Categorical constructions on computability models

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Outline of the talk

Computability models and examples

The category CompMod

Categories and computability bases

Computability models

Given data

- Set T
- Set $\mathbf{C}(t)$ for $t \in T$
- Set $\mathbf{C}[t, t']$ of partial functions from $\mathbf{C}(t)$ to $\mathbf{C}(t')$ for $t, t' \in T$

Properties

- $\mathbf{1}_{\mathbf{C}(t)} \in \mathbf{C}[t, t]$
- $f \in \mathbf{C}[t, t'], g \in \mathbf{C}[t', t''] \Rightarrow g \circ f \in \mathbf{C}[t, t'']$

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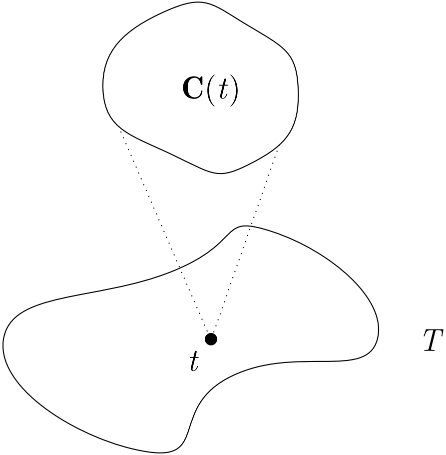

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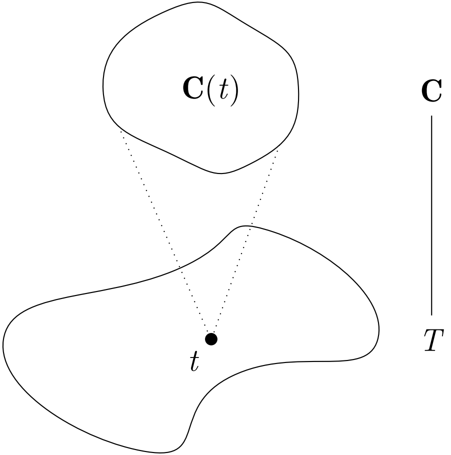

λ-calculus

$$M ::= x \mid \lambda x.M \mid MM$$

$\Lambda := \{\text{all terms}\}$
$T := \mathbb{1}$
$\mathbf{C}(\mathbb{0}) := \Lambda/\equiv_\beta$
$\mathbf{C}[\mathbb{0}, \mathbb{0}] := \{[M]_\beta \mapsto [PM]_\beta \mid P \in \Lambda/\equiv_\beta\}$

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Turing machines

$T = \mathbb{1}$
$\mathbf{C}(\mathbb{0}) := \mathbb{N}$
$\mathbf{C}[\mathbb{0},\mathbb{0}] := \{f : \mathbb{N} \to \mathbb{N} \mid f \text{ Turing computable}\}$

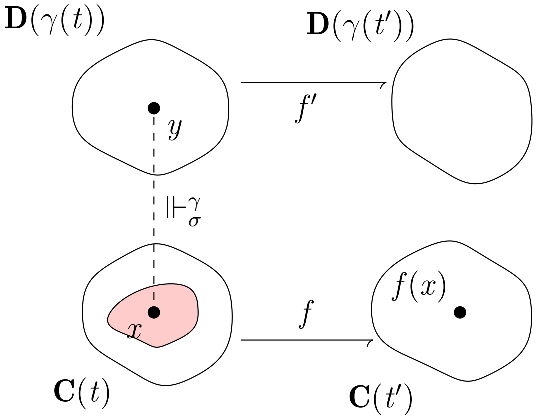
Simulations

Data

- Class map $\gamma \colon T \to U$ (Total!)
- Tracking relations $\Vdash_t^\gamma \subseteq \mathbf{D}(\gamma(t)) \times \mathbf{C}(t)$

Properties

- $\forall_{x \in \mathbf{C}(t)} \exists_{y \in \mathbf{D}(\gamma(t))} \, y \Vdash_t^\gamma x$
- $\forall_{f \in \mathbf{C}[t,t']} \exists_{f' \in \mathbf{D}[\gamma(t),\gamma(t')]} f' \Vdash_{(t,t')}^\gamma f$



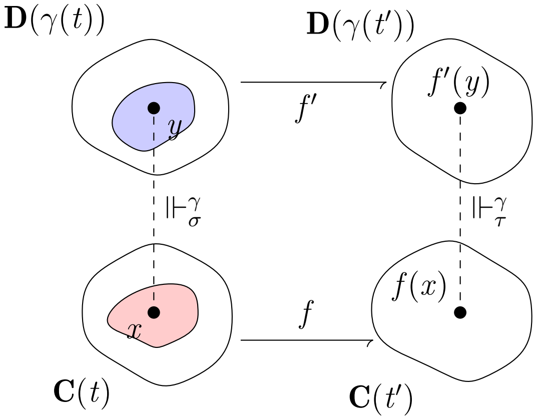
Simulations

Data

- Class map $\gamma\colon T \to U$ (Total!)
- Tracking relations $\Vdash^{\gamma}_{t} \subseteq \mathbf{D}(\gamma(t)) \times \mathbf{C}(t)$

Properties

- $\forall_{x \in \mathbf{C}(t)} \exists_{y \in \mathbf{D}(\gamma(t))} y \Vdash^{\gamma}_{t} x$
- $\forall_{f \in \mathbf{C}[t,t']} \exists_{f' \in \mathbf{D}[\gamma(t),\gamma(t')]} f' \Vdash^{\gamma}_{(t,t')} f$



Reminder

$$\mathbf{C}(\mathbb{0}) := \Lambda/\equiv_\beta \qquad \mathbf{C}[\mathbb{0},\mathbb{0}] := \{[M]_\beta \mapsto [PM]_\beta \mid P \in \Lambda/\equiv_\beta\}$$
$$\mathbf{D}(\mathbb{0}) := \mathbb{N} \qquad \mathbf{D}[\mathbb{0},\mathbb{0}] := \{f\colon \mathbb{N} \to \mathbb{N} \mid f \text{ Turing computable}\}$$

$\boldsymbol{\gamma}\colon \mathbb{1} \to \mathbb{1}$ identity

$\Vdash_{\mathbb{0}}^{\gamma} \subseteq \mathbb{N} \times \Lambda/\equiv_\beta$ defined by

$$m \Vdash_{\mathbb{0}}^{\gamma} M \Leftrightarrow m = \ulcorner M' \urcorner \text{ for } M' \equiv_\beta M$$

$\ulcorner - \urcorner$ Gödel numbering of terms.

Transformability

$\boldsymbol{\gamma}, \boldsymbol{\delta} \colon \mathbf{C} \rightharpoondown \mathbf{D}$. $\boldsymbol{\gamma} \preceq \boldsymbol{\delta}$ iff for all $t \in T$ we have $f_t \in \mathbf{D}[\gamma(t), \delta(t)]$ such that

$$\forall_x \forall_y (y \Vdash_t^{\gamma} x \Rightarrow f_t(y) \Vdash_t^{\delta} x)$$

Write $\boldsymbol{\delta} \sim \boldsymbol{\gamma}$ iff $\boldsymbol{\delta} \preceq \boldsymbol{\gamma}$ and $\boldsymbol{\gamma} \preceq \boldsymbol{\delta}$.
$\mathbf{C} \sim \mathbf{D}$ iff there exist $\boldsymbol{\gamma} \colon \mathbf{C} \rightharpoondown \mathbf{D}$ and $\boldsymbol{\delta} \colon \mathbf{D} \rightharpoondown \mathbf{C}$ such that

$$\boldsymbol{\gamma} \circ \boldsymbol{\delta} \sim \mathrm{id}_{\mathbf{D}} \;\text{ and }\; \boldsymbol{\delta} \circ \boldsymbol{\gamma} \sim \mathrm{id}_{\mathbf{C}}$$

The category CompMod

	Sets	**Cat**	CompMod
1 complete			
3 cocomplete			
4 ccc			
5 type-cat			
6 regular			
7 topos			

The category CompMod

	Sets	**Cat**	CompMod
1 complete	✓	✓	✓
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- **Given:** $\mathbf{C}_s$ for $s \in S$

  $$\begin{array}{c}\mathbf{C}_s \\ | \\ T_s\end{array}$$

- **Type names:** $\prod_{s\in S} T_s$
- **Data types:**
  $\left(\prod_{s\in S}\mathbf{C}\right)\left((t_s)_{s\in S}\right) = \prod_{s\in S}\mathbf{C}_s(t_s)$.
- **Computable functions:** $\left(\prod_{s\in S}\mathbf{C}\right)\left[(t_s)_{s\in S}, (t'_s)_{s\in S}\right] = \prod_{s\in S}\mathbf{C}_s[t_s, t'_s]$.

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- **Given:** $\begin{array}{ccccc} \mathbf{B} & \xrightarrow{\boldsymbol{\gamma}} & \mathbf{D} & \xleftarrow{\boldsymbol{\delta}} & \mathbf{C} \\ \mid & & \mid & & \mid \\ S & & U & & T \end{array}$
- **Type names:** $\gamma^{-1}(T)$
- **Data types:** $(\boldsymbol{\gamma}^{-1}(\mathbf{C}))(s,t)$
  $= \{(x,y) \in \mathbf{B}(s) \times \mathbf{C}(t)$
  $\big| \exists_{z \in \mathbf{C}(\delta(t))} z \Vdash^{\gamma}_{t} x \wedge z \Vdash^{\delta}_{v} y\}$

- **Computable functions:**

$$(\boldsymbol{\gamma}^{-1}(\mathbf{C}))\big[(s,t),(s',t')\big]$$
$$= \big\{(f_1,f_2) \in \mathbf{B}[s,s'] \times \mathbf{C}[t,t'] \;\big|\; \forall_{(x,y)\in(\boldsymbol{\gamma}^{-1}(\mathbf{D}))(s,t)} (x,y) \in \mathrm{dom}(f_1) \times \mathrm{dom}(f_2)$$
$$\Rightarrow \big(f_1(x), f_2(y)\big) \in (\boldsymbol{\gamma}^{-1}(\mathbf{C}))(s',t')\big\}.$$

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- **Type names:** $\gamma_*(T) = (S \amalg T)/\sim$

$$\begin{cases} T' := T \setminus \gamma(U), \\ S' := S \setminus \delta(U) \end{cases}$$

- **Data types:**

$$\boldsymbol{\gamma}_*(\mathbf{C})(x) = \begin{cases} \mathbf{B}(x) & \text{if } x \in S' \\ \mathbf{C}(x) & \text{if } x \in T', \\ \mathbf{B}(\gamma(u)) \amalg \mathbf{C}(\delta(u)) & \text{if } x = \gamma(u) \text{ for } u \in U \end{cases}$$

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- **Type names:** $\gamma_*(T) = (S \amalg T)/\sim$

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- **Computable functions:**

$$(\boldsymbol{\gamma}_*(\mathbf{D}))[x, y] = \begin{cases} \emptyset & \text{if } x \in S', y \in T' \text{ or } y \in S', x \in T', \\ \mathbf{B}[x, y] & \text{if } x \in T, y \in S' \text{ or } y \in T, x \in S', \\ \mathbf{C}[x, y] & \text{if } x \in V, y \in T' \text{ or } y \in V, x \in T', \\ \mathbf{B}[\gamma(u), \gamma(u')] \amalg \mathbf{C}[\delta(u), \delta(u')] & \text{if } x = \gamma(u), x' = \gamma(u') \end{cases}$$

The category CompMod

	Sets	Cat	CompMod
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- **Known:** Near-exponential structure (Transpose only unique if simulation single-valued)
  Def. Let $[\mathbf{C} \multimap_U \mathbf{D}]$ be the set families $\{\boldsymbol{\gamma}\colon \mathbf{C} \multimap \mathbf{D} \mid \gamma = U\}$
- **Def.** $A \in [\mathbf{C} \multimap_U \mathbf{D}]$ **uniformly tracked** if for all $f \in \mathbf{C}[t, t']$ there exists $f' \in \mathbf{D}[U(t), U(t')]$ such that for all $\boldsymbol{\gamma} \in A$ $f' \Vdash^{\gamma}_{(t,t')} f$

- $\mathbb{F} := \bigcup_U [\mathbf{C} \multimap_U \mathbf{D}]$
- **Def.** $A \in [\mathbf{C} \multimap_U \mathbf{D}]$, $B \in [\mathbf{C} \multimap_V \mathbf{D}]$, then $R \subseteq A \times B$ is a **uniform transformation** if for all $t \in T$ there exists $f_t \in \mathbf{D}[U(t), V(t)]$ such that for all $(\boldsymbol{\gamma}, \boldsymbol{\delta}) \in R$

$$y \Vdash^{\gamma}_{t} x \Rightarrow y \in \mathrm{dom}(f_t) \wedge f_t(y) \Vdash^{\gamma}_{t} x$$

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- $$\begin{array}{c}[\mathbf{C},\mathbf{D}]\\ \Big| \\ \mathbb{F}\end{array} \quad \begin{cases}[\mathbf{C},\mathbf{D}](G) = G, \\ [\mathbf{C},\mathbf{D}](G, G') = \{R \text{ uniform trafo } G \to G'\}\end{cases}$$

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- **"Computability model" Sets:**
  - Type names are sets
  - $\mathbf{Sets}(S) = S$
  - Computable functions are all partial functions
- **Family arrows:** $\mathrm{fHom}(\mathbf{C}) = \{\boldsymbol{\gamma} \colon \mathbf{C} \twoheadrightarrow \mathbf{Sets}\}$

- **Type names:** $\sum_{t \in T} \boldsymbol{\gamma}(t) := \big\{(t, b) \mid t \in T \text{ and } b \in \gamma(t)\big\}$
- **Data types:** $\left(\sum_{\mathbf{C}} \boldsymbol{\gamma}\right)(t, b) := \{y \in \mathbf{C}(t) \mid b \Vdash_t^{\gamma} y\}$
- **Computable functions:** $\left(\sum_{\mathbf{C}} \boldsymbol{\gamma}\right)[(s, a), (t, b)] =$
  $\left\{f \in \mathbf{C}[s, t] \mid \forall_{x \in \mathrm{dom}(f)} \left(x \in \left(\sum_{\mathbf{C}} \boldsymbol{\gamma}\right)(s, a) \Rightarrow f(x) \in \left(\sum_{\mathbf{C}} \boldsymbol{\gamma}\right)(t, b)\right)\right\}$

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- $\mathsf{pr}_1 : \sum_{t \in T} \boldsymbol{\gamma}(t) \to T$, defined by $(t, b) \mapsto t$, and for every $(t, b) \in \sum_{t \in T} \boldsymbol{\gamma}(t)$

$$y' \Vdash^{\mathsf{pr}_1}_{(t,b)} y :\Leftrightarrow y' = y,$$

- $\sum_{\boldsymbol{\delta}} \boldsymbol{\gamma}$, underlying set-function $\sum_{\delta} \gamma \colon \sum_{t \in T} \boldsymbol{\gamma}(t) \to \sum_{u \in U} \boldsymbol{\delta}(u)$ defined by $(t, b) \mapsto \big(\gamma(t), b\big)$

  Forcing relations defined by $x' \Vdash^{\sum_{\delta} \gamma}_{(t,b)} x :\Leftrightarrow x' \Vdash^{\gamma}_{t} x$.

The category CompMod

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5 type-cat	✓	✓	✓
6 regular	✓	✗	✗
7 topos			

$$\begin{array}{c}\mathbf{A}\\ \Big| \\ \mathbb{4}\end{array} \quad \begin{cases} \mathbf{A}(x) = \mathbb{1}, \text{for all } x \in \mathbb{4} \\ \mathbf{A}[\mathbb{0},\mathbb{1}] = \{\mathbf{1}_{\mathbb{1}}\} = \mathbf{A}[\mathbb{2},\mathbb{3}] \end{cases}$$

$$\begin{array}{c}\mathbf{B}\\ \Big| \\ \mathbb{3}\end{array} \quad \begin{cases} \mathbf{B}(x) = \mathbb{1}, \text{for all } x \in \mathbb{3} \\ \mathbf{B}[\mathbb{0},\mathbb{1}] = \mathbf{B}[\mathbb{0},\mathbb{2}] = \mathbf{B}[\mathbb{1},\mathbb{2}] = \{\mathbf{1}_{\mathbb{1}}\} \end{cases}$$

$$\begin{array}{c}\mathbf{C}\\ \Big| \\ \mathbb{2}\end{array} \quad \begin{cases} \mathbf{C}(x) = \mathbb{1} \qquad \text{for all } x \in \mathbb{2} \\ \mathbf{B}[\mathbb{0},\mathbb{1}] = \{\mathbf{1}_{\mathbb{1}}\} \end{cases}$$

The category CompMod

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6 regular	✓	✗	✗
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- $\gamma : \mathbf{A} \twoheadrightarrow \mathbf{B}$

$$\begin{cases} \mathbb{0} \mapsto \mathbb{0}, & \mathbb{1} \mapsto \mathbb{1} \\ \mathbb{2} \mapsto \mathbb{1}, & \mathbb{3} \mapsto \mathbb{2} \end{cases}$$

  regular epi

- $\delta : \mathbf{C} \twoheadrightarrow \mathbf{B}$

$$\begin{cases} \mathbb{0} \mapsto \mathbb{0}, \mathbb{1} \mapsto \mathbb{1} \end{cases}$$

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- $\gamma\colon \mathbf{A} \twoheadrightarrow \mathbf{B}$

$$\begin{cases} \mathbb{0} \mapsto \mathbb{0}, & \mathbb{1} \mapsto \mathbb{1} \\ \mathbb{2} \mapsto \mathbb{1}, & \mathbb{3} \mapsto \mathbb{2} \end{cases}$$

  regular epi

- $\delta\colon \mathbf{C} \twoheadrightarrow \mathbf{B}$

$$\begin{cases} \mathbb{0} \mapsto \mathbb{0}, \mathbb{1} \mapsto \mathbb{1} \end{cases}$$

- $\delta^{-1}\gamma$ **not** regular epi

The category CompMod

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7 topos	✓	✗	✗

Bases of computability

Data

- Category $\mathscr{C}$
- Subsets $B(c) \subseteq \mathrm{Mon}(-, c)$ for all $c \in \mathscr{C}$

Properties

- B contains all identities
- B closed under composition and pullback along arbitrary morphisms

Examples

- Total base tot
$$\mathrm{tot}(c) = \{\mathbf{1}_c\}$$
- Partial base prt ($\mathscr{C}$ cartesian closed)
$$\mathrm{prt}(c) = \mathrm{Mon}(-, c)$$
- Isomorphism base I
$$I(c) = \{f\colon b \to c \mid f \text{ iso }\}$$

1988 — ROSOLINI: Continuity and Effectiveness in topoi

- Partial product through bifunctor $- \times - : \mathscr{C} \times \mathscr{C} \to \mathscr{C}$
- ⇝ allows definition of domain as endomorphism
- ⇝ Category of domain $\mathsf{Dom}(\mathscr{C})$
- ⇝ embedding $\mathscr{C} \to \mathsf{P}(\mathsf{Dom}(\mathscr{C}), D)$

Bases of computability
in other work

- 1988 — ROSOLINI: Continuity and Effectiveness in topoi
- 1992 — MULRY: Partial map classifiers and partial cartesian closed categories

- Partial map classifiers in topoi as monads
- Partial cartesian closed categories

Bases of computability
in other work

- 1988 — ROSOLINI: Continuity and Effectiveness in topoi
- 1992 — MULRY: Partial map classifiers and partial cartesian closed categories
- 1998 — COCKETT & LACK: Restriction categories

- Domains as endomorphisms/Idempotents
- Subsumes p-categories
- 2-equivalence

$$\mathsf{rCat} \cong \mathsf{CatBaseComp}$$

The category CatBaseComp

	Sets	**Cat**	CatBaseComp
1 complete	✓	✓	✓
3 cocomplete	✓	✓	✓
4 ccc	✓	✓	✓
5 type-cat	✓	✓	✓
6 regular	✓	✗	✗
7 topos	✓	✗	✗

- $[\mathscr{C}, B](F) = \{\eta \colon F' \Rightarrow F \mid \forall_{c \in \mathscr{C}} \eta_c \in B\big(F(c)\big)\}$
- $\left(\sum_{\mathscr{C}} B\right)(c, x) = \left\{ i \in B(c) \;\middle|\; i \in \mathrm{Hom}_{\sum_{\mathscr{C}} S}\left(-, (c, x)\right) \right\}$

The associated computability model

Data

- Category $\mathscr{C}$
- Base of computability B on $\mathscr{C}$
- Presheaf $P \colon \mathscr{C} \to \mathsf{Sets}$ respecting pullbacks in B

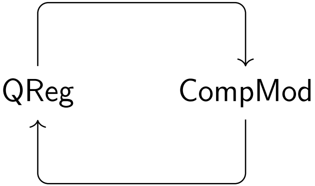
The model $\mathbf{CM}^B(\mathscr{C}; P)$

- Type names: $\mathscr{C}_0$, the objects of $\mathscr{C}$
- Data types $\mathbf{CM}^B(\mathscr{C}; P)(c) = P(c)$
- Computable functions

$$\mathbf{CM}^B(\mathscr{C}; P)[c, c'] = \Big\{ \big(S(i), S(f)\big) \mid (i, f) \colon c \rightharpoonup c', m \in B(c) \Big\}.$$

Questions

1



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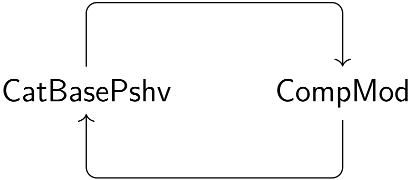
1. 

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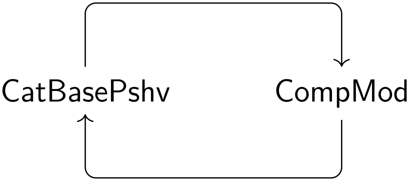
1. 

2. dependent arrow structure on CompMod, CatBaseComp

- Luis Gambarte and Iosif Petrakis. “The Grothendieck Computability Model”.
  In: *ICTCS’24: Italian Conference on Theoretical Computer Science. Torino, Italy*
- Luis Gambarte and Iosif Petrakis. “Categories with a base of computability
  in preparation
- Iosif Petrakis. “Strict Computability Models over Categories and Presheaves”.
  In: *Journal of Logic and Computation 32.8 (Dec. 2022), pp. 1815–1838.*
  ISSN: 0955-792X. DOI:10.1093/logcm/exac077.

Andrew M. Pitts. “Categorical Logic”.
In: *Handbook of Logic in Computer Science: Volume 5: Logic and Algebraic Methods.*
USA: Oxford University Press, Inc., Apr. 2001, pp. 39–123. ISBN: 978-0-19-853781-6.

John Longley and Dag Normann. ”Higher-order computability”
Springer Verlag, 2015
ISBN: 978-3-662-47991-9

Data

- For each $c \in \mathscr{C}$ a collection $\mathrm{fHom}(c)$ of family arrows
- Composition rules $\circ\colon\ \mathrm{fHom}(c) \times \mathrm{Hom}(c, b) \to \mathrm{fHom}(b)$

Properties

- Identity rule: $\lambda \circ \mathbf{1}_c = \lambda$ for $\lambda \in \mathrm{fHom}(c)$
- Associativity: $\lambda \circ (g \circ f) = (\lambda \circ g) \circ f$

Type categories

The Σ-objects

Data

- for every $c \in \mathscr{C}$: Operations $\sum_c \colon \mathrm{fHom}(c) \to \mathscr{C}_0$, $\lambda \mapsto \sum_c \lambda$, $\mathsf{pr}_1 \colon \mathrm{fHom}(c) \to \mathscr{C}_0$, $\lambda \mapsto \left(\mathsf{pr}_1^\lambda \colon \sum_c \lambda \to c\right)$
- for every $c, d \in \mathscr{C}$ and $f \in \mathrm{Hom}(d, c)$: Operation $\sum_{\_} f \colon \mathrm{fHom}(c) \to \mathscr{C}_1, \lambda \mapsto \left(\sum_\lambda f \colon \sum_d \lambda \circ f \to \sum_c \Lambda\right)$

Properties

- The square

$$\begin{array}{ccc} \sum_d(\lambda \circ f) & \xrightarrow{\sum_\lambda f} & \sum_c \lambda \\ \downarrow{\scriptstyle \mathsf{pr}_1^{\lambda\circ f}} & & \downarrow{\scriptstyle \mathsf{pr}_1^{\lambda}} \\ d & \xrightarrow{f} & c \end{array}$$

is a pullback square

$$\sum_\lambda \mathbf{1}_c = \mathbf{1}_{\sum_c \lambda}$$

$$\sum_\lambda (g \circ f) = \left(\sum_\lambda g\right) \circ \left(\sum_{\lambda \circ g} f\right)$$

Regular categories

1. $\mathscr{C}$ must be finitely complete
2. Pullback squares

$$\begin{array}{ccc} f^{-1}(c) & \xrightarrow{f^{-1}f} & c \\ \downarrow{\scriptstyle f^{-1}f} & & \downarrow{\scriptstyle f} \\ c & \xrightarrow{f} & d \end{array}$$

admit coequalisers

$$f^{-1}(c) \underset{f^{-1}f}{\overset{f^{-1}f}{\rightrightarrows}} c \longrightarrow \mathrm{Coeq}(f^{-1}f, f^{-1}f)$$

3. Pullbacks of regular epimorphisms along arbitrary morphisms are epimorphisms

Reminder: $f \colon c \to d$ regular epimorphism iff there exists a coequaliser diagram

$$b \underset{h}{\overset{g}{\rightrightarrows}} c \xrightarrow{f} d$$