Generically split projective homogeneous varieties. II

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Abstract

This article gives a complete classification of generically split projective homogeneous varieties. This project was begun in our previous article [PS10], but here we remove all restrictions on the characteristic of the base field, give a new uniform proof that works in all cases and in particular includes the case PGO^+_{2n} which was missing in [PS10].

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1 Introduction

Let G denote a simple algebraic group over a field k and let X denote a projective homogeneous G-variety. The variety X is called *generically split*, if G splits over the field of rational functions $k(X)$. In the present article we give a complete classification of such varieties.

We started this research in [PS10], where we mainly focused on groups of exceptional types and did not consider groups of type D_n corresponding to algebras with orthogonal involutions. In this article we give a different proof which is uniform, works in all characteristics of the base field k and includes groups of classical types as well. Our main result is Theorem 3.3,

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where we express the condition that a variety X is generically split in terms of the *J-invariant* of the group G. The latter is a discrete invariant, which measures the motivic decomposition of the variety of Borel subgroups of G. It was introduced and explicitly computed in [PSZ08].

The paper is organized as follows. In Section 2 we describe Chow rings of split reductive algebraic groups. We use these computations in Section 3, where we prove our main result. Finally, in Corollary 3.4 we give an application of our classification to the Rost invariant for groups of type E_8.

2 Chow rings of reductive groups

We follow our notation from [PS10].

2.1. Let G_0 be a split reductive algebraic group defined over a field k. We fix a split maximal torus T in G_0 and a Borel subgroup B of G_0 containing T and defined over k. We denote by Φ the root system of G_0, by Π the set of simple roots of Φ with respect to B, and by $\widehat{T}$ the group of characters of T. Enumeration of simple roots follows Bourbaki.

Any projective G_0-homogeneous variety X is isomorphic to G_0/P_Θ, where P_Θ stands for the (standard) parabolic subgroup corresponding to a subset $\Theta \subset \Pi$. We let P_i denote the maximal parabolic subgroup $P_{\Pi\setminus\{\alpha_i\}}$ *of type i*.

Let $c\colon S(\widehat{T}) \to \mathrm{CH}^*(G_0/B)$ denote the characteristic map from the symmetric algebra of $\widehat{T}$ to the Chow ring of G_0/B given in [PS10, 2.7], and let R^* denote its image. According to [Gr58, Rem. 2°] (see [Bri97, Ex. 2] for the proof), the ring $\mathrm{CH}^*(G_0)$ can be presented as the quotient of $\mathrm{CH}^*(G_0/B)$ modulo the ideal generated by the non-constant elements of R^*.

2.2 Lemma. *The pullback map*

$$\mathrm{CH}^*(G_0) \to \mathrm{CH}^*([G_0, G_0])$$

is an isomorphism.

Proof. Indeed, $B' = B \cap [G_0, G_0]$ is a Borel subgroup of $[G_0, G_0]$, the map

$$[G_0, G_0]/B' \to G_0/B$$

is an isomorphism, and the map $S(\widehat{T}) \to \mathrm{CH}^*(G_0/B)$ factors through the surjective map $S(\widehat{T}) \to S(\widehat{T'})$, where $T' = T \cap [G_0, G_0]$. □

Let P be a parabolic subgroup of G_0. Denote by L the Levi subgroup of P and set $H_0 = [L, L]$. We have

2.3 Lemma. *The pullback map*

$$\mathrm{CH}^*(P) \to \mathrm{CH}^*(H_0)$$

is an isomorphism.

Proof. The quotient map $P \to L$ is a Zariski locally trivial affine fibration, therefore the pullback map $\mathrm{CH}^*(L) \to \mathrm{CH}^*(P)$ is an isomorphism. Since the composition $L \to P \to L$ is the identity map, the pullback map $\mathrm{CH}^*(P) \to \mathrm{CH}^*(L)$ is an isomorphism as well. It remains to apply Lemma 2.2. □

2.4 Lemma. *The pullback map*

$$\mathrm{CH}^*(G_0) \to \mathrm{CH}^*(P)$$

is surjective.

Proof. Applying [Gr58, Prop. 3] to the natural map $G_0/B \to G_0/P$, we see that the map $\mathrm{CH}^*(G_0/B) \to \mathrm{CH}^*(P/B)$ is surjective. Since $P/B \simeq H_0/(B \cap H_0)$, by [Gr58, Rem. 2°] the pullback $\mathrm{CH}^*(P/B) \to \mathrm{CH}^*(H_0)$ is surjective. Therefore by Lemma 2.3, the map $\mathrm{CH}^*(P/B) \to \mathrm{CH}^*(P)$ is also surjective. Moreover, it fits into the commutative diagram

$$\begin{array}{ccc} \mathrm{CH}^*(G_0/B) & \twoheadrightarrow & \mathrm{CH}^*(P/B) \\ \downarrow & & \downarrow \\ \mathrm{CH}^*(G_0) & \longrightarrow & \mathrm{CH}^*(P). \end{array}$$

Thus, the bottom map $\mathrm{CH}^*(G_0) \to \mathrm{CH}^*(P)$ is surjective. □

2.5 (Definition of σ)**.** Now we restrict to the situation when G_0 is simple. Let p be a prime integer. Let $\mathrm{Ch}^*(-)$ denote the Chow ring with $\mathbb{F}_p$-coefficients. Explicit presentations of Chow rings with $\mathbb{F}_p$-coefficients of split semisimple algebraic groups are given in [Kc85, Theorem 3.5].

For G_0 and H_0 they look as follows:

$$\mathrm{Ch}^*(G_0) = \mathbb{F}_p[x_1, \ldots, x_r]/(x_1^{p^{k_1}}, \ldots, x_r^{p^{k_r}}) \text{ with } \deg x_i = d_i, 1 \le d_1 \le \ldots \le d_r,$$
$$\mathrm{Ch}^*(H_0) = \mathbb{F}_p[y_1, \ldots, y_s]/(y_1^{p^{l_1}}, \ldots, y_s^{p^{l_s}}) \text{ with } \deg y_m = e_m, 1 \le e_1 \le \ldots \le e_s$$

for certain integers k_i, l_i, d_i, and e_i depending on the Dynkin types of G_0 and H_0.

We remark that since G_0 is simple, by [Kc85] (or by [PSZ08, Table 4.13]) all integers $d_i > 1$ are pairwise distinct.

By Lemma 2.3 and 2.4 the pullback $\varphi\colon \operatorname{Ch}^*(G_0) \to \operatorname{Ch}^*(H_0)$ is surjective. For a graded ring S^*, let S^+ denote the ideal generated by the non-constant elements of S^*. The induced map

$$\tilde{\varphi}\colon \operatorname{Ch}^+(G_0)/\operatorname{Ch}^+(G_0)^2 \to \operatorname{Ch}^+(H_0)/\operatorname{Ch}^+(H_0)^2$$

is also surjective. Moreover, for any m with $e_m > 1$ there exists a unique i such that $d_i = e_m$. We write $\sigma(m)$ for this i. The surjectivity implies that

$$\varphi(x_{\sigma(m)}) = cy_m + \text{lower terms}, \quad c \in \mathbb{F}_p^\times.$$

3 Generically split varieties

Let G_0 be a split semisimple algebraic group over k and let $G = {}_\gamma G_0$ be the twisted form of G_0 given by the class of a 1-cocycle $\gamma \in H^1(k, G_0)$. For a prime number p, let

$$J_p(\gamma) = (j_1(\gamma), \dots, j_r(\gamma))$$

denote its *J-invariant* defined in [PSZ08].

We recall some notation from [PS10]. For a group G or a projective homogeneous G-variety X, we write $\operatorname{CH}^*(\overline{G})$, $\operatorname{Ch}^*(\overline{G})$, $\operatorname{CH}^*(\overline{X})$ and $\operatorname{Ch}^*(\overline{X})$ for the respective Chow rings over a splitting field. These Chow rings do not depend on the choice of a splitting field.

Let $\overline{\operatorname{CH}}^*(X)$, resp. $\overline{\operatorname{Ch}}^*(X)$, denote the image of the restriction map $\operatorname{CH}^*(X) \xrightarrow{\text{res}} \operatorname{CH}^*(\overline{X})$, resp. $\operatorname{Ch}^*(X) \xrightarrow{\text{res}} \operatorname{Ch}^*(\overline{X})$.

Finally, we let $\overline{\operatorname{Ch}}^*(G)$ denote the image of the composite map

$$\operatorname{Ch}^*(Y) \xrightarrow{\text{res}} \operatorname{Ch}^*(\overline{Y}) \to \operatorname{Ch}^*(\overline{G}),$$

where Y is the variety of Borel subgroups of G.

3.1 Theorem. *Let G_0 be a split simple algebraic group over k, $G = {}_\gamma G_0$ be the twisted form of G_0 given by the class of a 1-cocycle $\gamma \in H^1(k, G_0)$, $X = {}_\gamma(G_0/P)$ be the twisted form of G_0/P, and $Y = {}_\gamma(G_0/B)$ be the twisted form of G_0/B. The following conditions are equivalent:*

1. *X is generically split.*
2. *The composition map*
$$\overline{\mathrm{CH}}^*(Y) \to \mathrm{CH}^*(G_0) \to \mathrm{CH}^*(P)$$
is surjective.
3. *For every prime p the composition map*
$$\overline{\mathrm{Ch}}^1(Y) \to \mathrm{Ch}^1(G_0) \to \mathrm{Ch}^1(P)$$
is surjective, and
$$j_{\sigma(m)}(\gamma) = 0 \textit{ for all } m \textit{ with } d_{\sigma(m)} > 1.$$

Proof. 1⇒2. The same argument as in the proof of Lemma 2.4 (with Y instead of G_0/B and X instead of G_0/P).

2⇒3. Clearly, the composition
$$\overline{\mathrm{Ch}}^*(Y) \to \mathrm{Ch}^*(G_0) \to \mathrm{Ch}^*(P)$$
is surjective for every p. For a generator $y_m \in \mathrm{Ch}^{d_{\sigma(m)}}(P)$ there exists b in the image of $\overline{\mathrm{Ch}}^{d_{\sigma(m)}}(Y) \to \mathrm{Ch}^{d_{\sigma(m)}}(G_0)$ with $\varphi(b) = y_m$. Let $c \in \mathbb{F}_p$ denote the coefficient of b at $x_{\sigma(m)}$ and set $a := b - c \cdot x_{\sigma(m)}$.

When $d_{\sigma(m)} > 1$, the element a is decomposable, i.e., $a \in \mathrm{Ch}^+(G_0)^2$, because all $d_i > 1$ are pairwise distinct (see 2.5). The coefficient c is not 0, since $\tilde{\varphi}(a) = 0$, and therefore we can assume $c = 1$. Since a is strictly less than $x_{\sigma(m)}$ in the DegLex order, it follows by definition of the J-invariant that $j_{\sigma(m)}(\gamma) = 0$.

3⇒1. $G_{k(X)}$ has a parabolic subgroup of type P; denote the derived group of its Levi subgroup by H. We want to prove that H is split. Let H_0 be the corresponding split semisimple group. Then the cocycle $\gamma_{k(X)}$ comes from an element $\chi \in H^1(k(X), H_0)$. By [PS10, Proposition 3.9(3)] it suffices to show that $J_p(\chi)$ is trivial for every p.

Denote the variety of complete flags of H by Z. It follows from the commutative diagram

$$\begin{array}{ccc} \mathrm{Ch}^*(Y_{k(X)}) & \longrightarrow & \mathrm{Ch}^*(Z) \\ \downarrow & & \downarrow \\ \mathrm{Ch}^*(\overline{G}) & \longrightarrow & \mathrm{Ch}^*(\overline{H}) \end{array}$$

that $j_m(\chi) \le j_{\sigma(m)}(\gamma)$ if $d_{\sigma(m)} > 1$. Therefore

$$j_m(\chi) \le j_{\sigma(m)}(\gamma_{k(X)}) \le j_{\sigma(m)}(\gamma) = 0$$

when $d_{\sigma(m)} > 1$.

It remains to show that the remaining entries $j_i(\chi)$ (corresponding to generators of degree 1) are also zero. Consider the following commutative diagram

$$\begin{array}{ccccc} \mathrm{Ch}^1(Y) & \longrightarrow & \mathrm{Ch}^1(Y_{k(X)}) & \longrightarrow & \mathrm{Ch}^1(Z) & & \\ & & \downarrow & & \downarrow & & \\ & & \mathrm{Ch}^1(\overline{G}) & \longrightarrow & \mathrm{Ch}^1(\overline{H}) & = & \mathrm{Ch}^1(P). \end{array}$$

Since by hypothesis the composite map $\mathrm{Ch}^1(Y) \to \mathrm{Ch}^1(P)$ is surjective, the map $\mathrm{Ch}^1(Z) \to \mathrm{Ch}^1(\overline{H})$ is surjective as well. Therefore by the very definition of the J-invariant, all $j_i(\chi)$ of degree 1 are 0. □

3.2 Remark.

- If all $e_m > 1$, then the condition on $\overline{\mathrm{Ch}}^1(Y)$ is void.
- If G_0 is different from PGO^+_{2n} and $e_1 = 1$ (resp. $G_0 = \mathrm{PGO}^+_{2n}$ and $e_1 = e_2 = 1$), then in view of [PS10, Proposition 4.2] the condition on $\overline{\mathrm{Ch}}^1(Y)$ is equivalent to the condition that all Tits algebras of G are split. The latter is also equivalent to the condition that $j_1(\gamma) = 0$ (resp. $j_1(\gamma) = j_2(\gamma) = 0$).
- If $G_0 = \mathrm{PGO}^+_{2n}$ and there is exactly one m with $e_m = 1$, then there are exactly two fundamental weights among $\omega_1, \omega_{n-1}, \omega_n$ whose image with respect to the composition $\mathrm{Ch}^1(\overline{Y}) \to \mathrm{Ch}^1(\overline{G}) \to \mathrm{Ch}^1(\overline{H})$ equals y_1. Then the condition on $\overline{\mathrm{Ch}}^1(Y)$ is equivalent to the condition that at least one of the Tits algebras corresponding to these fundamental weights in the preimage of y_1 is split.

For the class of a cocycle $\gamma \in H^1(k, G_0)$, we let A_l denote its Tits algebra of fundamental weight ω_l.

3.3 Theorem. *Let G be a group given by the class of a 1-cocycle $\gamma \in H^1(k, G_0)$, where G_0 stands for the split adjoint group of the same type as G, and let X be the variety of the parabolic subgroups of G of type i.*

The variety X is generically split if and only if

G_0	i	*conditions on* G
PGL_n	*any* i	$\gcd(\exp A_1, i) = 1$
PGSp_{2n}	*any* i	i *is odd or* G *is split*
O^+_{2n+1}	*any* i	$j_m(\gamma) = 0$ *for all* $1 \le m \le \left[\frac{n+1-i}{2}\right]$
PGO^+_{2n}	i *is odd,* $i < n-1$	$[A_{n-1}] = 0$ *or* $[A_n] = 0$, *and* $j_m(\gamma) = 0$ *for all* $2 \le m \le \left[\frac{n+2-i}{2}\right]$
PGO^+_{2n}	i *is even,* $i < n-1$	$j_m(\gamma) = 0$ *for all* $1 \le m \le \left[\frac{n+2-i}{2}\right]$
PGO^+_{2n}	$i = n-1$ *or* $i = n$, n *is odd*	*none*
PGO^+_{2n}	$i = n-1$, n *is even*	$[A_1] = 0$ *or* $[A_n] = 0$
PGO^+_{2n}	$i = n$, n *is even*	$[A_1] = 0$ *or* $[A_{n-1}] = 0$
E_6	$i = 3, 5$	*none*
E_6	$i = 2, 4$	$J_3(\gamma) = (0, *)$
E_6	$i = 1, 6$	$J_2(\gamma) = (0)$
E_7	$i = 2, 5$	*none*
E_7	$i = 3, 4$	$J_2(\gamma) = (0, *, *, *)$
E_7	$i = 6$	$J_2(\gamma) = (0, 0, *, *)$ ($J_2(\gamma) = (0, 0, 0, 0)$ *if* $\mathrm{char}\, k \neq 2$)
E_7	$i = 1$	$J_2(\gamma) = (0, 0, 0, *)$ ($J_2(\gamma) = (0, 0, 0, 0)$ *if* $\mathrm{char}\, k \neq 2$)
E_7	$i = 7$	$J_3(\gamma) = (0)$ *and* $J_2(\gamma) = (*, 0, *, *)$ ($J_2(\gamma) = (*, 0, 0, 0)$ *if* $\mathrm{char}\, k \neq 2$)
E_8	$i = 2, 3, 4, 5$	*none*
E_8	$i = 6$	$J_2(\gamma) = (0, *, *, *)$ ($J_2(\gamma) = (0, 0, 0, *)$ *if* $\mathrm{char}\, k \neq 2$)
E_8	$i = 1$	$J_2(\gamma) = (0, 0, *, *)$ ($J_2(\gamma) = (0, 0, 0, *)$ *if* $\mathrm{char}\, k \neq 2$)
E_8	$i = 7$	$J_3(\gamma) = (0, *)$ *and* $J_2(\gamma) = (0, *, *, *)$ ($J_3(\gamma) = (0, 0)$ *if* $\mathrm{char}\, k \neq 3$, $J_2(\gamma) = (0, 0, 0, *)$ *if* $\mathrm{char}\, k \neq 2$)
E_8	$i = 8$	$J_3(\gamma) = (0, *)$ *and* $J_2(\gamma) = (0, 0, 0, *)$ ($J_3(\gamma) = (0, 0)$ *if* $\mathrm{char}\, k \neq 3$)
F_4	$i = 1, 2, 3$	*none*
F_4	$i = 4$	$J_2(\gamma) = (0)$
G_2	*any* i	*none*

("$*$" means "any value").

Proof. Follows immediately from Theorem 3.1 and [PSZ08, Table 4.13]. A detailed exposition of the trialitarian case PGO_8^+ is given in [QSZ]. □

This theorem allows us to give a shortened proof of the main result of [Ch10]:

3.4 Corollary. Let G_0 be the split group of type E_8 over a field k with $\operatorname{char} k \neq 3$ and $\gamma \in H^1(k, G_0)$. If the 3-component of the Rost invariant of γ is zero, then γ splits over a field extension of degree coprime to 3.

Proof. Let K/k be a field extension of degree coprime to 3 such that the 2-component of the Rost invariant of γ_K is zero.

Consider the variety X of parabolic subgroups of ${}_{\gamma}(G_0)_K$ of type 7. The Rost invariant of the semisimple anisotropic kernel of $\gamma_{K(X)}$ is zero. Therefore $\gamma_{K(X)}$ splits, and thus X is generically split.

By Theorem 3.3, $J_3(\gamma_K) = (0, 0)$. Hence by [PS10, Proposition 3.9(3)], γ_K splits over a field extension of degree coprime to 3. This implies the corollary. □

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