2) allows us to compute the integers j_i from (4) as follows: Let R_ξ denote the subring

$$\mathrm{Im}\big(\mathrm{Ch}^*({}_\xi\mathfrak{X}_0) \xrightarrow{\pi \circ \mathrm{res}_{\mathrm{Ch}}} \mathrm{Ch}^*(G_0) \xrightarrow{(2)} \mathbb{F}_p[x_1, \dots, x_r]/(x_1^{p^{k_1}}, \dots, x_r^{p^{k_r}})\big), \tag{5}$$

where the restriction map is defined in (2.3). Following [PSZ08, Definition 4.6] we introduce the deglex order on the set of generators $\{x_1, \dots, x_r\}$ with $x_1 < \ldots < x_r$. Define j_i to be the smallest non-negative integer a such that $x_i^{p^a}$ modulo smaller terms lies in R_ξ. We get an ordered r-tuple of integers $(j_1, \dots, j_r)$, whose elements precisely are the indices $j_1, \dots, j_r$ of (4). We call this tuple the *J-invariant* of ξ with respect to (2). In particular, the first element j_1 of the J-invariant is the smallest non-negative integer a such that $x_1^{p^a}$ belongs to R_ξ. Hence, $j_1 = 0$ if and only if $x_1 \in R_\xi$.

3.4 **Example.** One can check from the values given in the table [Kc85, Table II] (see also [PSZ08, §4]), that for simple groups G, except if $p = 2$ and

G is adjoint of type D_n with n even, the degrees d_i are pairwise distinct. If so, the increasing hypothesis $d_1 < \cdots < d_r$ determines a canonical ordering of the generators, and we obtain a well defined invariant $(j_1, j_2, \dots, j_r)$ of the group G which doesn't depend on a choice of a cocycle ξ with $G \simeq {}_\xi G_0$ and an isomorphism (2). We denote it by $J_p(G)$.

3.5 **Example.** Assume that $p = 2$ and G_0 is adjoint of type D_{2m}, with $m \geq 1$, that is $G_0 = \mathrm{PGO}^+_{4m}$. In this case $s = 2$ and an isomorphism (2) is uniquely determined by a choice of generators x_1 and x_2 of degree one. In view of (1) choosing two degree 1 generators for $\mathrm{Ch}^*(G_0)$ amounts to the choice of two generators of the cocenter $\Lambda_\omega/\Lambda_r = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ of the group. We set

$$x_1 = \pi(h_1),\ x_2 = \pi(h_{2m}) \text{ if } m \neq 1 \text{ and } x_1 = \pi(h_1 + h_2),\ x_2 = \pi(h_2) \text{ if } m = 1, \tag{6}$$

where $h_i = c_1(\mathcal{L}(\omega_i))$ as in Section 2. Observe that this is compatible with [PSZ08], since the relations $x_1^{2^{k_1}} = x_2^{2^{k_2}} = 0$ are fulfilled. By definition j_2 is then the smallest integer b such that

$$x_2^{2^b} + \sum_{0<i\leq 2^b} a_i x_1^i x_2^{2^b-i} \in R_\xi \text{ for some } a_i \in \mathbb{F}_2.$$

3.6 *The J-invariant of an algebra with involution.* Let $G_0 = \mathrm{PGO}^+_{4m}$ and consider $\xi \in Z^1(F, G_0)$. By [KMRT, p. 409] the class of ξ corresponds to a central simple algebra A of degree $4m$ with orthogonal involution σ and a designation of the two components of its Clifford algebra. Note that, if $m \geq 3$, A is the Tits algebra A_{ω_1}. Consider the two cocycles $\xi^+ = \xi$ and ξ^- corresponding to the opposite designation of the components of the Clifford algebra.

We claim that the J-invariants of ξ^+ and ξ^- with respect to (6) are equal. Indeed, since the two cocycles lead to the same group $\mathrm{PGO}^+(A,\sigma)$, the motivic point of view recalled in § 3.2 shows that the two tuples can only differ by a permutation of j_1 and j_2, which both correspond to generators of degree 1. Hence, it suffices to compute the first entry j_1 of the J-invariants of ξ^+ and ξ^-. The images of the restriction maps defined by ξ_+ and ξ_- differ by an automorphism of $\mathrm{Ch}^*(\mathfrak{X}_0)$, which permutes h_{2m-1} and h_{2m} and, hence, leaves the ideal $I(\mathfrak{c})$ invariant. This induces a ring automorphism of $\mathrm{Ch}^*(G_0)$, which fixes x_1. Hence, $x_1^{2^a} \in R_{\xi^+}$ if and only if $x_1^{2^a} \in R_{\xi^-}$ and, thus, j_1 for ξ^+ and ξ^- are equal.

So the tuple $(j_1, \dots, j_r)$ is an invariant of the algebra with involution (A, σ), denoted by $J(A,\sigma)$. If we are not in the trialitarian case, i.e., if $m \neq 2$, the algebra with involution (A,σ) is uniquely determined by its automorphism group; therefore, $(j_1, \dots, j_r)$ also is an invariant of the group G, which does not depend on the choice of a cocycle, and denoted by $J_2(G)$.

3.7 **Remark.** We could also take $x_1 = \pi(h_1)$ and $x_2 = \pi(h_{2m-1})$ in (6), this would not affect the value of the J-invariant.

3.8 *The trialitarian case.* In the trialitarian case, i.e., if $m = 2$, the twisted group $G = {}_\xi G_0$ can be described as the connected component of the automorphism group of three possibly non-isomorphic degree 8 algebras with orthogonal involution, which are the Tits algebras of the weights ω_1, ω_3 and ω_4 (see [KMRT, §42])

$$G \simeq \mathrm{PGO}^+(A, \sigma_A) \simeq \mathrm{PGO}^+(B, \sigma_B) \simeq \mathrm{PGO}^+(C, \sigma_C).$$

Therefore, the J-invariant does depend in this case of the choice of a cocycle. Precisely, picking a cocycle determines an ordering of those three algebras with involution, and the J-invariant of the cocycle is the J-invariant of the first algebra with involution in the corresponding triple. Nevertheless, since the automorphism group is the same, the variety $\mathfrak{X}$ and its motive do not depend on this choice. Hence, the multiset $\{(1, j_1), (1, j_2), (2, j_3)\}$ is the same for all three algebras with involution, so that if $J(A, \sigma_A) = (j_1, j_2, j_3)$, then

$$J(B, \sigma_B), J(C, \sigma_C) \in \{(j_1, j_2, j_3), (j_2, j_1, j_3)\}.$$

In Theorem 6.3 and Example 6.8 below, we give a more precise statement, and provide explicit examples of algebras with involution having isomorphic automorphism groups and different J-invariants.

4 The J-invariant in degree one and indices of the Tits algebras

In this section, we prove the main results of the paper, which give connections between the indices of the J-invariant corresponding to generators of degree 1 and indices of Tits algebras of the group G (cf. [PS10, §4]).

4.1 *Notation.* From now on, we let s be the dimension over $\mathbb{F}_p$ of $\Lambda_\omega/\hat{T}_0 \otimes \mathbb{F}_p \simeq \mathrm{Ch}^1(G_0)$, and we fix G_0 and a prime p so that $s \geq 1$. We fix a cocycle ξ, and a presentation (2) of the Chow group of G_0. The J-invariant in this section refers to the J-invariant of ξ with respect to (2). Moreover, we assume throughout this section that the degree 1 generators are given by $x_\ell = \pi(h_{i_\ell}) \in \mathrm{Ch}^1(G_0)$ for some integers $i_1, \ldots, i_s$, chosen so that the classes ω_{i_ℓ}, $\ell = 1, \ldots, s$, generate $\Lambda_\omega/\hat{T}_0 \otimes \mathbb{F}_p$.

Note that one can always choose generators x_ℓ in such a way, though this conflicts the convention we made in (6) to define the J-invariant of a degree 4 algebra with involution.

Consider the special elements g_i, $i = 1 \ldots n$ of the Steinberg basis of $K_0(\mathfrak{X}_0)$ (see Definition 2.9). Since $c_1(g_i) = h_i - c_1(\mathcal{L}(\alpha_i)) \in \mathrm{Ch}^1(\mathfrak{X}_0)$, we have

$$\pi(c_1(g_i)) = \pi(h_i) - \pi(c_1(\mathcal{L}(\alpha_i))) = \pi(h_i) \in \mathrm{Ch}^1(G_0).$$

Hence $x_\ell = \pi(c_1(g_{i_\ell}))$. In view of the isomorphism (1), it follows that for any $g \in \mathrm{Pic}(\mathfrak{X}_0)$ its Chern class modulo p can be written as

$$c_1(g) = \sum_{\ell=1}^{s} a_\ell c_1(g_{i_\ell}) \mod \mathrm{im}(\mathfrak{c}^{(1)}) \in \mathrm{Ch}^1(\mathfrak{X}_0) \tag{7}$$

As an immediate consequence of rationality of cycles introduced in Lemma 2.12 we obtain a different proof of the first part of [PS10, Prop. 4.2]:

4.2 **Corollary.** *For $\ell = 1, \ldots, s$ the first entries j_ℓ of the J-invariant are bounded*

$$j_\ell \leq \mathsf{i}_{i_\ell},$$

by the p-adic valuation i_{i_ℓ} of the index of the Tits algebra $A_{\omega_{i_\ell}}$ associated with ω_{i_ℓ}.

Proof. We apply lemma 2.12 to the weight $\rho_{s_{i_\ell}} = \omega_{i_\ell} - \alpha_{i_\ell}$. As noticed in 2.10, the index $i(\rho_{s_{i_\ell}})$ is equal to the index $i(\omega_{i_\ell})$ of the Tits algebra $A_{\omega_{i_\ell}}$. Hence, the cycle $c_1(g_{i_\ell})^{p^{\mathsf{i}_{i_\ell}}}$ is rational, and its image $x_\ell^{p^{\mathsf{i}_{i_\ell}}} \in \mathrm{Ch}^*(G_0)$ belongs to $R := R_\xi$. The inequality then follows from the definition of j_ℓ (see 3.3). □

Assume now that $p = 2$ and G_0 is adjoint of type D_{2m}, with $m \geq 2$. As explained in 3.5, we take $i_1 = 1$, $i_2 = 2m$, and define x_1 and x_2 as in (6).

4.3 **Corollary.** *If $p = 2$ and G is adjoint of type D_{2m} with $m \geq 2$, we have $s = 2$,*

$$j_1 \leq \mathsf{i}_1 \text{ and } j_2 \leq \min\{\mathsf{i}_{2m-1}, \mathsf{i}_{2m}\},$$

where i_ℓ is the 2-adic valuation of the index of the Tits algebra A_{ω_ℓ}.

The next result, which gives an inequality in the other direction, uses the notion of common index, which we introduce now.

4.4 **Definition.** Consider the Tits algebras $A_{\omega_{i_\ell}}$ associated to the fundamental weights ω_{i_ℓ}, for $1 \leq \ell \leq s$, where i_ℓ are as in 4.1. We define their *common index* i_J to be the p-adic valuation of the greatest common divisor of all the indices $\mathrm{ind}(A_{\omega_{i_1}}^{\otimes a_1} \otimes \ldots \otimes A_{\omega_{i_s}}^{\otimes a_s})$, where at least one of the a_i is coprime to p.

4.5 **Example.** If $s = 1$, then i_J is the p-adic valuation i_{i_1} of the index of the Tits algebra $A_{\omega_{i_1}}$. Assume for instance that G is adjoint of type D_{2m+1}. As recalled in 1.7, we may take $i_1 = 2m$ or $i_1 = 2m + 1$, so that i_J is the 2-adic valuation of any component $\mathcal{C}_+$ or $\mathcal{C}_-$ of the Clifford algebra of (A, σ). From the fundamental relations 1.8, we know that the two components have the same index.

4.6 **Example.** If $p = 2$ and G_0 is adjoint of type D_{2m} with $m \geq 2$, then $s = 2$. Using 1.8, one can check that i_J is the p-adic valuation of the greatest common divisor of the indices of A_{ω_1}, $A_{\omega_{2m-1}}$ and $A_{\omega_{2m}}$, that is

$$\mathsf{i}_J = \min\{\mathsf{i}_1, \mathsf{i}_{2m-1}, \mathsf{i}_{2m}\}.$$

We will prove:

4.7 **Theorem.** *Let i_J be the common index of the Tits algebras $A_{\omega_{i_\ell}}$, $\ell = 1 \ldots s$.*

If $\mathsf{i}_J > 0$, then $j_\ell > 0$ for every ℓ, $1 \leq \ell \leq s$.

If $\mathsf{i}_J > 1$ and $p = 2$, then for every ℓ such that $k_\ell > 1$ we have $j_\ell > 1$.

Proof. Consider the ideal $I(\mathrm{res}_{\mathrm{Ch}})$ of $\mathrm{Ch}^*(\mathfrak{X}_0)$ generated by non-constant rational elements. For any integer i, we let $I(\mathrm{res}_{\mathrm{Ch}})^{(i)} \subset \mathrm{Ch}^i(\mathfrak{X}_0)$ be the homogeneous part of degree i. Since the image of the characteristic map consists of rational elements, we have $I(\mathfrak{c}) \subset I(\mathrm{res}_{\mathrm{Ch}})$. The theorem follows immediately from the following lemma:

4.8 **Lemma.** *If $\mathfrak{i}_J > 0$, then $I(\mathrm{res}_{\mathrm{Ch}})^{(1)} = I(\mathfrak{c})^{(1)} \subset \mathrm{Ch}^1(\mathfrak{X}_0)$.*
If $\mathfrak{i}_J > 1$ and $p = 2$, then $I(\mathrm{res}_{\mathrm{Ch}})^{(2)} = I(\mathfrak{c})^{(2)} \subset \mathrm{Ch}^2(\mathfrak{X}_0)$.

Indeed, let us assume first that $\mathfrak{i}_J > 0$. By the lemma, any element in $\mathrm{im}(\mathrm{res}^{(1)}_{\mathrm{Ch}}) = I(\mathrm{res}_{\mathrm{Ch}})^{(1)}$ belongs to $I(\mathfrak{c})^{(1)}$, which is in the kernel of π. Therefore, the image of the composition

$$R^{(1)}_\xi = \mathrm{im}\big(\mathrm{Ch}^1(\mathfrak{X}) \xrightarrow{\mathrm{res}^{(1)}_{\mathrm{Ch}}} \mathrm{Ch}^1(\mathfrak{X}_0) \xrightarrow{\pi} \mathrm{Ch}^1(G_0)\big)$$

is trivial, $R^{(1)}_\xi = \{0\}$. From the definition 3.3, 3.5 of j_1 and j_2, this implies that they are both strictly positive.

The proof of the second part follows the same lines. We write it in details for $s = 2$ and $k_1, k_2 > 1$. Assume that $\mathfrak{i}_J > 1$. Since the image $\mathrm{im}(\mathrm{res}_{\mathrm{Ch}})^{(2)}$ is contained in $I(\mathrm{res}_{\mathrm{Ch}})^{(2)}$, the lemma again implies that $R^{(2)}_\xi = \{0\}$. On the other hand, the hypothesis on k_1 and k_2 guarantees that in the truncated polynomial algebra $\mathbb{F}_2[x_1, x_2]/(x_1^{2^{k_1}}, x_2^{2^{k_2}}) \subset \mathrm{Ch}^*(G_0)$, the elements x_1^2 and $x_2^2 + a_1x_1x_2 + a_2x_1^2$ are all non-trivial. Hence they do not belong to R_ξ, and we get $j_1, j_2 > 1$. □

The rest of the section is devoted to the proof of Lemma 4.8. The main tool is the Riemann-Roch theorem, which we now recall.

4.9 *Filtrations of K_0 and the Riemann-Roch Theorem.* Let X be a smooth projective variety over F. Consider the topological filtration on $K_0(X)$ given by

$$\tau^i K_0(X) = \langle [\mathcal{O}_V],\ \mathrm{codim}\, V \geq i \rangle,$$

where $\mathcal{O}_V$ is the structure sheaf of the closed subvariety V in X. There is an obvious surjection

$$\psi\colon\ \mathrm{CH}^i(X) \to \tau^{i/i+1} K_0(X) = \tau^i K_0(X)/\tau^{i+1} K_0(X),$$

given by $V \mapsto [\mathcal{O}_V]$. By the Riemann-Roch theorem without denominators [Ful, §15], the i-th Chern class induces a map in the opposite direction

$$c_i\colon \tau^{i/i+1} K_0(X) \to \mathrm{CH}^i(X)$$

and the composite $c_i \circ \psi$ is the multiplication by $(-1)^{i-1}(i-1)!$. In particular, it is an isomorphism for $i \leq 2$ (see [Ful, Ex. 15.3.6]).

The topological filtration can be approximated by the so-called γ-filtration. Let $c_i^{K_0}$ be the i-th Chern class with values in K_0 (see [Ful, Ex. 3.2.7(b)], or [Ka98, §2]). We use the convention $c_1^{K_0}([\mathcal{L}]) = 1-[\mathcal{L}^v]$ for any line bundle $\mathcal{L}$, where $\mathcal{L}^v$ is the dual of $\mathcal{L}$. Similarly, one can compute the second Chern class

$$c_2\big(c_1^{K_0}([\mathcal{L}_1])c_1^{K_0}([\mathcal{L}_2])\big) = -c_1(\mathcal{L}_1)c_1(\mathcal{L}_2). \tag{8}$$

The γ-filtration on $K_0(X)$ is given by the subgroups (cf. [GZ10, §1])

$$\gamma^i K_0(X) = \langle c_{n_1}^{K_0}(b_1)\cdot\ldots\cdot c_{n_m}^{K_0}(b_m) \mid n_1+\ldots+n_m \geq i,\ b_l \in K_0(X)\rangle,$$

(see [Ful, Ex.15.3.6], [FL, Ch.3,5]). We let $\gamma^{i/i+1}(K_0(X)) = \gamma^i K_0(X)/\gamma^{i+1}K_0(X)$ be the respective quotients, and $\gamma^*(X) = \oplus_{i\geq 0}\gamma^{i/i+1}(K_0(X))$ the associated graded ring.

By [Ka98, Prop. 2.14], $\gamma^i(K_0(X))$ is contained in $\tau^i(K_0(X))$, and they coincide for $i \leq 2$. Hence, by the Riemann-Roch theorem, the Chern class c_i with values in $\mathrm{CH}^i(X)$ vanishes on $\gamma^{(i+1)}K_0(X)$, and induces a map

$$c_i\colon \gamma^{i/i+1}(K_0(X)) \to \mathrm{CH}^i(X).$$

In codimension 1 we get an isomorphism

$$c_1\colon \gamma^{1/2}(K_0(X)) \xrightarrow{\simeq} \mathrm{CH}^1(X)$$

which sends for a line bundle $\mathcal{L}$ the class $c_1^{K_0}(\mathcal{L})$ to $c_1(\mathcal{L})$. In codimension 2 the map

$$c_2\colon \gamma^{2/3}(K_0(X)) \twoheadrightarrow \mathrm{CH}^2(X),$$

is surjective and has torsion kernel [Ka98, Cor. 2.15].

Let us now apply this to the varieties $\mathfrak{X}_0$ and $\mathfrak{X}$ of Borel subgroups of G_0 and G respectively. Since $K_0(\mathfrak{X}_0)$ is generated by the line bundles $g_w = [\mathcal{L}(\rho_w)]$ for $w \in W$, one can check that $\gamma^{i/i+1}(\mathfrak{X}_0)$ is generated by the products

$$\{c_1^{K_0}(g_{w_1})\ldots c_1^{K_0}(g_{w_i}),\ w_1,\ldots,w_i \in W\}.$$

Moreover, the restriction map commutes with Chern classes, so it induces

$$\mathrm{res}_\gamma\colon \gamma^*(\mathfrak{X}) \to \gamma^*(\mathfrak{X}_0).$$

Using Panin's description of the image of the restriction map $\mathrm{res}_{\mathrm{K_0}}$ we obtain that the image of $\mathrm{res}_\gamma^{(1)}\colon \gamma^{1/2}(\mathfrak{X}) \to \gamma^{1/2}(\mathfrak{X}_0)$ is generated by the elements $c_1^{K_0}(i(\rho_w)g_w) = i(\rho_w)c_1^{K_0}(g_w)$, for any $w \in W$, while the image of $\mathrm{res}_\gamma^{(2)}$ is generated by

$$i(\rho_{w_1})i(\rho_{w_2})c_1^{K_0}(g_{w_1})c_1^{K_0}(g_{w_2}) \text{ and } c_2^{K_0}(i(\rho_w)g_w) \text{ for } w_1,w_2,w \in W.$$

If the index $i(\rho_w)$ is 1, then $c_2^{K_0}(i(\rho_w)g_w) = 0$. Otherwise, the Whitney sum formula gives

$$c_2^{K_0}(i(\rho_w)g_w) = \binom{i(\rho_w)}{2}c_1^{K_0}(g_w)^2.$$

Applying the morphisms c_1 and c_2, and using (8), we now get

4.10 **Lemma.** *The subgroup* $c_1\big(\mathrm{im}(\mathrm{res}_\gamma^{(1)})\big) \in \mathrm{CH}^1(\mathfrak{X}_0)$ *is generated by* $i(\rho_w)c_1(g_w)$, *for all* $w \in W$. *The subgroup* $c_2\big(\mathrm{im}(\mathrm{res}_\gamma^{(2)})\big) \in \mathrm{CH}^2(\mathfrak{X}_0)$ *is generated by the elements* $i(\rho_{w_1})i(\rho_{w_2})c_1(g_{w_1})c_1(g_{w_2})$ *and* $\binom{i(\rho_w)}{2}c_1(g_w)^2$ *for all* $w_1, w_2, w \in W$.

Proof of Lemma 4.8. Since the image of the characteristic map consists of rational elements (see 2.3), we already know that $I(\mathfrak{c}) \subset I(\mathrm{res}_{\mathrm{Ch}})$. We now prove the reverse inclusions for the homogeneous parts of degree 1 and 2 under the relevant hypothesis on the common index $\mathfrak{i}_J$. Note that since c_1 and c_2 are both surjective, and commute with restriction maps, one has

$$\mathrm{im}\big(\mathrm{res}_{\mathrm{Ch}}^{(k)}\big) = c_k\big(\mathrm{im}(\mathrm{res}_\gamma^{(k)})\big), \text{ for } k = 1, 2.$$

In degree 1, we have $I(\mathrm{res}_{\mathrm{Ch}})^{(1)} = \mathrm{im}(\mathrm{res}_{\mathrm{Ch}}^{(1)})$, so to prove the first part of the lemma, we have to prove that if $\mathfrak{i}_J > 0$, then for any $w \in W$, the element $i(\rho_w)c_1(g_w)$ belongs, after tensoring with $\mathbb{F}_p$, to $I(\mathfrak{c})^{(1)} = \mathrm{im}\,\mathfrak{c}^{(1)}$. Let us write

$$c_1(g_w) = \sum_{\ell=1}^{s} a_\ell c_1(g_{i_\ell}) \mod \mathrm{im}\,\mathfrak{c}^{(1)},$$

as in (7). If all the $a_\ell \in \mathbb{F}_p$ are trivial, we are done, so we may assume at least one of them is invertible in $\mathbb{F}_p$. The weights ρ_w and ρ_{i_ℓ} satisfy the same relation

$$\rho_w = \sum_{\ell=1}^{s} a_\ell \rho_{i_\ell} \mod \hat{T}_0 \otimes_{\mathbb{Z}} \mathbb{F}_p.$$

Applying the morphism β, we get that the p-primary part of the Brauer class $\beta(\bar{\rho}_w)$ coincides with the p-primary part of the Brauer class of $\otimes_{\ell=1}^{s} A_{\omega_{i_\ell}}^{a_\ell}$ (see 2.10). The hypothesis on $\mathfrak{i}_J$ guarantees that this index of this algebra is divisible by p. Hence $i(\rho_w)$, which is the index of $\beta(\bar{\rho}_w)$, is also divisible by p, so that $i(\rho_w)c_1(g_w) = 0$ in the Chow group $\mathrm{Ch}^1(\mathfrak{X}_0)$ modulo p, and we are done.

Let us now assume that $p = 2$ and $\mathfrak{i}_J > 1$. The homogeneous part $I(\mathrm{res}_{\mathrm{Ch}})^{(2)}$ decomposes as

$$I(\mathrm{res}_{\mathrm{Ch}})^{(2)} = \mathrm{im}\big(\mathrm{res}_{\mathrm{Ch}}^{(1)}\big)\,\mathrm{Ch}^1(\mathfrak{X}_0) + \mathrm{im}\big(\mathrm{res}_{\mathrm{Ch}}^{(2)}\big).$$

By the first part of the Lemma, we already know that

$$\mathrm{im}\big(\mathrm{res}_{\mathrm{Ch}}^{(1)}\big)\,\mathrm{Ch}^1(\mathfrak{X}_0) \subset I(\mathfrak{c}).$$

Hence it remains to prove that $\mathrm{im}(\mathrm{res}_{\mathrm{Ch}}^{(2)}) = c_2(\mathrm{im}\,\mathrm{res}_\gamma^{(2)}) \subset I(\mathfrak{c})^{(2)}$. The proof for the degree 1 part already shows that $i(\rho_{w_1})i(\rho_{w_2})c_1(g_{w_1})c_1(g_{w_2})$ belongs to $I(\mathfrak{c})^{(2)}$. The same argument extends to $\binom{i(\rho_w)}{2}c_1(g_w)^2$. Indeed, if the coefficients a_ℓ are not all trivial modulo 2, the condition on the common index now implies that 4 divides $i(\rho_w)$, so that $\binom{i(\rho_w)}{2}$ is zero modulo 2. □

5 Applications to quadratic forms and algebras with orthogonal involutions

Let φ be a non-degenerate quadratic form of even dimension $2n$. We always assume that φ has trivial discriminant, so that its special orthogonal group $\mathrm{O}^+(\varphi)$ satisfies condition 1.3. We define the J-invariant of φ as

$$J(\varphi) = J_2\big(\mathrm{O}^+(\varphi)\big).$$

Let φ_0 be any non-degenerate subform of φ of codimension 1. Since φ has trivial discriminant, φ and φ_0 have the same splitting fields. In particular, each of them splits over the function field of the maximal orthogonal Grassmannian of the other. Therefore by the comparison lemma [PSZ08, 5.18(iii)], the corresponding indecomposable motives $R_2\big(\mathrm{O}^+(\varphi_0)\big)$ and $R_2\big(\mathrm{O}^+(\varphi)\big)$ are isomorphic. Hence they have the same Poincaré polynomial, and by (3), it follows that $\mathrm{O}^+(\varphi_0)$ and $\mathrm{O}^+(\varphi)$ have the same J-invariant. Since any odd-dimensional form can be embedded in an even-dimensional form with trivial discriminant, we only consider the even-dimensional case in the sequel.

Theorem 4.7 immediately implies:

5.1 **Corollary.** *Let φ be a $2n$-dimensional quadratic form with trivial discriminant. The 2-adic valuation i_S of its Clifford algebra and the first index j_1 of its J-invariant are related as follows:*

(1) $j_1 \leq \mathsf{i}_S$; (2) *If $n \geq 2$, and $\mathsf{i}_S > 0$, then $j_1 > 0$;*

(3) If $n \geq 3$ and $\mathsf{i}_S > 1$, then $j_1 > 1$.

Let now (A, σ) be a degree $2n$ central simple algebra over F, endowed with an involution of orthogonal type and trivial discriminant. In particular, this implies that A has exponent 2, so that it has index 2^{i_A} for some integer i_A. The connected component $\mathrm{PGO}^+(A, \sigma)$ of the automorphism group of (A, σ) is an adjoint group of type D_n. Because of the discriminant hypothesis, it is an inner twisted form of PGO^+_{2n}.

The J-invariant $J(A, \sigma)$ was defined in section 3. From the table [PSZ08, 4.13] (see also the appendix below), $J(A, \sigma)$ is an r-tuple $(j_1, j_2, \dots, j_r)$, with $r = m + 1$ if $n = 2m$ and $r = m$ if $n = 2m + 1$. Note that our notation slightly differs from the notation in the table, where in the n-odd case, there is an additional index, but which is bounded by $k_1 = 0$. So, for n odd, our $(j_1, \dots, j_r)$ coincides with $(j_2, \dots, j_{r+1})$ in [PSZ08]. In particular, the indices corresponding to generators of degree 1 are j_1 if n is odd and j_1 and j_2 if n is even.

Since σ has trivial discriminant, its Clifford algebra splits as a direct product $\mathcal{C}(A, \sigma) = \mathcal{C}_+ \times \mathcal{C}_-$ of two central simple algebras over F. We let i_A (respectively i_+, i_-) be the 2-adic valuation of the index of A (respectively $\mathcal{C}_+$, $\mathcal{C}_-$). From Examples 4.5 and 4.6, the common index i_J is

$$\mathsf{i}_J = \begin{cases} \mathsf{i}_+ = \mathsf{i}_- & \text{if } n \text{ is odd,} \\ \min\{\mathsf{i}_A, \mathsf{i}_+, \mathsf{i}_-\} & \text{if } n \text{ is even.} \end{cases}$$

Hence, Corollaries 4.2 and 4.3 and Theorem 4.7 translate as follows:

5.2 **Corollary.** *Depending on the parity of* $n = \deg(A)/2$ *we have*

n is even, $n \neq 2$ and $\mathsf{i}_J = \min\{\mathsf{i}_A, \mathsf{i}_+, \mathsf{i}_-\}$

(1) $j_1 \leq \mathsf{i}_A$;
(2) $j_2 \leq \min\{\mathsf{i}_+, \mathsf{i}_-\}$;
(3) *If* $\mathsf{i}_J > 0$, *then* $j_1 > 0$ *and* $j_2 > 0$.
(4) *If* $\deg(A) \equiv 0[8]$ *and* $\mathsf{i}_J > 1$, *then* $j_1 > 1$.
(5) *If* $\deg(A) \geq 8$ *and* $\mathsf{i}_J > 1$, *then* $j_2 > 1$.

n is odd and $\mathsf{i}_S = \mathsf{i}_+ = \mathsf{i}_-$

(1) $j_1 \leq \mathsf{i}_S$;
(2) *If* $\mathsf{i}_S > 0$, *then* $j_1 > 0$;
(3) *If* $\deg(A) \geq 6$ *and* $\mathsf{i}_S > 1$, *then* $j_1 > 1$.

The additional conditions on the degrees are obtained from the table [PSZ08, 4.13], and guarantee that $k_1 > 1$ or $k_2 > 1$.

We say that an algebra (A, σ) with $\deg(A) \equiv 0[4]$ is half-spin, if one of the components of its Clifford algebra is split. As explained in [Ga09, 4.1], this happens if and only if one of the cocycles $\xi \in Z^1(F, \mathrm{PGO}^+_{4m})$ associated to (A, σ) lifts to the half-spin group. Therefore, we can refine the inequalities in this case by applying Theorem 4.7 to the corresponding twisted group $\mathrm{Spin}^+(A, \sigma)$. The common index is $\mathsf{i}_J = \mathsf{i}_A$, and we get:

5.3 **Corollary.** *If* $\deg(A) \equiv 0[4]$ *and* (A, σ) *is half-spin, then:*

(1) If $\mathsf{i}_A > 0$, *then* $j_1 > 0$. (2) *If* $\mathsf{i}_A > 1$, *then* $j_1 > 1$.

5.4 **Remark.** Using [PS10, Prop. 4.2] one can show that (A, σ) is half-spin iff $j_2 = 0$, and A is split iff $j_1 = 0$.

6 The trialitarian case

From now on, we assume that (A, σ) has degree 8. The J-invariant of (A, σ) is a triple $J(A, \sigma) = (j_1, j_2, j_3)$ with $0 \leq j_1, j_2 \leq 2$ and $0 \leq j_3 \leq 1$. In this section, we will explain how to compute $J(A, \sigma)$. As a consequence of our results, we will prove:

6.1 **Corollary.** *(i) There is no algebra of degree* 8 *with orthogonal involution with trivial discriminant having J-invariant equal to* $(1, 2, 0)$, $(2, 1, 0)$ *or* $(2, 2, 0)$.
(ii) All other possible values do occur.

In particular, this shows that the restrictions described in the table [PSZ08, 4.13] (see also §8), which were obtained by applying the Steenrod operations on $\mathrm{Ch}^*(G_0)$ (*loc. cit.* 4.12), are not the only ones.

Recall that the group $\mathrm{PGO}^+(A, \sigma)$ is of type D_4. To complete the classification in this case, we need to understand the action of the symmetric group S_3 on the J-invariant (see 3.8). Let (B, τ) and (C, γ) be the two components of the Clifford algebra $\mathcal{C}(A, \sigma)$, each endowed with its canonical involution. It follows from the structure theorems [KMRT, (8.10) and (8.12)] that both are degree 8 algebras with orthogonal involutions. The triple $\big((A, \sigma), (B, \tau), (C, \gamma)\big)$ is a trialitarian triple in the sense of *loc.cit.* §42.A, and in particular, the Clifford algebra of any of those three algebras with involution is the direct product of the other 2. Hence, if one of them, say (A,σ) is split, then the other two are half-spin.

6.2 **Definition.** The trialitarian triple $\big((A,\sigma),(B,\tau),(C,\gamma)\big)$ is said to be *ordered by indices* if the indices of the algebras A, B and C satisfy

$$\operatorname{ind}(A)\le\operatorname{ind}(B)\le\operatorname{ind}(C).$$

In the next theorem we compute the J-invariant of such a triple. We remark that we don't know any elementary proof, which does not use both inequalities from Theorem 4.7.

6.3 **Theorem.** *Let $\big((A,\sigma),(B,\tau),(C,\gamma)\big)$ be a trialitarian triple ordered by indices, so that $\mathsf{i}_A\le\mathsf{i}_B\le\mathsf{i}_C$. The J-invariants are given by*

$$J(A,\sigma)=(j,j',j_3)\quad\textit{and}\quad J(B,\tau)=J(C,\gamma)=(j',j,j_3),$$

where $j=\min\{\mathsf{i}_A,2\}$ and $j'=\min\{\mathsf{i}_B,\mathsf{i}_C,2\}=\min\{\mathsf{i}_B,2\}$.

Moreover, the third index j_3 is 0 if the involution is isotropic and 1 otherwise.

6.4 **Remark.** (i) The first index of the J-invariant of a degree 8 algebra with involution (D,ρ) is $\min\{\mathsf{i}_D,2\}$ if D is not of maximal index in its triple. But if $\operatorname{ind}D$ is maximal, then j_1 might be strictly smaller. In Example 6.9 below, we will give an explicit example where $j_1<\mathsf{i}_D=2$.

(ii) By 3.8, we already know that j_3 does not depend on the choice of an element of the triple. On the other hand, as explained in [Ga99], the involutions σ, τ and γ are either all isotropic or all anisotropic. The triple is said to be isotropic or anisotropic accordingly.

Proof. To start with, let us compute the first two indices j_1 and j_2 of the J-invariant of (A,σ). Since we are in degree 8, they are both bounded by 2. Moreover, the triple being ordered by indices, the common index is given by $\mathsf{i}_J=\mathsf{i}_A$. So the equality $j_1=j$ follows directly from the inequalities of Corollary 5.2. If additionally $j'=j$, the very same argument gives $j_2=j'$. Assume now that j and j' are different, that is $j<j'$. If so, $j=0$ or $j=1$. In the first case, we have $\mathsf{i}_A=0$ so that the algebra A is split, and the result follows from Corollary 5.1.

The only remaining case is $j=\mathsf{i}_A=1$ and $\mathsf{i}_B\ge 2$, so that $j'=2$. Consider the function field F_A of the Severi-Brauer variety of A, which is a generic splitting field of A. By the fundamental relations 1.8, the algebra C is Brauer equivalent to $A\otimes B$. Hence Merkurjev's index reduction formula [Me91] says

$$\operatorname{ind}(B_{F_A})=\min\{\operatorname{ind}(B),\operatorname{ind}(B\otimes A)\}=\operatorname{ind}(B).$$

So the values of i_B and j' are the same over F and F_A. We know the result holds over F_A by reduction to the split case. Since the index j_2 can only decrease under scalar extension, we get $j_2\ge j'=2$, which concludes the proof in this case.

So the J-invariant of (A,σ) is given by $J(A,\sigma)=(j,j',j_3)$ for some integer j_3. Let us now compute the J-invariant of (B,τ) and (C,γ). Recall from 3.8

that (j, j', j_3) and (j', j, j_3) are the only possible values. So, if $j = j'$, there is no choice and we are done. Again, there are two remaining cases. Assume first that $j = \mathfrak{i}_A = 0$ and $j' \geq 1$, so that $J(A, \sigma) = (0, j', j_3)$. Since A is split, (B, τ) and (C, γ) are half-spin, so they have trivial j_2 and this gives the result. Assume now that $j = 1$ and $j' = 2$, so that $J(A, \sigma) = (1, 2, j_3)$. By the previous case, over the field F_A, both (B, τ) and (C, γ) have J-invariant $(2, 0, j_3)$. So the value over F has to be $(2, 1, j_3)$.

To conclude the proof, it only remains to compute j_3. If the involution is anisotropic, then by [Ka00] in the division case, by [Si05, Prop. 3] in index 4, and by [PSS01, Cor. 3.4] in index 2 (see also [Ka09]) the triple remains anisotropic after scalar extension to a generic splitting field F_A of the algebra A, and the J-invariant over F_A is $(0, *, *)$. Now extending scalars to a generic splitting field F_C of the Clifford algebra, by [La96, Thm. 4] the respective quadratic form still remains anisotropic, and the J-invariant equals $(0, 0, 1)$. Hence j_3 is equal to 1 over the generic splitting fields, and this implies $j_3 = 1$ over F.

If σ is isotropic, then obviously $J(A, \sigma)$ equals $(0, 2, 0)$, if the semisimple anisotropic kernel of the respective group is of type A_3, equals $(1, 1, 0)$, if the anisotropic kernel is of type $3\mathrm{A}_1$, and equals $(0, 1, 0)$, if it is of type $2\mathrm{A}_1$. □

The first part of Corollary 6.1 follows from Theorem 6.3. Indeed, if one of j_1, j_2 is 2 and the other one is ≥ 1, then the algebras A, B and C are all three non-split, and B and C have index ≥ 4. By 1.6, since A and B are non-split, the involution γ on C is not hyperbolic, so it is anisotropic, and the theorem gives $j_3 = 1$.

Explicit examples

We now prove the second part of Corollary 6.1. Obviously, if A is split, and σ is adjoint to a quadratic form φ, then $J(A, \sigma) = (0, J(\varphi))$, and any triple with $j_1 = 0$ is obtained for a suitable choice of φ. Considering the components of the even Clifford algebra of those quadratic forms, we also obtain all triples with $j_2 = 0$ by Theorem 6.3. The maximal value $(2, 2, 1)$ is obtained from a generic cocycle; such a cocycle exists by [KM06, Thm. 6.4(ii)]. Hence, it only remains to prove that the values $(1, 1, 0)$, $(1, 1, 1)$, $(1, 2, 1)$ and $(2, 1, 1)$ occur. For each of those, we will produce an explicit example, inspired by the trialitarian triple constructed in [QT10, Lemma 6.2]

Our construction uses the notion of direct sum for algebras with involution, which was introduced by Dejaiffe [Dej98]. Consider two algebras with involution (E_1, θ_1) and (E_2, θ_2) which are Morita-equivalent, that is E_1 and E_2 are Brauer equivalent and the involutions θ_1 and θ_2 are of the same type. Dejaiffe defined a notion of Morita equivalence data, and explains how to associate to any such data an algebra with involution (A, σ), which is called a direct sum of (E_1, θ_1) and (E_2, θ_2). In the split orthogonal case, if θ_1 and θ_2 are respectively adjoint to the quadratic forms φ_1 and φ_2, any direct sum of (E_1, θ_1) and (E_2, θ_2) is adjoint to $\varphi_1 \oplus \langle\lambda\rangle\varphi_2$ for some $\lambda \in F^\times$, and the choice of a Morita-equivalence data precisely amounts to the choice of a scalar λ. In general, as the split case shows, there exist non-isomorphic direct sums of two given algebras with involution. We will use the following characterization of direct sums [QT10, Lemma 6.3]:

6.5 **Lemma.** *The algebra with involution (A,σ) is a direct sum of (E_1,θ_1) and (E_2,θ_2) if and only if there is an embedding of the direct product $(E_1,\theta_1)\times(E_2,\theta_2)$ in (A,σ) and $\deg(A)=\deg(E_1)+\deg(E_2)$.*

Slightly extending Garibaldi's 'orthogonal sum lemma' [Ga01, Lemma 3.2], we get:

6.6 **Proposition.** *Let Q_1,Q_2,Q_3 and Q_4 be quaternion algebras such that $Q_1\otimes Q_2$ and $Q_3\otimes Q_4$ are isomorphic. If (A,σ) is a direct sum of $(Q_1,\bar{\ })\otimes(Q_2,\bar{\ })$ and $(Q_3,\bar{\ })\otimes(Q_4,\bar{\ })$, then one of the two components of the Clifford algebra of (A,σ) is a direct sum of $(Q_1,\bar{\ })\otimes(Q_3,\bar{\ })$ and $(Q_2,\bar{\ })\otimes(Q_4,\bar{\ })$, while the other is a direct sum of $(Q_1,\bar{\ })\otimes(Q_4,\bar{\ })$ and $(Q_2,\bar{\ })\otimes(Q_3,\bar{\ })$.*

6.7 **Remark.** If one of the four quaternion algebras is split, as we assumed in [QT10], then all three direct sums have a hyperbolic component. Hence they are uniquely defined. This is not the case anymore in the more general setting considered here. The algebra with involution (A,σ) does depend on the choice of an equivalence data. Nevertheless, once such a choice is made, its Clifford algebra is well defined. So the equivalence data defining the other two direct sums are determined by the one we have chosen.

Proof. Denote $(E_1,\theta_1)=(Q_1,\bar{\ })\otimes(Q_2,\bar{\ })$ and $(E_2,\theta_2)=(Q_3,\bar{\ })\otimes(Q_4,\bar{\ })$. By [KMRT, (15.12)], their Clifford algebras with canonical involution are respectively $(Q_1,\bar{\ })\times(Q_2,\bar{\ })$, and $(Q_3,\bar{\ })\times(Q_4,\bar{\ })$. The embedding of the direct product $(E_1,\theta_1)\times(E_2,\theta_2)$ in (A,σ) induces an embedding of the tensor product of their Clifford algebras in the Clifford algebra of (A,σ):

$$\big((Q_1,\bar{\ })\times(Q_2,\bar{\ })\big)\otimes\big((Q_3,\bar{\ })\times(Q_4,\bar{\ })\big)\hookrightarrow(\mathcal{C}(A,\sigma),\underline{\sigma}).$$

This tensor product splits as a direct product of four tensor products of quaternion algebras with canonical involution; for degree reasons, two of them embed in each component of $\mathcal{C}(A,\sigma)$. To identify them, it is enough to look at their Brauer classes. From the hypothesis, we have Brauer equivalences $Q_1\otimes Q_3\sim Q_2\otimes Q_4$ and $Q_1\otimes Q_4\sim Q_2\otimes Q_3$. If $Q_1\otimes Q_3$ and $Q_1\otimes Q_4$ are not Brauer equivalent, that is if A is non-split, this concludes the proof. Otherwise, all four tensor products are isomorphic, and the result is still valid. □

With this in hand, we now give explicit examples of algebras with involution having J-invariant $(1,2,1)$, $(2,1,1)$, $(1,1,1)$ and $(1,1,0)$.

6.8 **Example.** Let $F=K(x,y,z,t)$ be a function field in 4 variables over a field K, and consider the following quaternion algebras over F:

$$Q_1=(x,zt),\ Q_2=(y,zt),\ Q_3=(xy,z)\text{ and }Q_4=(xy,t).$$

We let (A,σ) be a direct sum of $(Q_1,\bar{\ })\otimes(Q_2,\bar{\ })$ and $(Q_3,\bar{\ })\otimes(Q_4,\bar{\ })$ as in Proposition 6.6, and denote by (B,τ), and respectively (C,γ), the component of

$\mathcal{C}(A,\sigma)$ Brauer equivalent to $Q_1 \otimes Q_3 \sim (x,t) \otimes (y,z)$ and $Q_1 \otimes Q_4 \sim (x,z) \otimes (y,t)$. The algebras A, B and C have index 2, 4 and 4, so that $\big((A,\sigma),(B,\tau),(C,\gamma)\big)$ is a trialitarian triple ordered by indices. By Theorem 6.3, we get $J(A,\sigma) = (1,2,j_3)$ and $J(B,\tau) = J(C,\gamma) = (2,1,j_3)$ for some j_3. Finally, assertion (i) of Corollary 6.1 implies $j_3 = 1$; in other words, this triple is anisotropic.

6.9 **Example.** This example is obtained from the previous one by scalar extension. Consider the Albert form $\varphi = \langle x, t, -xt, -y, -z, yz \rangle$ associated to the biquaternion algebra $Q_1 \otimes Q_3$. We let F' be its function field, $F' = F(\varphi)$, and denote by (A',σ'), (B',τ') and (C',γ') the extensions of (A,σ), (B,τ) and (C,γ) to F'. Since B is Brauer equivalent to $Q_1 \otimes Q_3$, the algebra B' has index 2. On the other hand, it follows from Merkurjev's index reduction formula [Me91, Thm. 3] that the indices of A and C are preserved by scalar extension to F', so that A' and C' have indices 2 and 4 respectively. Hence $\big((A',\sigma'),(B',\tau'),(C',\gamma')\big)$ is again a trialitarian triple ordered by indices and Theorem 6.3 now gives $J(A',\sigma') = J(B',\tau') = J(C',\gamma') = (1,1,j_3)$ for some j_3. The same argument as in the proof of the first assertion of Corollary 6.1 applies here: since A' and B' are non-split and C' has index 4, the involutions are anisotropic and Theorem 6.3 gives $j_3 = 1$. Note that, in particular, we have $J(C',\gamma') = (1,1,1)$, even though C' has index $4 = 2^2$.

6.10 **Example.** We now produce another example of an anisotropic trialitarian triple having J-invariant $(1,1,1)$ in which all three algebras have index 2. Namely, consider the F-quaternion algebras

$$Q_1 = (x,y),\ Q_2 = (x,z),\ Q_3 = (x,t) \text{ and } Q_4 = (x,yzt).$$

Pick an arbitrary orthogonal involution ρ on $H = (x,yz)$ over F. Since $Q_1 \otimes Q_2$ is isomorphic to 2 by 2 matrices over H, the tensor product of the canonical involutions of Q_1 and Q_2 is adjoint to a 2-dimensional hermitian form h_{12} over (H,ρ). Similarly, $(Q_3,\bar{\ }) \otimes (Q_4,\bar{\ })$ is isomorphic to $M_2(H)$ endowed with the adjoint involution with respect to some hermitian form h_{34}. Since h_{12} and h_{34} are both anisotropic, the hermitian form $h = h_{12} \oplus \langle u \rangle h_{34}$ over $H'' = H \otimes F(u)$, for some indeterminate u, is also anisotropic. We define

$$(A,\sigma) = (M_4(H''), \mathrm{ad}_h).$$

It is clear from the definition that (A,σ) is a direct sum of $(Q_1,\bar{\ }) \otimes (Q_2,\bar{\ })$ and $(Q_3,\bar{\ }) \otimes (Q_4,\bar{\ })$. Hence, by Proposition 6.6, the two components (B,τ) and (C,γ) of its Clifford algebra are Brauer equivalent to (x,yt) and (x,zt). This shows that all three algebras have index 2. Since the involutions are anisotropic, by Theorem 6.3, their J-invariant is $(1,1,1)$.

6.11 **Remark.** Note that there are many other examples, and not all of them can be described as in Proposition 6.6. In particular, any triple which includes a division algebra cannot be obtained from this proposition. Consider for instance the algebra with involution (A,σ) described in [QT02, Example 3.6], and let (B,τ) and (C,γ) be the two components of its Clifford algebra. As explained there, A is a indecomposable division algebra, and one component of its Clifford algebra, say B, has index 2. Since A is Brauer equivalent to $B \otimes C$, its indecomposability guarantees that C is division, and we get $J(A,\sigma) = J(C,\gamma) = (2,1,1)$ and $J(B,\tau) = (1,2,1)$.

To produce examples of algebras with involution having J-invariant $(1,1,0)$, we now construct examples of isotropic non-split and non-half-spin triples. As opposed to the previous examples, they can always be described using Proposition 6.6. Indeed, we get the following explicit description for such triples (cf. Garibaldi's [Ga98, Thm. 0.1])

6.12 **Proposition.** *If $\big((A,\sigma),(B,\tau),(C,\gamma)\big)$ is an isotropic trialitarian triple with A, B and C non-split, then there exists division quaternion algebras Q_1, Q_2 and Q_3 such that $Q_1 \otimes Q_2 \otimes Q_3$ is split and the triple is described as in Proposition 6.6 with $Q_4 = M_2(k)$.*

Proof. Since B and C are non-split, the involution σ is not hyperbolic by 1.6. Hence A has index 2, $A = M_4(Q_1)$ for some quaternion algebra Q_1 over F. Fix an orthogonal involution ρ_1 on Q_1; the involution σ is adjoint to a hermitian form $h = h_0 \oplus h_1$ over (Q_1, ρ_1), with h_0 hyperbolic, h_1 anisotropic and both of dimension 2 and trivial discriminant. Therefore, (A,σ) is a direct sum of $(M_2(Q_1), \mathrm{ad}_{h_0})$ and $(M_2(Q_1), \mathrm{ad}_{h_1})$. Since the first summand is hyperbolic, it is isomorphic to $(M_2(k), ^-) \otimes (Q_1, ^-)$. The second is $(Q_2, ^-) \otimes (Q_3, ^-)$, where Q_2 and Q_3 are the two components of the Clifford algebra ad_{h_1}, and this concludes the proof. □

We refer the reader to [QT10, §6] for a more precise description of those triples. They are the only ones for which the J-invariant is $(1,1,0)$.

7 Generic properties

In the present section we investigate the relationship between the values of the J-invariant of an algebra with involution (A,σ) and the J-invariant of the respective adjoint quadratic form φ_σ over the function field F_A of the Severi-Brauer variety of A, which is a generic splitting field of A.

7.1 **Definition.** We say (A,σ) is generically Pfister if φ_σ is a Pfister form. Observe that in this case $\deg A$ is always a power of 2 and the J-invariant over F_A has the form:

$$J\big((A,\sigma)_{F_A}\big) = (0,\dots,0,*)$$

(all zeros except possibly the last entry which is 0 or 1).

We say (A,σ) is in I^s, $s > 2$, if φ_σ belongs to the s-th power $I^s(F_A)$ of the fundamental ideal $I(F_A) \subset W(F_A)$ of the Witt ring of F_A.

7.2 **Theorem.** *Let (A,σ) be an algebra of degree $2n$ with orthogonal involution with trivial discriminant.*

(a) If (A,σ) is in I^s, $s > 2$, then $J(A,\sigma) = (j_1, \underbrace{0,\dots,0,}_{2^{s-2}-1 \text{ times}} *,\dots,*)$.

(b) In particular, if (A,σ) is generically Pfister, then $J(A,\sigma)=(*,0,\dots,0,*)$.

Proof. (a) Let $X=\mathrm{D}_n/P_i$ be the variety of maximal parabolic subgroups of type $i:=2\cdot[\frac{n+1}{2}]-2^{s-1}+1$ (For parabolic subgroups we use notation from [PS10, 2.1]). Since i is odd, $A_{F(X)}$ splits, and therefore the quadratic form φ_σ is defined over $F(X)$. By assumption $\varphi_\sigma\in I^s\big(F(X)\big)$. The Witt index of φ_σ is at least i. Therefore the anisotropic part of φ_σ has dimension at most $2(n-i)<2^s$. Thus, by the Arason-Pfister theorem φ_σ is hyperbolic. In particular, the variety X is generically split. Therefore by [PS11, Theorem 2.3] we obtain the desired expression for the J-invariant.

(b) Finally, if (A,σ) is generically Pfister, then $\varphi_\sigma\in I^s\big(F(X)\big)$, where $2^s=2n$ and (b) follows from (a). □

7.3 **Remark.** Let $(j_2,\dots,j_r)$ be the J-invariant of φ_σ over F_A, $r=[\frac{n+2}{2}]$. In view of the theorem one can conjecture that the J-invariant of (A,σ) is obtained from $J(\varphi_\sigma)$ just by adding an arbitrary left term, i.e.

$$J(A,\sigma)=(*,j_2,\dots,j_r).$$

For example, if φ_σ is excellent, then the J-invariant should be equal to

$$J(A,\sigma)=(*,0,\dots,0,*,0,\dots,0),$$

where the second $*$ has degree 2^s-1 for some s and equals either 0 or 1.

By the results of Section 6, observe that this holds for algebras of degree 8.

8 Appendix

8.1 The following table provides the values of the parameters of the J-invariant for all orthogonal groups (here $p=2$).

G_0	r	d_i	k_i	restrictions on j_i
O_n^+	$[\frac{n+1}{4}]$	$2i-1$	$[\log_2\frac{n-1}{d_i}]$	if $d_i+l=2^sd_m$ and $2\nmid\binom{d_i}{l}$, then $j_m\le j_i+s$
$\mathrm{Spin}_{2n}^{\pm}$, $2\mid n$	$\frac{n}{2}$	$1,\ i=1$ $2i-1,\ i\ge 2$	$2^{k_1}\parallel n$ $[\log_2\frac{2n-1}{d_i}]$	the same restrictions
Spin_n	$[\frac{n-3}{4}]$	$2i+1$	$[\log_2\frac{n-1}{d_i}]$	the same restrictions
PGO_{2n}^+	$[\frac{n+2}{2}]$	$1,\ i=1,2$ $2i-3,\ i\ge 3$	$2^{k_1}\parallel n$ $[\log_2\frac{2n-1}{d_i}]$	the same restrictions assuming $i,m\ge 2$

Note that this table coincides with [PSZ08, Table 4.13] except of the last column which in our case contains more restrictive conditions. For $s=0$ and 1 the restrictions in the last column are equivalent to those in [PSZ08, Table 4.13].

The conditions of the last column are simply translation of [Vi05, Prop. 5.12] from the language of Vishik's J-invariant to ours.

All values of the J-invariant which satisfy the restrictions given in the table are called admissible.

8.2 In his paper [Ho98], Hoffmann classified quadratic forms of small dimension in terms of their splitting pattern. Using his classification, one can give a precise description of quadratic forms of dimension 8 with trivial discriminant, depending on the value of their J-invariant. The results are summarized in the table below.

The notation $J_v(\varphi)$ stands for Vishik's J-invariant, as defined in [EKM, §88]. The index i is the 2-adic valuation of the greatest common divisor of the degrees of the splitting fields of φ. In the explicit description, Pf_k stands for a k-fold Pfister form, $s_{l/k}(Pf_2)$ for the Scharlau transfer of a 2-fold Pfister form with respect to a quadratic field extension, and Al_6 for an Albert form.

$J(\varphi)$	$J_v(\varphi)$	i_S, i	Splitting Pattern	Description
(0)	$\emptyset$	$\mathsf{i}_S = \mathsf{i} = 0$	(4)	hyperbolic
(1,0)	$\{1\}$	$\mathsf{i}_S = \mathsf{i} = 1$	(2,4)	$Pf_2 \perp 2\mathbb{H}$
(2,0)	$\{1,2\}$	$\mathsf{i}_S = \mathsf{i} = 2$	(1,2,4)	$Al_6 \perp \mathbb{H}$
(0,1)	$\{3\}$	$\mathsf{i}_S = 0$; $\mathsf{i} = 1$	(0,4)	Pf_3
(1,1)	$\{1,3\}$	$\mathsf{i}_S = \mathsf{i} = 1$	(0,2,4)	$q = \langle 1, -a\rangle \otimes q'$
(2,1)	$\{1,2,3\}$	$\mathsf{i}_S = \mathsf{i} = 2$	(0,1,2,4)	$Pf_2 \perp Pf_2$ or $s_{l/k}(Pf_2)$
		$\mathsf{i}_S = \mathsf{i} = 3$	"	generic

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