ON THE BRAUER GROUP OF THE PRODUCT OF A TORUS AND A SEMISIMPLE ALGEBRAIC GROUP

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Abstract. Let T be a torus (not assumed to be split) over a field F, and denote by ${}_n\mathrm{H}^2_{\mathrm{et}}(X,\mathbb{G}_{\mathrm{m}})$ the subgroup of elements of exponent dividing n in the cohomological Brauer group of a scheme X over the field F. We provide conditions on X and n for which the pull-back homomorphism ${}_n\mathrm{H}^2_{\mathrm{et}}(T,\mathbb{G}_{\mathrm{m}}) \longrightarrow {}_n\mathrm{H}^2_{\mathrm{et}}(X\times_F T,\mathbb{G}_{\mathrm{m}})$ is an isomorphism. We apply this to compute the Brauer group of some reductive groups and of non singular affine quadrics.

Apart from this, we investigate the p-torsion of the Azumaya algebra defined Brauer group of a regular affine scheme over a field F of characteristic $p>0$.

0. Introduction

Let G be a simply connected group over an algebraically closed field F of characteristic zero. Magid [22] has shown that the pull-back homomorphism

$$p_T^*\colon\ \mathrm{Br}(T) \longrightarrow \mathrm{Br}(G\times_F T)$$

along the projection $p_T\colon G\times_F T\longrightarrow T$ is an isomorphism for all F-tori T. Here and in the following we denote by $\mathrm{Br}(X)$ the Brauer group of equivalence classes of Azumaya algebras over a scheme X, and by ${}_n\mathrm{Br}(X)$ the subgroup of elements whose exponent divides the integer $n\geq 2$.

Magid's proof uses the Lefschetz principle to reduce to the case that F is the field of complex numbers, and then Grothendieck's deep comparison result for étale and singular cohomology with finite coefficients, which finally reduces everything to computations of singular cohomology groups.

In the present article we give a purely algebraic proof of Magid's result and generalize it to groups over arbitrary fields, see Corollary 3.9 below. This gives in particular an algebraic proof that the symbol algebras for pairs of independent characters of a split torus over an algebraically closed field of characteristic zero generate the Brauer group of the torus. This has been proven before by Magid [22] using the topological methods described above.

We also generalize the main theorem of [14] which asserts that the natural homomorphism ${}_n\mathrm{Br}(F)\longrightarrow {}_n\mathrm{Br}(G)$ is an isomorphism for any semisimple algebraic

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group G over a field F if $\operatorname{char} F$ does not divide n, and if n is coprime to the order of the fundamental group of G.

Actually, we prove a more general result, see Theorem 2.11. Let $\mathrm{H}^2_{\mathrm{et}}(X, \mathbb{G}_{\mathrm{m}})$ be the cohomological Brauer group of the scheme X, and denote by ${}_n\mathrm{H}^2_{\mathrm{et}}(X, \mathbb{G}_{\mathrm{m}})$ the subgroup of elements whose order divides n. We give conditions under which for a regular and integral scheme X of finite type over a field F the natural homomorphisms

$$ {}_n\mathrm{Br}(F) \longrightarrow {}_n\mathrm{H}^2_{\mathrm{et}}(X, \mathbb{G}_{\mathrm{m}}) \qquad \text{and} \qquad {}_n\mathrm{Br}(T) \longrightarrow {}_n\mathrm{H}^2_{\mathrm{et}}(T \times X, \mathbb{G}_{\mathrm{m}}) \,, $$

where T is a form of a torus, are isomorphisms. (Recall that as T is affine and smooth over a field we have $\mathrm{Br}(T) = \mathrm{H}^2_{\mathrm{et}}(T, \mathbb{G}_{\mathrm{m}})$.)

As an application we get from Theorem 2.11 some computations of Brauer groups of reductive groups and affine quadrics. For instance if $\operatorname{char} F = 0$ we show in Corollary 3.14 that if H is a reductive group, such that the derived group $G = [H, H]$ is adjoint, then ${}_n\mathrm{Br}(H) \simeq {}_n\mathrm{Br}(\mathrm{Rad}(H))$ for all integers $n \geq 2$ which are coprime to the order of the fundamental group of G. (Here $\mathrm{Rad}(H)$ stands for the radical of H and $[H, H]$ denotes the commutator subgroup of H.)

Theorem 2.11 also implies that if X is a non singular affine quadric of Krull dimension ≥ 3 over a field F of characteristic $\neq 2$ then the pull-back of the structure morphism

$$ {}_n\mathrm{Br}(F) \longrightarrow {}_n\mathrm{Br}(X) $$

is an isomorphism for all integers n which are not divisible by $\operatorname{char} F$. In particular we have an isomorphism $\mathrm{Br}(F) \xrightarrow{\simeq} \mathrm{Br}(X)$ for such affine quadrics over a field F of characteristic zero.

In the last section we investigate the p-torsion of the Brauer group, where p is the characteristic of the base field. We show that if X is the spectrum of a regular geometrically integral algebra of finite type and of Krull dimension ≥ 2 over a field F of characteristic $p > 0$ then the cokernel of the pull-back $\mathrm{Br}(F) \longrightarrow \mathrm{Br}(X)$ contains a non trivial p-torsion and p-divisible (hence infinite) abelian group. In particular if G is a linear algebraic group of Krull dimension ≥ 2 over a field F of characteristic $p > 0$ then the pull-back $\mathrm{Br}(F) \longrightarrow \mathrm{Br}(G)$ is never surjective.

This gives an explanation that our computational results for the n-torsion subgroup of the Brauer group of certain affine schemes over a field F are only true if $\operatorname{char} F$ does not divide n.

Note that our result on the non triviality of the cokernel $\mathrm{Br}(F) \longrightarrow \mathrm{Br}(X)$ for a geometrically integral affine scheme X over a field F of positive characteristic is an "affine" result. There are examples of geometrically integral smooth and projective varieties Y over F, such that $\mathrm{Br}(F) \longrightarrow \mathrm{Br}(Y)$ is surjective (and hence an isomorphism if Y has a F-rational point) even if $\operatorname{char} F > 0$. For instance, Merkurjev and Tignol [24, Thm. B] have shown that for a projective homogeneous variety Y for a semisimple algebraic group G over an arbitrary field F the homomorphism $\mathrm{Br}(F) \longrightarrow \mathrm{Br}(Y)$ is surjective.

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1. Preliminaries on the Brauer group

1.1. *Notation.* Let F be a field with separable closure F_s. We denote by $\Gamma_F = \mathrm{Gal}(F_s/F)$ the absolute Galois group of F. If X is an F-scheme we set $X_s := F_s \times_F X$.

By $\mathrm{H}^i(F, -)$ we denote the Galois cohomology of the field F.

We denote by $R^\times$ the multiplicative group of units of a commutative ring R.

Let X be an F-scheme. Then we denote by $F[X]$ its ring of regular functions, and in case X is integral by $F(X)$ its function field.

For a prime number p we denote by $\mathbb{Z}(p^\infty)$ the infinite abelian group $\mathbb{Z}[\frac{1}{p}]/\mathbb{Z}$. This is a p-divisible and p-torsion group, and any p-divisible and p-torsion group is a direct sum of copies of $\mathbb{Z}(p^\infty)$, see *e.g.* [19].

1.2. *Brauer groups.* Let R be a commutative ring (with 1) and $X = \mathrm{Spec}\, R$. The Brauer group $\mathrm{Br}(R)$ of the commutative ring R (also denoted $\mathrm{Br}(X)$) has been introduced by Auslander and Goldman [2]. It classifies Azumaya algebras over R and generalizes the classical Brauer group of a field.

If A is an R-Azumaya algebra we denote by $[A]$ the class of A in $\mathrm{Br}(R)$.

There is also a cohomological Brauer group of R, the second étale cohomology group $\mathrm{H}^2_{\mathrm{et}}(X, \mathbb{G}_{\mathrm{m}})$. By Grothendieck [15] we have a natural homomorphism

$$j_X \colon \ \mathrm{Br}(X) \longrightarrow \mathrm{H}^2_{\mathrm{et}}(X, \mathbb{G}_{\mathrm{m}})\,.$$

If $X = \mathrm{Spec}\, R$ is smooth over a field F, then j_X is an isomorphism as shown by Hoobler [17]. In this case we identify $\mathrm{Br}(X)$ with $\mathrm{H}^2_{\mathrm{et}}(X, \mathbb{G}_{\mathrm{m}})$. (There is a more general identification result due to Gabber but we do not need this.)

We denote the subgroup of elements of order dividing n in $\mathrm{Br}(X)$ by ${}_n\mathrm{Br}(X)$, and similar we denote by ${}_n\mathrm{H}^2_{\mathrm{et}}(X, \mathbb{G}_{\mathrm{m}})$ the subgroup of elements of order n in the cohomological Brauer group of X.

If $\mathrm{char}\, F$ does not divide n we have an exact sequence of étale sheaves

$$\mu_n \rightarrowtail \mathbb{G}_{\mathrm{m}} \xrightarrow{\cdot n} \mathbb{G}_{\mathrm{m}}\,, \tag{1}$$

i.e. μ_n is (by definition) the kernel of $\mathbb{G}_{\mathrm{m}} \xrightarrow{\cdot n} \mathbb{G}_{\mathrm{m}}$. It follows from the associated long exact cohomology sequence that the homomorphism

$$\mathrm{H}^2_{\mathrm{et}}(X, \mu_n) \longrightarrow {}_n\mathrm{H}^2_{\mathrm{et}}(X, \mathbb{G}_{\mathrm{m}})$$

is an isomorphism if $\mathrm{Pic}\, X$ is n-divisible (*e.g.* if $\mathrm{Pic}\, X$ is trivial).

1.3. *The Hochschild-Serre spectral sequence.* This spectral sequence is a useful tool to compute Brauer groups. Let $X' \longrightarrow X$ be a Galois cover with (finite) group Π. Then, see [25, Chap. III, Thm. 2.20], there is a convergent spectral sequence

$$E_2^{p,q} := \mathrm{H}^p(\Pi, \mathrm{H}^q_{\mathrm{et}}(X', \mathbb{G}_{\mathrm{m}})) \quad\Longrightarrow\quad \mathrm{H}^{p+q}_{\mathrm{et}}(X, \mathbb{G}_{\mathrm{m}})\,,$$

where $\mathrm{H}^i(\Pi, -)$ denotes the group cohomology of Π.

If X is a finite type F-scheme we get from this spectral sequence for finite Galois covers by a limit argument, see [25, Rem. 2.21], the following spectral sequence

$$E_2^{p,q} := \mathrm{H}^p(F, \mathrm{H}^q_{\mathrm{et}}(X_s, \mathbb{G}_{\mathrm{m}})) \quad\Longrightarrow\quad \mathrm{H}^{p+q}_{\mathrm{et}}(X, \mathbb{G}_{\mathrm{m}})\,, \tag{2}$$

where (as in 1.1) we have set $X_s = F_s \times_F X$ and $\mathrm{H}^i(F, -)$ denotes Galois cohomology of the field F.

For later use we recall that if $\operatorname{Pic} X_s = 0$ we get from the spectral sequence (2) above an exact sequence

$$0 \longrightarrow \mathrm{H}^2(F, F_s[X_s]^\times) \longrightarrow \mathrm{H}^2_{\mathrm{et}}(X, \mathbb{G}_{\mathrm{m}}) \longrightarrow \mathrm{H}^2_{\mathrm{et}}(X_s, \mathbb{G}_{\mathrm{m}})^{\Gamma_F} \longrightarrow \mathrm{H}^3(F, F_s[X_s]^\times)\,,$$

where Γ_F is the absolute Galois group of the field F.

1.4. *A class of examples of "cyclic" Azumaya algebras.* Let R be an integral domain and $x, y \in R^\times$. Let further $n \geq 2$ be an integer and assume that R contains $\frac{1}{n}$ and a primitive n-th root of unity ξ.

We define an associative R-algebra $A = A^R_\xi(x, y)$ as follows: The R-algebra A is generated by elements α and β which are subject to the following relations:

$$\alpha^n = x\,, \quad \beta^n = y\,, \text{ and } \; x \cdot y = \xi(y \cdot x)\,.$$

If $\mathfrak{m}$ is a maximal ideal of the ring R, then by Milnor [27, §15] we know that $R/\mathfrak{m} \otimes_R A^R_\xi(x, y)$ is a central simple $R/\mathfrak{m}$-algebra and so $A^R_\xi(x, y)$ is an R-Azumaya algebra.

Assume now that R is a regular integral domain. Let K be its quotient field. Then the natural morphism $\mathrm{Br}(R) \longrightarrow \mathrm{Br}(K)$ is injective as proven by Auslander and Goldman [2, Thm. 7.2]. Since the class of $A^K_\xi(x, y)^{\otimes n} = K \otimes_R A^R_\xi(x, y)^{\otimes n}$ is trivial in $\mathrm{Br}(K)$, see [27, Expl. 15.3] we see that for regular integral domains R the classes of the algebras $A^R_\xi(x, y)$ lie in ${}_n\mathrm{Br}(R)$.

We will also use the notation $A^X_\xi(x, y)$ for $A^R_\xi(x, y)$, where $X = \operatorname{Spec} R$.

1.5. *The norm residue homomorphism.* Let $n \geq 2$ be an integer and F a field whose characteristic does not divide n. We denote by $\mathrm{K}^M_i(F)$ the i-th Milnor K-group of F as defined by Milnor in [26].

As usual we denote "pure" symbols in $\mathrm{K}^M_i(F)$ by $\{c_1, \ldots, c_i\}$, where $c_i \in F^\times$.

By Hilbert 90 we get from the long exact cohomology sequence associated with the short exact sequence (1) an isomorphism

$$R^F_{1,n} \,:\, F^\times / F^{\times\, n} \xrightarrow{\;\simeq\;} \mathrm{H}^1(F, \mu_n)\,, \quad \{x\} \longmapsto (x)\,,$$

where $\mu_n \subset F_s^\times$ is the subgroup of n-th roots of unity. Taking the cup product this map induces the so called norm residue homomorphism

$$R^F_{2,n} \,:\, \mathrm{K}^M_2(F)/n \cdot \mathrm{K}^M_2(F) \longrightarrow \mathrm{H}^2(F, \mu_n^{\otimes 2})\,, \quad \{x, y\} + n \cdot \mathrm{K}^M_2(F) \longmapsto (x) \cup (y)\,,$$

which is by the Merkurjev-Suslin Theorem [23] an isomorphism.

1.6. Example. Assume that F contains a primitive n-th root of unity ξ. Then we have isomorphisms

$$ {}_n\mathrm{Br}(F) \xrightarrow{\simeq} \mathrm{H}^2(F,\mu_n) \xrightarrow{\tau_\xi^F} \mathrm{H}^2(F,\mu_n^{\otimes 2}) \,. $$

The isomorphism τ_ξ^F is induced by the Γ_F-module isomorphism $\mu_n \xrightarrow{\simeq} \mu_n \otimes_{\mathbb{Z}} \mu_n$, $\eta \longmapsto \eta \otimes \xi$, and so depends on the choice of the primitive n-th root of unity ξ. One can show, see *e.g.* [13, Prop. 4.7.1], that the composition of these isomorphisms maps the class of the algebra $A_\xi^F(x,y)$ to $(x)\cup(y)$.

1.7. If the base field F contains a primitive n-th root of unity ξ we have also the isomorphism of étale sheaves $\mu\!\!\mu_n \xrightarrow{\simeq} \mu\!\!\mu_n \otimes \mu\!\!\mu_n$. It induces an isomorphisms

$$ \tau_\xi^X \;:\; \mathrm{H}^2_{\mathrm{et}}(X,\mu\!\!\mu_n) \xrightarrow{\simeq} \mathrm{H}^2_{\mathrm{et}}(X,\mu\!\!\mu_n^{\otimes 2}) $$

for all F-schemes X, which also depends on the choice of the primitive n-th root of unity ξ.

2. On the Brauer group of the product of a torus with certain regular and integral schemes

2.1. *Cycle modules.* Let F be a field and

$$ \mathrm{K}_*^M(F) \;:=\; \bigoplus_{i\geq 0} \mathrm{K}_i^M(F) $$

the Milnor K-theory ring of F which is a graded ring. This is the prototype of a cycle module in the sense of Rost whose definition we recall here briefly.

A cycle module M_* over the field F is a covariant functor

$$ E \longmapsto \mathrm{M}_*(E) \;=\; \bigoplus_{i\in\mathbb{Z}} \mathrm{M}_i(E) $$

from the category of field extensions of F into the category of graded abelian groups, such that (see Rost [30, p. 329, **D2**–**D4**])

(i) $\mathrm{M}_*(E)$ is a graded $\mathrm{K}_*^M(E)$-module for all fields $E\supseteq F$;

(ii) if $L\supseteq E\supseteq F$ are field extensions with $L\supseteq E$ finite then there is a transfer or corestriction homomorphism $\mathrm{cor}_{L/E}:\mathrm{M}_*(L)\longrightarrow \mathrm{M}_*(E)$; and

(iii) if v is a valuation on $E\supseteq F$ which is trivial on F then there is a (so called) second residue map $\partial_v:\mathrm{M}_*(E)\longrightarrow \mathrm{M}_{*-1}(E(v))$, where $E(v)$ is the residue field of v.

These data are subject to a long list of axioms, as *e.g.* a projection formula, and all of them have the Milnor K-theory counterparts. We have to refer to Rost's paper [30, pp. 329-331 and p. 337] for more details.

Example. The cycle module which plays here a major role is Milnor K-theory modulo n

$$ E \longmapsto \mathrm{K}_*^M(E)/n\cdot \mathrm{K}_*^M(E) \;:=\; \bigoplus_{i\geq 0} \mathrm{K}_i^M(E)/n\cdot\mathrm{K}_i^M(E) $$

for some integer $n\geq 2$. For brevity we denote this cycle module by K_*^M/n.

2.2. *Cycle complexes.* If X is an integral scheme of finite type over F (or more general an integral and excellent scheme) Rost [30] has constructed following Kato's construction [20] for Milnor K-theory a complex $\mathrm{C}^\bullet(X,\mathrm{M}_l)$:

$$\mathrm{M}_l(F(X)) \xrightarrow{d_X^0} \bigoplus_{x\in X^{(1)}} \mathrm{M}_{l-1}(F(x)) \xrightarrow{d_X^1} \bigoplus_{x\in X^{(2)}} \mathrm{M}_{l-2}(F(x)) \longrightarrow \cdots,$$

for all integers l, where $X^{(i)}$ denotes the set of points of codimension i in X, $F(x)$ the residue field of $x\in X$, and the differentials d_X^i are induced by the second residue maps. We consider $\mathrm{C}^\bullet(X,\mathrm{M}_l)$ as a cohomological complex with $\bigoplus\limits_{x\in X^{(i)}} \mathrm{M}_{l-i}(F(x))$ in degree i, and denote the cohomology group in degree i by $\mathrm{H}^i(X,\mathrm{M}_l)$.

We will make use of the fact proven by Rost [30, Prop. 8.6] that these groups are homotopy invariant, *i.e.* the pull-back $\mathrm{H}^i(X,\mathrm{M}_l)\longrightarrow \mathrm{H}^i(X\times_F\mathbb{A}^1_F,\mathrm{M}_l)$ along the projection $X\times_F\mathbb{A}^1_F\longrightarrow X$ is an isomorphism for all $i\geq 0$.

2.3. *The* K_2^M/n*-unramified cohomology of a split torus.* Let $n\geq 2$ be an integer and $T=\operatorname{Spec} F[t_1^{\pm 1},\dots,t_d^{\pm 1}]$ a split torus of rank $d\geq 1$. It follows from homotopy invariance and the localization sequence that

$$\mathrm{H}^0(T,\mathrm{K}_2^M/n) \simeq q_T^*\big(\mathrm{K}_2^M(F)/n\cdot\mathrm{K}_2^M(F)\big) \bigoplus \left(\bigoplus_{i=1}^d F^\times/F^{\times\, n}\cdot\{t_i\}\right) \bigoplus \left(\bigoplus_{1\leq i<j\leq d} \mathbb{Z}/\mathbb{Z}n\cdot\{t_i,t_j\}\right), \quad (3)$$

where $q_T: T\longrightarrow \operatorname{Spec} F$ is the structure morphism, and $F^\times/F^{\times\, n}\cdot\{t_i\}$ denotes subgroup of $\mathrm{K}_2^M(F(T))/n\cdot\mathrm{K}_2^M(F(T))$ generated by all $\{a,t_i\}$ with $a\in F^\times$.

2.4. *A commutative diagram.* Let X be a regular and integral F-scheme of finite type and $n\geq 2$ be an integer which is not divisible by $\operatorname{char} F$. We set here for brevity $\mathrm{k}_i^M(E):=\mathrm{K}_i^M(E)/n\cdot\mathrm{K}_i^M(E)$ for all field extensions $E\supseteq F$.

We have then a commutative diagram, see Bloch [3, Thm. 2.3],

$$\begin{array}{ccccc} \mathrm{k}_2^M(F(X)) & \xrightarrow{d_X^0} & \bigoplus\limits_{x\in X^{(1)}} \mathrm{k}_1^M(F(x)) & \xrightarrow{d_X^1} & \bigoplus\limits_{x\in X^{(2)}} \mathbb{Z}/\mathbb{Z}n \\ {\scriptstyle R_{2,n}^{F(X)}}\downarrow\simeq & & {\scriptstyle (R_{1,n}^{F(x)})_{x\in X^{(1)}}}\downarrow\simeq & & \downarrow = \\ \mathrm{H}^2(F(X),\mu_n^{\otimes 2}) & \xrightarrow[\delta_X^0]{} & \bigoplus\limits_{x\in X^{(1)}} \mathrm{H}^1(F(x),\mu_n) & \xrightarrow[\delta_X^1]{} & \bigoplus\limits_{x\in X^{(2)}} \mathbb{Z}/\mathbb{Z}n \end{array}, \quad (4)$$

where the lower row comes from the Bloch-Ogus resolution [4]. By the Merkurjev-Suslin Theorem, see 1.5, we know that $R_{2,n}^{F(X)}$, which maps a symbol $\{x,y\}$ to the product $(x)\cup(y)$, is an isomorphism. Hence all vertical maps in this diagram are isomorphisms.

Consequently, the norm residue homomorphism induces an isomorphism between the kernel of $d_X^0:\mathrm{k}_2^M(F(X))\longrightarrow \bigoplus\limits_{x\in X^{(1)}}\mathrm{k}_1^M(F(x))$ and the kernel of δ_X^0 : $\mathrm{H}^2(F(X),\mu_n^{\otimes 2})\longrightarrow \bigoplus\limits_{x\in X^{(1)}}\mathrm{H}^1(F(x),\mu_n)$.

Therefore we know by (3) that the kernel of δ^0_T is equal the direct sum

$$q_T^*\big(\mathrm{H}^2(F,\mu_n^{\otimes 2})\big) \bigoplus \left(\bigoplus_{i=1}^{d}(F^\times/F^{\times\, n})\cup(t_i)\right) \bigoplus \left(\bigoplus_{1\le i<j\le d}\mathbb{Z}/\mathbb{Z}n\cdot(t_i)\cup(t_j)\right),$$

where $(F^\times/F^{\times\, n})\cup(t_i)$ means the subgroup of $\mathrm{H}^2(F(T),\mu^{\otimes 2})$ generated by all cup products $(a)\cup(t_i)$, $a\in F^\times$, and $q_T : T \longrightarrow \operatorname{Spec} F$ is the structure morphism.

2.5. *The unramified K_2^M/n-cohomology of the product of a split torus with certain regular schemes.* We fix a field F and an integer $n\ge 2$. We assume first that the field F contains a primitive n-th root of unity ξ, and that the torus T is split, *i.e.* either $T = \operatorname{Spec} F[t_1^{\pm1},\dots,t_d^{\pm1}]$, where $d=\operatorname{rank} T\ge 1$, or T is the "trivial" torus $\operatorname{Spec} F$ of rank $d=0$.

Let further X be a regular and integral F-scheme of finite type with structure morphism $q_X : X \longrightarrow \operatorname{Spec} F$. We make the following assumption on X:

(Hn) The natural homomorphism $q_X^* : \mathrm{K}_i^M(F)/n\cdot \mathrm{K}_i^M(F) \longrightarrow \mathrm{H}^0(X,\mathrm{K}_i^M/n)$ is an isomorphism for $i=1,2$.

2.6. Lemma. *Under these assumptions the pull-back homomorphism*

$$p_T^* \,:\, \mathrm{H}^0(T,\mathrm{K}_i^M/n) \longrightarrow \mathrm{H}^0(X\times_F T,\mathrm{K}_i^M/n)$$

along the projection $p_T : X\times_F T \longrightarrow T$ is an isomorphism for $i=1,2$. Hence by the commutative diagram (4) also the pull-back homomorphism

$$p_T^* \,:\, \operatorname{Ker}\delta_T^0 \longrightarrow \operatorname{Ker}\delta^0_{X\times_F T}\,.$$

is an isomorphism.

Proof. We prove this by induction on the rank of T. The rank zero case, *i.e.* $T=\operatorname{Spec} F$, is assumption **(Hn)**. So let $d=\operatorname{rank} T\ge 1$ and $S:=\operatorname{Spec} F[t_1^{\pm1},\dots,t_{d-1}^{\pm1}]$. Then we get from the localization sequences for the open embeddings $\iota : T\hookrightarrow S[t_d]$ and $\mathrm{id}_X\times\iota : X\times_F T\hookrightarrow X\times_F S[t_d]$ a commutative diagram whose rows are exact

$$\begin{array}{ccccc}
\mathrm{H}^0(S[t_d],\mathrm{K}_i^M/n) & \xrightarrow{\ \iota^*\ } & \mathrm{H}^0(T,\mathrm{K}_i^M/n) & \xrightarrow{\ \partial_T\ } & \mathrm{H}^0(S,\mathrm{K}_{i-1}^M/n) \\
\downarrow{\scriptstyle (p_S[t_d])^*} & & \downarrow{\scriptstyle p_T^*} & & \downarrow{\scriptstyle p_S^*} \\
\mathrm{H}^0(X\times_F S[t_d],\mathrm{K}_i^M/n) & \xrightarrow{(\mathrm{id}_X\times\iota)^*} & \mathrm{H}^0(X\times_F T,\mathrm{K}_i^M/n) & \xrightarrow{\ \partial_{X\times T}\ } & \mathrm{H}^0(X\times_F S,\mathrm{K}_{i-1}^M/n)\,,
\end{array}$$

where $p_S : X\times_F S \longrightarrow S$ is the projection. As indicated the left horizontal arrows ι^* and $(\mathrm{id}_X\times\iota)^*$ are injective and both connecting homomorphisms ∂_T and $\partial_{X\times T}$ are surjective. In fact these are split epimorphisms. The splitting is induced by multiplication with the symbol of length one $\{t_d\}$.

The right hand vertical arrow p_S^* is an isomorphism for $i=1$ since X and $X\times_F S$ are irreducible and so $\mathrm{H}^0(S,\mathrm{K}_0^M/n)=\mathrm{H}^0(X\times_F S,\mathrm{K}_0^M/n)=\mathbb{Z}/\mathbb{Z}n$, and for $i=2$ by induction. The left hand vertical arrow $(p_S[t_d])^*$ is an isomorphism by induction and homotopy invariance, and so it follows by the snake lemma that also the vertical arrow p_T^* in the middle of the diagram is an isomorphism. We are done. ☐

We use this to describe ${}_n\mathrm{Br}(T)$ and ${}_n\mathrm{H}^2_{\mathrm{et}}(X\times_F T,\mathbb{G}_{\mathrm{m}})$.

2.7. *The Brauer group of a split torus T.* Let T and F be as above. Since $\operatorname{Pic} T = 0$ as T is a split torus we know (*cf.* 1.2) that ${}_n\mathrm{Br}(T) = \mathrm{H}^2_{\mathrm{et}}(T, \boldsymbol{\mu}_n)$, and so by Auslander and Goldman [2, Thm. 7.2] the pull-back homomorphism

$$\iota_T^* \,:\, \mathrm{H}^2_{\mathrm{et}}(T, \boldsymbol{\mu}_n) \longrightarrow \mathrm{H}^2_{\mathrm{et}}(F(T), \boldsymbol{\mu}_n) = \mathrm{H}^2(F(T), \mu_n)$$

is injective, where $\iota_T : \operatorname{Spec} F(T) \longrightarrow T$ denotes the generic point. By the very definition of the Bloch-Ogus complex the image of $\iota_T^* : \mathrm{H}^2_{\mathrm{et}}(T, \boldsymbol{\mu}_n^{\otimes 2}) \longrightarrow \mathrm{H}^2(F(T), \boldsymbol{\mu}_n^{\otimes 2})$ lies in the kernel of δ^0_T. Hence we have a commutative diagram where all vertical arrows are monomorphisms

$$\begin{array}{ccccc}
\mathrm{H}^2_{\mathrm{et}}(F, \mu_n) & \xrightarrow[\simeq]{\tau_\xi^F} & \mathrm{H}^2(F, \mu_n^{\otimes 2}) & & \\
\big\downarrow q_T^* & & \big\downarrow q_T^* & & \\
\mathrm{H}^2_{\mathrm{et}}(T, \boldsymbol{\mu}_n) & \xrightarrow[\simeq]{\tau_\xi^T} & \mathrm{H}^2_{\mathrm{et}}(T, \boldsymbol{\mu}_n^{\otimes 2}) & \xrightarrow{\iota_T^*} & \operatorname{Ker} \delta^0_T \\
\big\downarrow \iota_T^* & & \big\downarrow \iota_T^* & \swarrow \subseteq & \\
\mathrm{H}^2(F(T), \mu_n) & \xrightarrow[\simeq]{\tau_\xi^{F(T)}} & \mathrm{H}^2(F(T), \mu_n^{\otimes 2})\,. & &
\end{array} \tag{5}$$

Therefore $\iota_T^* \circ \tau_\xi^T : \mathrm{H}^2_{\mathrm{et}}(T, \boldsymbol{\mu}_n) \longrightarrow \operatorname{Ker} \delta^0_T$ is injective. We claim that it is also surjective.

In fact, let x, y be units in $F[T]^\times$ then the class of the Azumaya algebra $A_\xi^T(x, y)$ is in ${}_n\mathrm{Br}(T) = \mathrm{H}^2_{\mathrm{et}}(T, \boldsymbol{\mu}_n)$, see 1.4. By Example 1.6 we have

$$\tau_\xi^{F(T)}\big(\,\iota_T^*([A_\xi^T(x, y)])\,\big) \;=\; (x) \cup (y)\,.$$

Since as seen above $\operatorname{Ker} \delta^0_T$ is the direct sum of $q_T^*\big(\mathrm{H}^2(F, \mu_2^{\otimes 2})\big)$ and the subgroup generated by all $(a) \cup (t_i)$ and $(t_i) \cup (t_j)$ with $a \in F^\times$ and $1 \le i, j \le d$ we get therefore from diagram (5) our claim.

We have proven the following theorem which is due to Magid [22] if F is an algebraically closed field of characteristic 0, see also P. Gille and Pianzola [12, Sect. 4] for another proof in this case.

2.8. Theorem. *Let $n \ge 2$ be an integer, $T = \operatorname{Spec} F[t_1^{\pm 1}, \dots, t_d^{\pm 1}]$ be a split torus of rank $d \ge 1$ over a field F which contains a primitive n-th root of unity ξ. Then ${}_n\mathrm{Br}(T)$ is isomorphic to*

$${}_n\mathrm{Br}(F) \oplus \bigoplus_{i=1}^{d} \sum_{a \in F^\times} \mathbb{Z}/\mathbb{Z}n \cdot [A_\xi^T(a, t_i)]$$

if $d = 1$, and to

$${}_n\mathrm{Br}(F) \,\oplus\, \bigoplus_{i=1}^{d} \sum_{a \in F^\times} \mathbb{Z}/\mathbb{Z}n \cdot [A_\xi^T(a, t_i)] \,\oplus\, \bigoplus_{1 \le i < j \le d} \mathbb{Z}/\mathbb{Z}n \cdot [A_\xi^T(t_i, t_j)]$$

if $d \ge 2$. In particular, if $F^\times = F^{\times\, n}$ we have a natural isomorphism ${}_n\mathrm{Br}(F) \xrightarrow{\simeq} {}_n\mathrm{Br}(T)$ if the rank of T is 1, and an isomorphism

$${}_n\mathrm{Br}(T) \,\simeq\, {}_n\mathrm{Br}(F) \oplus \bigoplus_{1 \le i < j \le d} \mathbb{Z}/\mathbb{Z}n \cdot [A_\xi^T(t_i, t_j)]$$

if the rank of T is $d \ge 2$.

2.9. Let (as above) $n \geq 2$ be an integer and F a field which contains a primitive n-th root of unity. Let further T be a split torus or equal $\operatorname{Spec} F$ and X a regular and integral F-scheme of finite type which satisfies the condition **(Hn)** in 2.5 and also

(nd) $\operatorname{Pic} X$ is n-divisible.

We set $Y = X \times_F T$ and denote $p_T : Y \longrightarrow T$ the projection (which is the structure morphism if $T = \operatorname{Spec} F$ is the “trivial” torus). Then as $\operatorname{Pic} X$ is n-divisible it follows by homotopy invariance and localization sequence that also $\operatorname{Pic} Y$ is n-divisible. Therefore $\mathrm{H}^2_{\mathrm{et}}(Y, \boldsymbol{\mu}_n) = {}_n\mathrm{H}^2_{\mathrm{et}}(Y, \mathbb{G}_{\mathrm{m}})$ and so by Grothendieck [15, Chap. II, Cor. 1.8], see also [25, Chap. III, Expl. 2.22], we know that the pull-back $\iota_Y^* : \mathrm{H}^2_{\mathrm{et}}(Y, \boldsymbol{\mu}_n) \longrightarrow \mathrm{H}^2(F(Y), \mu_n)$ along the generic point $\iota_Y : \operatorname{Spec} F(Y) \longrightarrow Y$ is a monomorphism.

We have now the following commutative diagram

$$
\begin{array}{ccccc}
\mathrm{H}^2_{\mathrm{et}}(T, \boldsymbol{\mu}_n) & \xrightarrow[\simeq]{\tau_\xi^T} & \mathrm{H}^2(T, \boldsymbol{\mu}_n^{\otimes 2}) & \xrightarrow[\simeq]{\iota_T^*} & \operatorname{Ker} \delta^0_T \\
\downarrow{\scriptstyle p_T^*} & & \downarrow{\scriptstyle p_T^*} & & \simeq \downarrow{\scriptstyle p_T^*} \\
\mathrm{H}^2_{\mathrm{et}}(Y, \boldsymbol{\mu}_n) & \xrightarrow[\simeq]{\tau_\xi^Y} & \mathrm{H}^2(Y, \boldsymbol{\mu}_n^{\otimes 2}) & \xrightarrow{\iota_Y^*} & \operatorname{Ker} \delta^0_Y \\
\downarrow{\scriptstyle \iota_Y^*} & & \downarrow{\scriptstyle \iota_Y^*} & \swarrow{\scriptstyle \subseteq} & \\
\mathrm{H}^2(F(Y), \mu_n) & \xrightarrow[\simeq]{\tau_\xi^{F(Y)}} & \mathrm{H}^2(F(Y), \mu_n^{\otimes 2}) \,. & &
\end{array}
$$

The vertical arrow on the right hand side $p_T^* : \operatorname{Ker} \delta^0_T \longrightarrow \operatorname{Ker} \delta^0_Y$ is an isomorphism by Lemma 2.6, and we have shown in 2.7 that also ι_T^* is an isomorphism. Hence since $\iota_Y^* : \mathrm{H}^2_{\mathrm{et}}(Y, \boldsymbol{\mu}_n) \longrightarrow \mathrm{H}^2(F(Y), \mu_n)$ is injective we get from this diagram that also $p_T^* : \mathrm{H}^2_{\mathrm{et}}(T, \boldsymbol{\mu}_n) \longrightarrow \mathrm{H}^2_{\mathrm{et}}(Y, \boldsymbol{\mu}_n)$ is an isomorphism.

We have proven:

2.10. Theorem. *Let n be an integer ≥ 2, F a field which contains a primitive n-th root of unity and T either a split torus or equal* $\operatorname{Spec} F$. *Let further X be a regular and integral F-scheme, such that (i)* $\operatorname{Pic} X$ *is n-divisible, and (ii) the natural homomorphism* $\mathrm{K}^M_i(F)/n \cdot \mathrm{K}^M_i(F) \longrightarrow \mathrm{H}^0(X, \mathrm{K}^M_i/n)$ *is an isomorphism for $i = 1, 2$. Then the pull-back homomorphism*

$$p_T^* \; : \; {}_n\mathrm{Br}(T) \; \longrightarrow \; {}_n\mathrm{H}^2_{\mathrm{et}}(X \times_F T, \mathbb{G}_{\mathrm{m}})$$

where $p_T : X \times_F T \longrightarrow T$ is the projection, is an isomorphisms.

Note that if $\mathrm{Br}(F) \longrightarrow \mathrm{H}^2_{\mathrm{et}}(X, \mathbb{G}_{\mathrm{m}})$ is injective (e.g. if $X(F) \neq \emptyset$) and the natural homomorphism ${}_n\mathrm{Br}(F) = \mathrm{H}^2(F, \mu_n) \longrightarrow \mathrm{H}^2_{\mathrm{et}}(X, \boldsymbol{\mu}_n)$ is an isomorphism then it follows from the long exact cohomology sequence for the exact sequence (1) that $\operatorname{Pic} X$ is n-divisible. Hence in this case condition **(nd)** is a necessary condition.

We consider now the case that the torus T is not split. Here we show the following generalization.

2.11. Theorem. *Let F be a field and $n \geq 2$ an integer which is not divisible by* $\operatorname{char} F$. *Let further X be a regular and geometrically integral F-scheme of finite type, and T a F-scheme, such that either T_s is isomorphic as F_s-scheme to a torus, or T is the “trivial” torus* $\operatorname{Spec} F$. *We assume that*

(i) $\operatorname{Pic} X_s = 0$,

(ii) $\mathrm{H}^0(X_s, \mathrm{K}_i^M/n) = 0$ *for* $i = 1, 2$, *and*

(iii) $F_s[X_s]^\times = F_s^\times$.

Then the natural homomorphism

$$p_T \;:\; {}_n\mathrm{Br}(T) \longrightarrow {}_n\mathrm{H}^2_{\mathrm{et}}(X \times_F T, \mathbb{G}_{\mathrm{m}})$$

along the projection $p_T : X \times_F T \longrightarrow T$ *is an isomorphism.*

Proof. We show first that ${}_n\mathrm{Br}(F) \longrightarrow {}_n\mathrm{Br}(X)$ is an isomorphism. If F is separably closed this is proven in 2.9 above. In the general case we get since $\operatorname{Pic} X_s = 0$ from the Hochschild-Serre spectral sequence, see 1.3, an exact sequence

$$0 \longrightarrow \mathrm{H}^2(F, F_s[X_s]^\times) \longrightarrow \mathrm{H}^2_{\mathrm{et}}(X, \mathbb{G}_{\mathrm{m}}) \longrightarrow \mathrm{H}^2_{\mathrm{et}}(X_s, \mathbb{G}_{\mathrm{m}})^{\Gamma_F} \;,$$

where Γ_F is the absolute Galois group of F. By the case of a separably closed base field we know that ${}_n\mathrm{H}^2_{\mathrm{et}}(X_s, \mathbb{G}_{\mathrm{m}}) = 0$, and by our assumption (iii) we have $F_s[X_s]^\times = F_s^\times$, and so $\mathrm{H}^2(F, F_s[X_s]^\times) = \mathrm{Br}(F)$. Hence we get from the exact sequence above the claimed isomorphism ${}_n\mathrm{Br}(F) \xrightarrow{\simeq} {}_n\mathrm{H}^2_{\mathrm{et}}(X, \mathbb{G}_{\mathrm{m}})$.

Assume now that T is not the "trivial" torus. Then if $F = F_s$ is separably closed this is the assertion of the theorem in 2.9. In the general case, we know by homotopy invariance and the localization sequence that $\operatorname{Pic} T_s = 0$ and by the same reasoning using our assumption (i) that also $\operatorname{Pic}(X \times_F T)_s = 0$. Hence we get from the Hochschild-Serre spectral sequence, see 1.3, a commutative diagram whose rows are exact

$$\begin{array}{ccccccc}
\mathrm{H}^2(F, F_s[T_s]^\times) & \overset{\alpha}{\rightarrowtail} & \mathrm{Br}(T) & \overset{\beta}{\longrightarrow} & \mathrm{Br}(T_s)^{\Gamma_F} & \overset{\gamma}{\longrightarrow} & \mathrm{H}^3(F, F_s[T_s]^\times) \\
\simeq \big\downarrow p^*_{T_s} & & \big\downarrow p^*_T & & \big\downarrow p^*_{T_s} & & \simeq \big\downarrow p^*_{T_s} \\
\mathrm{H}^2(F, F_s[Y_s]^\times) & \overset{\alpha'}{\rightarrowtail} & \mathrm{H}^2_{\mathrm{et}}(Y, \mathbb{G}_{\mathrm{m}}) & \overset{\beta'}{\longrightarrow} & \mathrm{H}^2_{\mathrm{et}}(Y_s, \mathbb{G}_{\mathrm{m}})^{\Gamma_F} & \overset{\gamma'}{\longrightarrow} & \mathrm{H}^3(F, F_s[Y_s]^\times)\,,
\end{array}$$

where we have set $Y = X \times_F T$ and $p_T : X \times_F T \longrightarrow T$ and $p_{T_s} : X_s \times_{F_s} T_s \longrightarrow T_s$ are the respective projections. In particular, the arrows α and α' on the left hand side are injective. As indicated the utmost left and right vertical arrows are isomorphisms, since by Rosenlicht [29, Thms. 1 and 2], see also [7, Lem. 10], and our assumption (iii), we know that the morphism p_{T_s} induces an isomorphism $F_s[T_s]^\times \xrightarrow{\simeq} F_s[Y_s]^\times$. By the split case $p^*_{T_s} : \mathrm{Br}(T_s) \longrightarrow \mathrm{H}^2_{\mathrm{et}}(Y_s, \mathbb{G}_{\mathrm{m}})$ induces an isomorphism between the subgroups of elements of order dividing n. This implies by a straightforward diagram chase that $p^*_T : {}_n\mathrm{Br}(T) \longrightarrow {}_n\mathrm{H}^2_{\mathrm{et}}(Y, \mathbb{G}_{\mathrm{m}})$ is injective.

To show that this map is also surjective, let $x' \in {}_n\mathrm{H}^2_{\mathrm{et}}(Y, \mathbb{G}_{\mathrm{m}})$. Then by the split case there is $y \in {}_n\mathrm{Br}(T_s)^{\Gamma_F}$, such that $p^*_{T_s}(y) = \beta'(x')$. We have $p^*_{T_s}(\gamma(y)) = \gamma'(p^*_{T_s}(y)) = \gamma'(\beta'(x')) = 0$ and so there exists a $z \in \mathrm{Br}(T)$, such that $\beta(z) = y$.

It follows then that $\beta'(x' - p^*_T(z)) = 0$ and so we get $x' = p^*_T(z + \alpha(w))$ for some $w \in \mathrm{H}^2(F, F_s[T_s]^\times)$. Since as observed above $p^*_T : {}_n\mathrm{Br}(T) \longrightarrow {}_n\mathrm{H}^2_{\mathrm{et}}(Y, \mathbb{G}_{\mathrm{m}})$ is a monomorphism the assumption $n \cdot x' = 0$ implies also $n \cdot (z + \alpha(w)) = 0$ and so the homomorphism $p^*_T : {}_n\mathrm{Br}(T) \longrightarrow {}_n\mathrm{H}^2_{\mathrm{et}}(Y, \mathbb{G}_{\mathrm{m}})$ is also surjective, hence an isomorphism. □

2.12. Corollary. *Let n and F be as above, and Z a regular and integral F-scheme which contains an open F-subscheme X of finite type which satisfies conditions (i)–(iii) in the theorem above, and let T be a F-scheme, such that either T_s is isomorphic to a torus as F_s-scheme, or $T = \operatorname{Spec} F$. Then ${}_n\mathrm{Br}(T) \longrightarrow {}_n\mathrm{H}^2_{\mathrm{et}}(Z \times_F T, \mathbb{G}_{\mathrm{m}})$ is an isomorphism.*

Proof. Since Z and so also $Z \times_F T$ are regular and integral schemes the natural homomorphisms ${}_n\mathrm{H}^2_{\mathrm{et}}(Z, \mathbb{G}_{\mathrm{m}}) \longrightarrow {}_n\mathrm{H}^2_{\mathrm{et}}(X, \mathbb{G}_{\mathrm{m}})$ and ${}_n\mathrm{H}^2_{\mathrm{et}}(Z \times_F T, \mathbb{G}_{\mathrm{m}}) \longrightarrow {}_n\mathrm{H}^2_{\mathrm{et}}(X \times_F T, \mathbb{G}_{\mathrm{m}})$ are injective by Grothendieck [15, Chap. II, Cor. 1.8]. □

2.13. Remark. If $\operatorname{char} F = p > 0$ and X is a regular affine and geometrically integral F-scheme of dimension ≥ 2 then the cokernel of the homomorphism $\mathrm{Br}(F) \longrightarrow \mathrm{Br}(X)$ contains a non trivial p-torsion and p-divisible subgroup, and so is in particular an infinite abelian group, see our Theorem 4.7.

2.14. *A geometric criterion.* We give here a geometric condition which implies the axiom **(Hn)** in 2.5. To formulate it we introduce the following notation. Let E be a field and $\mathbb{A}^1_E$ the affine line over E. We denote for $i = 0, 1$ by $s_i : \operatorname{Spec} E \longrightarrow \mathbb{A}^1_E$ the points 0 and 1, respectively.

Let F be a field and X a regular and integral F-scheme of finite type. Let $\iota_X : \operatorname{Spec} F(X) \longrightarrow X$ be the generic point and $q_X : X \longrightarrow \operatorname{Spec} F$ the structure morphism. Assume that there is a F-rational point $x : \operatorname{Spec} F \longrightarrow X$, and there are F-morphisms $h_j : \mathbb{A}^1_{F(X)} \longrightarrow X$, $j = 1, \ldots, l$, such that (a) $h_1 \circ s_0$ factors through x, (b) $h_j \circ s_1 = h_{j+1} \circ s_0$ for $j = 1, \ldots, l-1$, and (c) $h_l \circ s_1 = \iota_X$. Then

$$q_X^* \,:\, \mathrm{M}_k(F) \longrightarrow \mathrm{H}^0(X, \mathrm{M}_k)$$

is an isomorphism for all $k \in \mathbb{Z}$ and all cycle modules M_* over F.

In fact, let $q_{\mathbb{A}^1_{F(X)}} : \mathbb{A}^1_{F(X)} \longrightarrow \operatorname{Spec} F(X)$ be the structure morphism. Then by homotopy invariance the pull-back $q_{\mathbb{A}^1_{F(X)}}^* : \mathrm{M}_k(F(X)) \longrightarrow \mathrm{H}^0(\mathbb{A}^1_{F(X)}, \mathrm{M}_k)$ is an isomorphism. Since $q_{\mathbb{A}^1_{F(X)}} \circ s_i = \mathrm{id}_{\operatorname{Spec} F(X)}$ for $i = 0, 1$ we have $s_0^* = (q_{\mathbb{A}^1_{F(X)}}^*)^{-1} = s_1^*$. Therefore $(h_j \circ s_0)^* = (h_j \circ s_1)^*$ for all $j = 1, \ldots, l$, and so it follows from (b) that

$$(h_1 \circ s_0)^* \,=\, (h_l \circ s_1)^* \,:\, \mathrm{M}_k(F(X)) \longrightarrow \mathrm{H}^0(X, \mathrm{M}_k)\,.$$

But by (c) we have $h_l \circ s_1 = \iota_X$ and so by the very definition of the pull-back homomorphism on cycle complexes $(h_l \circ s_1)^*$ and hence also $(h_1 \circ s_0)^*$ is injective. This implies by (a) that $x^* : \mathrm{H}^0(X, \mathrm{M}_k) \longrightarrow \mathrm{M}_k(F)$ is injective and so $q_X^* : \mathrm{M}_k(F) \longrightarrow \mathrm{H}^0(X, \mathrm{M}_k)$ an isomorphism since $q_X \circ x = \mathrm{id}_{\operatorname{Spec} F}$.

3. Applications: Brauer groups of affine quadrics and some reductive algebraic groups

3.1. *The Brauer group of an affine quadric.* We assume here that F is a field of characteristic $\neq 2$. As above n is an integer ≥ 2 which is not divisible by $\operatorname{char} F$.

Let

$$q \,=\, \sum_{i=0}^{m} a_i x_i^2\,,\; a_i \in F^\times\,,\; m \geq 4\,,$$

be a regular quadratic form over the field F. Denote by $X_q \hookrightarrow \mathbb{P}^m_F$ the corresponding projective quadric, and by $X_{q,\mathrm{aff}} \subset X_q$ the open affine subscheme defined by $x_0 \neq 0$. This is (isomorphic to) the non singular affine quadric in $\mathbb{A}^m_F$ given by the equation $a_0 + \sum_{j=1}^{m} a_j x_j^2 = 0$, and so in particular a smooth and geometrically integral affine scheme. We want to apply Theorem 2.11 to compute the Brauer group of this F-scheme, *i.e.* we have to verify conditions (i)–(iii) there. Since these are assumptions on $F_s \times_F X_{q,\mathrm{aff}}$ we can assume for ease of notation that $F = F_s$ is separably closed. Then q is a split quadratic form and so we can assume that $q = a_0 x_0 + x_1 \cdot x_2 + \sum_{j=3}^{m} a_j x_j^2$. The variety $X_{q,\mathrm{aff}}$ is then isomorphic to the affine quadric in $\mathbb{A}^m_F$ given by the equation $a_0 + x_1 \cdot x_2 + \sum_{j=3}^{m} a_i x_i^2 = 0$.

Consider the open subscheme $x_1 \neq 0$ of $X_{q,\mathrm{aff}}$. It is isomorphic to the spectrum of $F[x_1^{\pm 1}, x_3, \ldots, x_m]$ and so every unit of this open subscheme is equal $a \cdot x_1^r$ for some $a \in F^\times$ and some $r \in \mathbb{Z}$. Since x_1 is not a unit in $F[X_{q,\mathrm{aff}}]$ we see that $F[X_{q,\mathrm{aff}}]^\times = F^\times$, hence (iii). To show (ii), *i.e.* $\operatorname{Pic} X_{q,\mathrm{aff}} = 0$, we recall that $\mathrm{CH}_{m-1}(X_q) \simeq \operatorname{Pic} X_q$ is generated by the class of the hyperplane section $x_0 = 0$. Therefore by the localization sequence for Chow groups, see [10, Prop. 1.8], we have $\operatorname{Pic} X_{q,\mathrm{aff}} \simeq \mathrm{CH}_{m-1}(X_{q,\mathrm{aff}}) = 0$.

Finally we have to check (ii), *i.e.* $\mathrm{H}^0(X_{q,\mathrm{aff}}, \mathrm{K}^M_i/n) = 0$ for $i = 1, 2$. For this we note first that the closed subscheme of X_q defined by $x_0 = 0$ is isomorphic to the projective quadric $X_{q'} \hookrightarrow \mathbb{P}^{m-1}_F$, where $q' = x_1 \cdot x_2 + \sum_{j=3}^{m} a_i x_i^2$. Hence we have an exact sequence of complexes

$$\mathrm{C}^\bullet(X_{q'}, \mathrm{K}^M_{i-1}/n)[-1] \rightarrowtail \mathrm{C}^\bullet(X_q, \mathrm{K}^M_i/n) \twoheadrightarrow \mathrm{C}^\bullet(X_{q,\mathrm{aff}}, \mathrm{K}^M_i/n)\,, \qquad (6)$$

for all $i \geq 1$, where we denote for a complex $K^\bullet$ by $K^\bullet[-1]$ the shifted by -1 complex: $(K^\bullet[-1])^j = K^{j-1}$.

To analyze the associated cohomology sequence for $i = 1$ and 2 we recall that for a projective quadric Q of (Krull-)dimension ≥ 1 over the separably closed field F we have $\mathrm{H}^0(Q, \mathrm{K}^M_j/l) = 0$ for all $j \geq 1$ and $l \geq 2$. In fact, the motive of Q is isomorphic to a direct sum of Tate motives $\bigoplus_{k=0}^{\dim X_q} \underline{\mathbb{Z}}(k)^{\oplus r_k}$, where the integers r_k are 1, except if $\dim X_q$ is even and $k = \frac{\dim X_q}{2}$, in which case it is 2. Hence we have

$$\mathrm{H}^0(Q, \mathrm{K}^M_j/l) \;=\; \mathrm{K}^M_j(F)/l \cdot \mathrm{K}^M_j(F)$$

for all j, and the group $\mathrm{K}^M_j(F)/l \cdot \mathrm{K}^M_j(F)$ is zero for $j \geq 1$ if $l \geq 2$ since F is separably closed.

Hence we get from the long exact cohomology sequence associated with (6) an exact sequence

$$0 \longrightarrow \mathrm{H}^0(X_{q,\mathrm{aff}}, \mathrm{K}^M_i/n) \xrightarrow{\partial} \mathrm{H}^0(X_{q'}, \mathrm{K}^M_{i-1}/n)$$

$$\longrightarrow \mathrm{H}^1(X_q, \mathrm{K}^M_i/n) \longrightarrow \mathrm{H}^1(X_{q,\mathrm{aff}}, \mathrm{K}^M_i/n)$$

for all $i \geq 1$. Since as observed above $\mathrm{H}^0(X_{q'}, \mathrm{K}_1^M/n) = 0$ we get form this exact sequence immediately $\mathrm{H}^0(X_{q,\mathrm{aff}}, \mathrm{K}_2^M/n) = 0$.

To show that also $\mathrm{H}^0(X_{q,\mathrm{aff}}, \mathrm{K}_1^M/n) = 0$ we note that

$$\mathrm{H}^i(Y, \mathrm{K}_i^M/n) \simeq \mathbb{Z}/\mathbb{Z}n \otimes_{\mathbb{Z}} \mathrm{CH}^i(Y)$$

for a smooth and integral F-scheme of finite type Y and so by what we have shown above this exact sequence of cohomology groups gives a short exact sequence

$$0 \longrightarrow \mathrm{H}^0(X_{q,\mathrm{aff}}, \mathrm{K}_1^M/n) \longrightarrow \mathbb{Z}/\mathbb{Z}n \longrightarrow \mathbb{Z}/\mathbb{Z}n \longrightarrow 0\,,$$

which in turn implies that $\mathrm{H}^0(X_{q,\mathrm{aff}}, \mathrm{K}_1^M/n)$ is trivial.

This proves by Theorem 2.11 and its corollary the following result.

3.2. Theorem. *Let F be a field of characteristic $\neq 2$, and $X_q \hookrightarrow \mathbb{P}_F^m$ be the projective quadric defined by the equation $q = \sum_{i=0}^{m} a_i x_i^2$ with $m \geq 4$ and $a_i \in F^\times$ for all $i = 0, 1, \ldots, m$. Let further $X_{q,\mathrm{aff}} \subset X_q$ be the open affine quadric defined by $x_0 \neq 0$ and $n \geq 2$ an integer which is not divisible by the characteristic of F. Then the pull-back homomorphisms*

$${}_n\mathrm{Br}(T) \longrightarrow {}_n\mathrm{Br}(X_{q,\mathrm{aff}} \times_F T) \quad \textit{and} \quad {}_n\mathrm{Br}(T) \longrightarrow {}_n\mathrm{H}^2_{\mathrm{et}}(X_q \times_F T, \mathbb{G}_{\mathrm{m}})$$

along the respective projections to T are isomorphisms for any F-scheme T, such that either T_s is isomorphic as F_s-scheme to a torus, or $T = \mathrm{Spec}\, F$.

3.3. Remark. Merkurjev and Tignol [24, Thm. B] have shown that for any projective homogeneous variety X for a semisimple algebraic group over a field F the natural homomorphism $\mathrm{Br}(F) \longrightarrow \mathrm{H}^2_{\mathrm{et}}(X, \mathbb{G}_{\mathrm{m}})$ is surjective, and moreover they have given a description of the kernel. In particular they proved, see [24, Cor. 2.9], that for a projective quadric X_q of odd dimension over a field F as above the natural homomorphism $\mathrm{Br}(F) \longrightarrow \mathrm{H}^2_{\mathrm{et}}(X_q, \mathbb{G}_{\mathrm{m}})$ is an isomorphism, and so in particular $\mathrm{Br}(X_q) = \mathrm{H}^2(X_q, \mathbb{G}_{\mathrm{m}})$.

3.4. *Simply connected algebraic groups.* Let G be a simply connected algebraic group over the field F. We assume that $\mathrm{char}\, F$ does not divide an integer $n \geq 2$. Then we have

$$\mathrm{H}^0(G_s, \mathrm{K}_i^M/n) = \mathrm{K}_i^M(F_s)/n \cdot \mathrm{K}_i^M(F_s) = 0$$

for $i = 1, 2$, and also $\mathrm{Pic}\, G_s = 0$, see e.g. [14, Thm. 1.5], or P. Gille [11, Prop. 2, Section 2.2] for a proof. Moreover by Rosenlicht [29] we know that $F_s[G_s]^\times = F_s^\times$, and so we conclude from Theorem 2.11 and its corollary the following.

3.5. Corollary. *Let G be an algebraic variety (not necessarily an algebraic group) over a field F such that the variety G_s is isomorphic to a simply connected algebraic group over F_s, and $n \geq 2$ an integer which is not divisible by the characteristic of F. Then the pull-back homomorphism*

$$p_T^* \,:\, {}_n\mathrm{Br}(T) \longrightarrow {}_n\mathrm{Br}(G \times_F T)$$

along the projection $p_T : G \times_F T \longrightarrow T$ is an isomorphism in the following cases:

(i) *$T = \mathrm{Spec}\, F$ and so p_T is the structure morphism $G \longrightarrow \mathrm{Spec}\, F$; or*

(ii) *the F-scheme T is a form of a torus, i.e. $T_s = F_s \times_F T$ is isomorphic as F_s-scheme to a torus over F_s.*

In particular, if $\operatorname{char} F = 0$ *we have an isomorphism*

$$p_T^* \,:\, \mathrm{Br}(T) \xrightarrow{\simeq} \mathrm{Br}(G \times_F T)$$

for such G *and* T.

Moreover, if $\overline{G}$ *is a smooth compactification of* G, *then the natural pull-back homomorphism* ${}_n\mathrm{Br}(F) \longrightarrow {}_n\mathrm{H}^2_{\mathrm{et}}(\overline{G}, \mathbb{G}_{\mathrm{m}})$ *is also an isomorphism.*

3.6. Remark. The case $T = \operatorname{Spec} F$ was proven by Iversen [18] using topological methods if F is an algebraically closed field of characteristic 0, and by the first author in [14] for arbitrary fields.

3.7. Example. Let F be a field and $\mathrm{GL}_d(F)$ the general linear group over F of rank d. As an F-variety (not as an algebraic group!) the general linear group $\mathrm{GL}_d(F)$ is isomorphic to $\mathrm{SL}_d(F) \times_F \mathrm{GL}_1(F)$. Since the special linear group is simply connected we get from Corollary 3.5 an isomorphism

$${}_n\mathrm{Br}(\mathrm{GL}_d(F)) \,\simeq\, {}_n\mathrm{Br}(F[t, t^{-1}])$$

for all $n \in \mathbb{N}$ which are not divisible by $\operatorname{char} F$.

3.8. *Product of a torus with an arbitrary semisimple algebraic group.* Let (as above) F be a field and $n \geq 2$ an integer which is not divisible by $\operatorname{char} F$.

Let G be an arbitrary connected semisimple algebraic group over F and T a F-scheme, such that T_s is isomorphic to a torus over F_s. Let $\pi\colon G_{\mathrm{sc}} \longrightarrow G$ be the simply connected cover, see Tits [31, Prop. 2]. We denote by Π_G the fundamental group of G, *i.e.* more precisely the fundamental group of $F_s \times_F G$. We assume that $\operatorname{char} F$ does not divide the order of Π_G. Then the induced morphism $F_s \times_F G_{\mathrm{sc}} \longrightarrow F_s \times_F G$ is a Galois cover with group Π_G. In particular, the function field extension $F_s(G_{\mathrm{sc}}) \supseteq F_s(G)$ is a Galois extension of degree $|\Pi_G|$. Since $F_s \times_F G$ is connected this implies that $F(G_{\mathrm{sc}})$ is a separable extension of $F(G)$ of degree $|\Pi_G|$.

The same applies to the morphism $\pi \times \mathrm{id}_T \colon G_{\mathrm{sc}} \times_F T \longrightarrow G \times_F T$. In particular, there exists a corestriction (or also called transfer) homomorphism

$$\mathrm{cor}_{F(G_{\mathrm{sc}} \times_F T)/F(G \times_F T)} \,:\, {}_n\mathrm{Br}(F(G_{\mathrm{sc}} \times_F T)) \longrightarrow {}_n\mathrm{Br}(F(G \times_F T))\,.$$

By the projection formula the composition

$$\mathrm{cor}_{F(G_{\mathrm{sc}} \times_F T)/F(G \times_F T)} \,\circ\, \mathrm{res}_{F(G_{\mathrm{sc}} \times_F T)/F(G \times_F T)}$$

is equal to the multiplication by $[F(G_{\mathrm{sc}} \times_F T) : F(G \times_F T)] = |\Pi_{G_s}|$ and so if the cardinality $|\Pi_G|$ of the fundamental group of G is coprime to n we get that the restriction homomorphism $\mathrm{res}_{F(G_{\mathrm{sc}} \times_F T)/F(G \times_F T)}$ is injective on the subgroup of elements of exponent dividing n.

We assume now that the integer n is coprime to $|\Pi_G|$ and consider the commutative diagram

$$\begin{array}{ccc}
{}_n\mathrm{Br}(F(G_{\mathrm{sc}} \times_F T)) & \xleftarrow{\mathrm{res}_{F(G_{\mathrm{sc}} \times_F T)/F(G \times_F T)}} & {}_n\mathrm{Br}(F(G \times_F T)) \\
\uparrow{\scriptstyle \iota_{\mathrm{sc}}^*} & & \uparrow{\scriptstyle \iota^*} \\
{}_n\mathrm{Br}(G_{\mathrm{sc}} \times_F T) & \xleftarrow{(\pi \times \mathrm{id}_T)^*} & {}_n\mathrm{Br}(G \times_F T) \\
& {\scriptstyle p_{\mathrm{sc}}^*} \nwarrow \quad \nearrow {\scriptstyle p_T^*} & \\
& {}_n\mathrm{Br}(T)\,, &
\end{array} \tag{7}$$

where ι and $\iota_{\rm sc}$ are the respective generic points, and $p_{\rm sc}$ and p_T denote the projection $G_{\rm sc}\times_F T \longrightarrow T$ and $G\times_F T\longrightarrow T$, respectively.

Since both $G\times_F T$ and $G_{\rm sc}\times_F T$ are smooth we know that $\iota_{\rm sc}^*$ and ι^* are both injective by Auslander and Goldman [2, Thm. 7.2], and so by the commutative diagram above we conclude that $(\pi\times{\rm id}_T)^* : {}_n{\rm Br}(G\times_F T)\longrightarrow {}_n{\rm Br}(G_{\rm sc}\times_F T)$ is injective, too. Since as shown above, see Corollary 3.5, the homomorphism $p_{\rm sc}^*$ is an isomorphism this implies by the diagram (7) that $p_T^* : {}_n{\rm Br}(T)\longrightarrow {}_n{\rm Br}(G\times_F T)$ is surjective and so also an isomorphism. We have proven:

3.9. Corollary. *Let F be a field and $n\geq 2$ an integer which is not divisible by* ${\rm char}\,F$. *Let further G be a connected semisimple algebraic group over a field F and T an F-scheme, such that T_s is isomorphic to a torus over F_s. Assume that n is coprime to the order of the fundamental group Π_G of G. If moreover* ${\rm char}\,F$ *does not divide $|\Pi_G|$ then the pull-back homomorphism*

$$p_T^* \,:\, {}_n{\rm Br}(T) \longrightarrow {}_n{\rm Br}(G\times_F T)$$

along the projection $p_T\colon G\times_F T\longrightarrow T$ is an isomorphism.

3.10. *Brauer groups of reductive groups over a separably closed field.* Let F be a separably closed field whose characteristic does not divide the integer $n\geq 2$, and H a reductive group over F. Then by [22, Lem. 8] there exists a (split) torus $T\hookrightarrow H$, such that we have an isomorphism

$$H\,\simeq\, G\times_F T \tag{8}$$

as F-schemes, where G is the commutator subgroup $[H,H]$. Let us recall the proof of the decomposition (8). Let T_1 be a maximal torus of $G=[H,H]$. It is contained in a maximal torus S of H, and since F is separably closed we have $S=T_1\times_F T$ for some torus $T\subset S$. The centralizer of T_1 in G is equal to T_1 and so we have $T(F_s)\cap G(F_s)=\{e\}$, hence the morphism of F-schemes $G\times_F T\longrightarrow H$, $(g,t)\mapsto g\cdot t$ is an isomorphism. (Note that if H is semisimple, *i.e.* $H=[H,H]$, then the torus T in (8) is the "trivial" torus ${\rm Spec}\,F$.)

If ${\rm char}\,F$ does not divide the order of the fundamental group Π_G of G and the integer n is coprime to this order, we conclude from this isomorphism of F-schemes and Corollary 3.5 that ${}_n{\rm Br}(H)\simeq {}_n{\rm Br}(T)$. If the integer n is not coprime to $|\Pi_G|$ we use the strategy of Magid [22, Thm. 9] who has computed the Brauer group of a reductive group over an algebraically closed field of characteristic zero. The same approach gives also some results for fields of positive characteristic, and also for fields which are not separably closed.

3.11. *Some remarks about Brauer groups of reductive groups over an arbitrary field.* Let F be a field (not assumed to be separably closed) and H a reductive group over F. Let $G=[H,H]$ and assume the following:

(i) There is an isomorphism of F-schemes

$$H\,\simeq\, G\times_F T\,,$$

where T is a split F-torus, and

(ii) the fundamental group Π_G of the semisimple algebraic group $G=[H,H]$ is constant, *i.e.* defined over the base field, and ${\rm char}\,F$ does not divide the order of Π_G. Hence the simply connected cover $\pi : G_{\rm sc}\longrightarrow G$ is a Galois cover with group G.

We have then a commutative diagram

$$\begin{array}{ccc} {}_n\mathrm{Br}(G_{\mathrm{sc}} \times_F T) & \xleftarrow{(\pi\times \mathrm{id}_T)^*} & {}_n\mathrm{Br}(G \times_F T) \\ & {\scriptstyle p_{\mathrm{sc}}^*} \nwarrow \qquad \nearrow {\scriptstyle p_T^*} & \\ & {}_n\mathrm{Br}(T) & \end{array}$$

where $p_{\mathrm{sc}} : G_{\mathrm{sc}} \times_F T \longrightarrow T$ and $p_T : G \times_F T \longrightarrow T$ are the respective projections. Since as remarked above $p_{\mathrm{sc}}^* : {}_n\mathrm{Br}(T) \longrightarrow {}_n\mathrm{Br}(G_{\mathrm{sc}} \times_F T)$ is an isomorphism we get from this diagram a direct sum decomposition

$${}_n\mathrm{Br}(G \times_F T) \simeq {}_n\mathrm{Br}(T) \oplus \mathrm{Ker}\left({}_n\mathrm{Br}(G \times_F T) \xrightarrow{(\pi\times\mathrm{id}_T)^*} {}_n\mathrm{Br}(G_{\mathrm{sc}} \times_F T) \right).$$

By our assumptions $\pi \times \mathrm{id}_T : G_{\mathrm{sc}} \times_F T \longrightarrow G \times_F T$ is a Galois cover with group Π_G. Hence, see 1.3, we have a convergent spectral sequence

$$E_2^{p,q} := \mathrm{H}^p(\Pi_G, \mathrm{H}^q_{\mathrm{et}}(G_{\mathrm{sc}} \times_F T, \mathbb{G}_{\mathrm{m}})) \implies \mathrm{H}^{p+q}_{\mathrm{et}}(G \times_F T, \mathbb{G}_{\mathrm{m}}),$$

where $\mathrm{H}^i(\Pi_G, -)$ denotes the group cohomology of the finite group Π_G. The associated five term exacts sequence reads as follows:

$$\mathrm{Pic}(G \times_F T) \longrightarrow \mathrm{Pic}(G_{\mathrm{sc}} \times_F T)^{\Pi_G} \longrightarrow \mathrm{H}^2(\Pi_G, F[G_{\mathrm{sc}} \times_F T]^\times) \longrightarrow$$
$$\mathrm{Ker}\left(\mathrm{Br}(G \times_F T) \xrightarrow{(\pi\times\mathrm{id}_T)^*} \mathrm{Br}(G_{\mathrm{sc}} \times_F T) \right) \longrightarrow \mathrm{H}^1(\Pi_G, \mathrm{Pic}(G_{\mathrm{sc}} \times_F T)).$$

(This is the Chase-Harrison-Rosenberg [6] exact sequence for the Galois extension $G_{\mathrm{sc}} \times_F T \longrightarrow G \times_F T$, see also [8, Chap. IV, Thm. 1.1].)

We have $\mathrm{Pic}\, G_{\mathrm{sc}} = 0$, see *e.g.* [14, Thm. 1.5], and so by localization and homotopy invariance also $\mathrm{Pic}(G_{\mathrm{sc}} \times_F T) = 0$ as T is a split torus by our assumptions. Therefore we get from this exact sequence an isomorphism

$${}_n\mathrm{Br}(G \times_F T) \simeq {}_n\mathrm{Br}(T) \oplus {}_n\mathrm{H}^2(\Pi_G, F[G_{\mathrm{sc}} \times_F T]^\times),$$

where ${}_n\mathrm{H}^i(\Pi_G, -)$ denotes the subgroup of $\mathrm{H}^i(\Pi_G, -)$ of elements of order dividing n.

By Rosenlicht [29] we have $F[G_{\mathrm{sc}} \times_F T]^\times = F^\times \times \widehat{T}$, where $\widehat{T} = \mathrm{Hom}(T, \mathbb{G}_{\mathrm{m}})$ is the group of characters of T. The group Π_G operates trivially on $\widehat{T}$, and therefore there is an isomorphism

$${}_n\mathrm{H}^2(\Pi_G, F[G_{\mathrm{sc}} \times_F T]^\times) \simeq {}_n\mathrm{H}^2(\Pi_G, F^\times) \oplus {}_n\mathrm{H}^2(\Pi_G, \mathbb{Z})^{\oplus d},$$

where d is the rank of the torus T. Since Π_G is a finite group operating trivially on $\mathbb{Z}$ we have $\mathrm{H}^1(\Pi_G, \mathbb{Z}) = 0$ and so by the long exact cohomology sequence for the short exact sequence $\mathbb{Z} \xrightarrow{\cdot n} \mathbb{Z} \longrightarrow \mathbb{Z}/\mathbb{Z}n$ we get ${}_n\mathrm{H}^2(\Pi_G, \mathbb{Z}) \simeq \mathrm{H}^1(\Pi_G, \mathbb{Z}/\mathbb{Z}n)$.

We arrive finally at the following result, which is due to Magid [22, Thm. 9] if the field F is algebraically closed of characteristic zero.

3.12. Theorem. *Let F be a field and H a (connected) reductive F-group. Assume that* $\mathrm{char}\, F$ *neither divides n nor the order of the the fundamental group Π_G of the semisimple F-group $G = [H, H]$. Assume further that*

(i) *there is an isomorphism of F-schemes $H \simeq G \times_F T$, where T is a split F-torus of rank d, and*

(ii) *the simply connected cover $\pi : G_{\mathrm{sc}} \longrightarrow G$ is a Galois cover with group Π_G.*

Then we have an isomorphism

$$ {}_n\mathrm{Br}(H) \simeq {}_n\mathrm{Br}(T) \oplus {}_n\mathrm{H}^2(\Pi_G, F^\times) \oplus \mathrm{H}^1(\Pi_G, \mathbb{Z}/\mathbb{Z}n)^{\oplus d} . $$

Remark. Assumptions (i) and (ii) are automatically satisfied if the field F is separably closed.

3.13. Let again H be a connected reductive linear algebraic group over a field F, which is not assumed to be separably closed. Let further (as above) n an integer which is coprime to the characteristic of F. We denote by $\bar{F}$ an algebraic closure of F.

By [5, §2] the derived group as well as the radical of H are defined over F and we have

$$ H(\bar{F}) = [H,H](\bar{F}) \cdot \mathrm{Rad}(H)(\bar{F}) , $$

where $[H,H]$ and $\mathrm{Rad}(H)$ denote the F-subgroup schemes of H which represent the commutator subgroup and the radical of H, respectively.

Moreover, the radical is a torus and the intersection of the $\bar{F}$-points of $[H,H]$ and $\mathrm{Rad}(H)$ is a finite group. Since this intersection is in the center of H the group H is isomorphic to $[H,H] \times_F \mathrm{Rad}(H)$ if $[H,H]$ is of adjoint type. We conclude from Corollary 3.9:

3.14. Corollary. *Let H be a connected reductive group over a field F, such that $[H,H]$ is of adjoint type. Assume that the characteristic of F does not divide the order of the fundamental group of the commutator subgroup of H. Then*

$$ {}_n\mathrm{Br}(\mathrm{Rad}(H)) \simeq {}_n\mathrm{Br}(H) $$

for all integers n which are not divisible by $\mathrm{char}\, F$ *and coprime to the order of the fundamental group of the commutator subgroup $[H,H]$.*

4. The Brauer group of geometrically integral and regular rings of finite type over a field of positive characteristic

4.1. *The Brauer group of polynomial rings over fields of positive characteristic.* We start with the following result due to Auslander and Goldman [2] if $d = 1$, and due to Knus, Ojanguren and Saltman [21] if $d \geq 2$, which can be also deduced from Hoobler [16].

4.2. Proposition. *Let F be a field of characteristic $p > 0$. Then if $d \geq 2$, or if $d \geq 1$ and F is not perfect, the natural split monomorphism*

$$ \mathrm{Br}(F) \longrightarrow \mathrm{Br}(F[T_1, \dots, T_d]) $$

is not surjective.

Proof. If F is not perfect we know by Knus, Ojanguren and Saltman [21, Thm. 5.5] that $\mathrm{Br}(F[T_1]) \simeq \mathrm{Br}(F) \oplus \mathbb{Z}(p^\infty)^{(I)}$ for some infinite set I, see also Auslander and Goldman [2, Thm. 7.5]. Since $\mathrm{Br}(F[T_1])$ is a direct summand of $\mathrm{Br}(F[T_1, \dots, T_d])$ the claim follows.

Assume now that F is perfect and $d \geq 2$. If F is moreover finite then this result has been proven by Knus, Ojanguren, and Saltman in [21, Proof of Thm.

5.7]. The same argument works for any perfect field. In fact, since F is perfect we have $\big(F[T_1^{\frac{1}{p}},\dots,T_d^{\frac{1}{p}}]\big)^p = F[T_1,\dots,T_d]$, and so by [21, Cor. 3.10] the subgroup ${}_p\mathrm{Br}(F[T_1,\dots,T_d])$ is isomorphic to the kernel of the natural homomorphism $\mathrm{Br}(F[T_1,\dots,T_d]) \longrightarrow \mathrm{Br}(F[T_1^{\frac{1}{p}},\dots,T_d^{\frac{1}{p}}])$. This kernel contains the kernel of $\mathrm{Br}(F[T_1,\dots,T_d]) \longrightarrow \mathrm{Br}(F[T_1,\dots,T_{n-1},T_d^{\frac{1}{p}}])$ which in turn contains a subgroup isomorphic to the quotient group $F[T_1,\dots,T_{d-1}]\big/(F[T_1,\dots,T_{d-1}])^p$ by [21, Prop. 5.3]. The latter group is not trivial since by assumption $d \geq 2$, and so also ${}_p\mathrm{Br}(F[T_1,\dots,T_d]) \neq 0$.

On the other hand, by a result of Albert [1, p. 109] any central simple F-algebra of exponent p is Brauer equivalent to a cyclic algebra and so trivial in $\mathrm{Br}(F)$ since F is perfect. Therefore we have ${}_p\mathrm{Br}(F) = 0$. We are done. □

4.3. Our aim is now to show that if F is not perfect, or if $d \geq 2$, then the cokernel of the pull-back homomorphism $\mathrm{Br}(F) \longrightarrow \mathrm{Br}(F[T_1,\dots,T_d])$ is not trivial, p-torsion, and p-divisible for all fields F of characteristic $p > 0$. We prove first the following general lemma; see also Orzech and Small [28, Proof of Cor. 8.8].

4.4. Lemma. *Let R be a regular integral domain of characteristic $p > 0$ and $l > 0$ an integer which is not divisible by p. Then the pull-back*

$$\pi_R^* : {}_l\mathrm{Br}(R) \longrightarrow {}_l\mathrm{Br}(R[T])$$

along the inclusion $\pi_R : R \longrightarrow R[T]$ is an isomorphism. In particular, if F is a field of characteristic p and p does not divide l then the pull-back

$${}_l\mathrm{Br}(F) \longrightarrow {}_l\mathrm{Br}(F[T_1,\dots,T_d])$$

is an isomorphism.

Proof. We denote for a ring S by a_S the zero section $S[T] \longrightarrow S$, $T \mapsto 0$. This is a left inverse of the inclusion $\pi_S : S \longrightarrow S[T]$, *i.e.* $a_S \circ \pi_S = \mathrm{id}_S$.

We have a commutative diagram

$$\begin{array}{ccc} {}_l\mathrm{Br}(R[T]) & \xrightarrow{a_R^*} & {}_l\mathrm{Br}(R) \\ \downarrow & & \downarrow \\ {}_l\mathrm{Br}(K[T]) & \xrightarrow{a_K^*} & {}_l\mathrm{Br}(K) \end{array} ,$$

where K is the quotient field of R and the vertical arrows are induced by the inclusions $R \hookrightarrow K$ and $R[T] \hookrightarrow K[T]$.

By a result of Grothendieck, see *e.g.* [14, Lem. 4.4], the pull-back homomorphism $\pi_K^* : {}_l\mathrm{Br}(K) \longrightarrow {}_l\mathrm{Br}(K[T])$ is an isomorphism since (by assumption) $p = \operatorname{char} K$ does not divide l. Hence $a_K^* = (\pi_K^*)^{-1} : {}_l\mathrm{Br}(K[T]) \longrightarrow {}_l\mathrm{Br}(K)$ is also an isomorphism. Since R is a regular integral domain we know from Auslander and Goldman [2, Thm. 7.2] that $\mathrm{Br}(R) \longrightarrow \mathrm{Br}(K)$ is injective and so above commutative diagram shows that $a_R^* : {}_l\mathrm{Br}(R[T]) \longrightarrow {}_l\mathrm{Br}(R)$ is a monomorphism. This implies the lemma since $a_R^* \circ \pi_R^*$ is equal to the identity of ${}_l\mathrm{Br}(R)$. □

We are in position to prove the following (slight) generalization of Knus, Ojanguren, and Saltman [21, Thm. 5.7].

4.5. Theorem. *Let F be a field of characteristic $p > 0$ and $d \geq 1$ an integer. Then we have*

$$\mathrm{Br}(F[T_1, \ldots, T_d]) \simeq \mathrm{Br}(F) \oplus \mathbb{Z}(p^\infty)^{(I)},$$

where the set I is non empty if (and only if) F is not perfect or $d \geq 2$.

Proof. We show first that $\mathrm{Coker}\Big(\mathrm{Br}(F) \longrightarrow \mathrm{Br}(F[T_1, \ldots, T_d])\Big)$ is p-torsion. Let for this $0 \neq x \in \mathrm{Br}(F[T_1, \ldots, T_d])$. Since the Brauer group of a commutative ring is torsion, see *e.g.* [28, Chap. 12], there exists $l \geq 2$, such that $l \cdot x = 0$. Write $l = p^m \cdot l'$ with l' an integer coprime to p and $m \geq 0$. Then $l' \cdot (p^m \cdot x) = 0$ and so $p^m \cdot x$ has exponent l'. Since p does not divide l' it follows from the lemma above that $p^m \cdot x$ is in the image of natural homomorphism $\mathrm{Br}(F) \longrightarrow \mathrm{Br}(F[T_1, \ldots, T_d])$.

But $\mathrm{Br}(F[T_1, \ldots, T_d])$ is p-divisible by [21, Cor. 4.4] and so is also the quotient $\mathrm{Br}(F[T_1, \ldots, T_d])\big/\mathrm{Br}(F)$. Therefore by the structure theorem for divisible abelian groups, see [19, Thm. 4], it is isomorphic to $\mathbb{Z}(p^\infty)^{(I)}$ for some set I. By Theorem 4.1 this set is not empty if $d \geq 1$ and F is not perfect, or if $d \geq 2$. We are done. ☐

4.6. Let F be a field of characteristic $p > 0$ and R a regular F-algebra of finite type. We assume that R is geometrically integral and has Krull dimension $d \geq 1$. Then, see *e.g.* [9, Cor. 16.18], there are $T_1, \ldots, T_d \in R$, such that

(i) R is finite over the (polynomial) ring $D := F[T_1, \ldots, T_d]$, and

(ii) the quotient field K of R is a separable extension of $E := F(T_1, \ldots, T_d)$.

(In other words, $T_1, \ldots, T_d \in R$ are a separating transcendence basis of K over F.)

We set now $\overline{\mathrm{Br}}(S) := \mathrm{Coker}\Big(\mathrm{Br}(F) \longrightarrow \mathrm{Br}(S)\Big)$ for an F-algebra S. With this notation we have a commutative diagram where all morphisms are induced by the ring inclusions except for $c_{K/E}$ which is induced by the corestriction

$$\begin{array}{ccc} \overline{\mathrm{Br}}(R) & \xrightarrow{\iota_R} & \overline{\mathrm{Br}}(K) \\ {\scriptstyle r_{R/D}}\uparrow & & {\scriptstyle r_{K/E}}\uparrow\;\downarrow{\scriptstyle c_{K/E}} \\ \overline{\mathrm{Br}}(D) & \xrightarrow[\iota_D]{} & \overline{\mathrm{Br}}(E) \end{array} \quad . \tag{9}$$

Since R and D are regular integral domains the pull-back maps $\mathrm{Br}(R) \longrightarrow \mathrm{Br}(K)$ and $\mathrm{Br}(D) \longrightarrow \mathrm{Br}(E)$ are monomorphisms by Auslander and Goldman [2, Thm. 7.2] and so also ι_R and ι_D are both injective.

Assume now that $\dim R \geq 1$ and F is not perfect, or that $\dim R \geq 2$. Then by Theorem 4.5 we have

$$\overline{\mathrm{Br}}(D) = \overline{\mathrm{Br}}(F[T_1, \ldots, T_n]) \simeq \mathbb{Z}(p^\infty)^{(I)}$$

for some non empty set I. Since $K \supseteq E$ is a finite separable extension we know that $c_{K/E} \circ r_{K/E}$ is equal to the multiplication by $m = [K : E]$. Therefore by the commutative diagram (9) we have

$$(c_{K/E} \circ \iota_R)\big[\, r_{R/D}(\overline{\mathrm{Br}}(D))\big] = m \cdot \iota_D(\overline{\mathrm{Br}}(D)) \simeq \mathbb{Z}(p^\infty)^{(I)}$$

for some non empty set I since $\mathbb{Z}(p^\infty)$ is divisible. Hence $\overline{\mathrm{Br}}(R)$ is not trivial.

We have shown the following theorem.

4.7. Theorem. *Let F be a field of characteristic $p > 0$. If R is a regular and geometrically integral F-algebra of finite type of dimension ≥ 1 if F is non perfect, respectively of dimension ≥ 2 if F is perfect, then the natural homomorphism*

$$\mathrm{Br}(F) \longrightarrow \mathrm{Br}(R)$$

is not surjective.

4.8. Examples. Let F be a field of characteristic $p > 0$.

(i) Let G be a (connected) simply connected semisimple linear algebraic group over F. Since $G(F) \neq \emptyset$ we have $\mathrm{Br}(G) \simeq \mathrm{Br}(F) \oplus M$. for some subgroup M which is not trivial by Theorem 4.7 above and p-divisible by [21, Cor. 4.4]. In [14], see also our Corollary 3.5, it has been shown that the pull-back homomorphism ${}_n\mathrm{Br}(F) \longrightarrow {}_n\mathrm{Br}(G)$ is an isomorphism for all $n \in \mathbb{N}$ which are not divisible by $p = \operatorname{char} F$ and so, *cf.* the proof of Theorem 4.5, the subgroup M is also p-torsion. Hence we have

$$\mathrm{Br}(G) \simeq \mathrm{Br}(F) \oplus \mathbb{Z}(p^\infty)^{(I)}$$

for some non empty set I.

(ii) We assume here that $p \neq 2$. Let $X_{q,\mathrm{aff}}$ be a non singular affine quadric of Krull dimension ≥ 3 over F as in 3.1. Then by Theorem 3.2 the canonical homomorphism ${}_n\mathrm{Br}(F) \longrightarrow {}_n\mathrm{Br}(X_{q,\mathrm{aff}})$ is an isomorphism for all $n \in \mathbb{N}$ not divisible by $\operatorname{char} F$. Therefore by the same arguments as in part (i) above we have

$$\mathrm{Br}(X_{q,\mathrm{aff}}) \simeq \mathrm{Br}(F) \oplus \mathbb{Z}(p^\infty)^{(I)}$$

for some non empty set I.

4.9. Remark. Theorem 4.7 is wrong for non affine schemes. For instance, a projective homogeneous variety X for a semisimple algebraic group over a field F (*e.g.* a projective quadric) is smooth and geometrically integral, but as shown by Merkurjev and Tignol [24, Thm. B] the natural map $\mathrm{Br}(F) \longrightarrow \mathrm{H}^2_{\mathrm{et}}(X, \mathbb{G}_{\mathrm{m}})$ is surjective even if $\operatorname{char} F > 0$.

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