7. Group Characters. Dirichlet L-series

7.1. Definition (Group characters). Let G be a group. A *character* of G is a group homomorphism

$$\chi : G \longrightarrow \mathbb{C}^*.$$

If G is a finite group (written multiplicatively), then every element $x \in G$ has finite order, say $r = \mathrm{ord}(x)$. It follows that

$$\chi(x)^r = \chi(x^r) = \chi(e) = 1,$$

hence $\chi(x)$ is a root of unity for all $x \in G$.

Example. Let G be a cyclic group of order r and $g \in G$ a generator of G, i.e.

$$G = \{e = g^0, g = g^1, g^2, g^3, \ldots, g^{r-1}\} =: \langle g \rangle, \qquad (g^r = e).$$

If $\chi : G \to \mathbb{C}^*$ is a character, $\chi(g)$ is an r-th root of unity, hence there exits an integer k, $0 \le k < r$, with $\chi(g) = e^{2\pi ik/r}$. Conversely, for any such k,

$$\chi_k(g^s) := e^{2\pi iks/r}$$

defines indeed a group character of G.

7.2. Theorem. *Let G be a group.*

a) *The set of all group characters $\chi : G \to \mathbb{C}^*$ is itself a group if one defines the multiplication of two characters χ_1, χ_2 by*

$$(\chi_1\chi_2)(x) := \chi_1(x)\chi_2(x) \quad \text{for all } x \in G.$$

This group is called the *character group* of G and is denoted by $\widehat{G}$.

b) *If G is a finite abelian group, then the character group $\widehat{G}$ is isomorphic to G.*

Proof. a) The easy verification is left to the reader.

b) Consider first the case when $G = \langle g \rangle$ is a cyclic group of order r. Let

$$E_r := \{e^{2\pi ik/r} : 0 \le k < r\}$$

be the group of r-th roots of unity. E_r is itself a cyclic group of order r and the map

$$\widehat{G} \longrightarrow E_r, \quad \chi \mapsto \chi(g),$$

is easily seen to be an isomorphism. To prove the general case, we use the fact that every finite abelian group G is isomorphic to a direct product of cyclic groups:

$$G \cong C_1 \times \ldots \times C_m.$$

From $\widehat{G} \cong \widehat{C}_1 \times \ldots \times \widehat{C}_m$ the assertion follows.

7.3. Theorem. *Let G be a finite abelian group of order r.*
a) *Let $\chi \in \widehat{G}$ be a fixed character. Then*

$$\sum_{x \in G} \chi(x) = \begin{cases} r, & \text{if } \chi \text{ is the unit character } \chi \equiv 1, \\ 0 & \text{else.} \end{cases}$$

b) *Let $x \in G$ be a fixed element. Then*

$$\sum_{\chi \in \widehat{G}} \chi(x) = \begin{cases} r, & \text{if } x = e, \\ 0 & \text{else.} \end{cases}$$

Proof. a) The formula is trivial for the unit character. If χ is any group character different from the unit character, there exists an $x_0 \in G$ with $\chi(x_0) \neq 1$. If x runs through all group elements, also $x_0 x$ runs through all group elements. Therefore

$$\sum_{x \in G} \chi(x) = \sum_{x \in G} \chi(x_0 x) = \chi(x_0) \sum_{x \in G} \chi(x).$$

It follows

$$(1 - \chi(x_0)) \sum_{x \in G} \chi(x) = 0 \quad \Longrightarrow \quad \sum_{x \in G} \chi(x) = 0, \quad \text{q.e.d.}$$

b) The formula is trivial for the unit element e. If x is a group element different from e, there exists a group character $\psi \in \widehat{G}$ with $\psi(x) \neq 1$. Otherwise all group characters would be constant on the subgroup $H \subset G$ generated by x, hence could be regarded as characters of the quotient group G/H, which contradicts theorem 7.2.b). If χ runs through all elements of $\widehat{G}$, so does $\psi\chi$. Hence

$$\sum_{\chi \in \widehat{G}} \chi(x) = \sum_{\chi \in \widehat{G}} (\psi\chi)(x) = \psi(x) \sum_{\chi \in \widehat{G}} \chi(x).$$

It follows

$$(1 - \psi(x)) \sum_{\chi \in \widehat{G}} \chi(x) = 0 \quad \Longrightarrow \quad \sum_{\chi \in \widehat{G}} \chi(x) = 0, \quad \text{q.e.d.}$$

7.4. Definition (Dirichlet characters). Let m be an integer ≥ 2. An arithmetical function $\chi : \mathbb{N}_1 \longrightarrow \mathbb{C}$ is called a Dirichlet character modulo m, if χ is induced by a group character

$$\tilde{\chi} : (\mathbb{Z}/m)^* \longrightarrow \mathbb{C}^*,$$

which means that

$$\chi(n) = \begin{cases} \tilde{\chi}(\overline{n}), & \text{if } \gcd(n,m) = 1, \\ 0, & \text{if } \gcd(n,m) > 1. \end{cases}$$

(Here $\overline{n}$ denotes the residue class of n modulo m).

The *principal* Dirichlet character modulo m is the Dirichlet character induced by the unit character $1 : (\mathbb{Z}/m)^* \to \mathbb{C}$. We denote this principal character by χ_{0m} or briefly by χ_0, if the value of m is clear by the context. Hence we have

$$\chi_{0m}(n) = \begin{cases} 1, & \text{if } \gcd(n,m) = 1, \\ 0, & \text{if } \gcd(n,m) > 1. \end{cases}$$

It is clear that a Dirichlet character is completely multiplicative. It is easy to see that an arithmetical function $f : \mathbb{N}_1 \to \mathbb{C}$ is a Dirichlet character modulo m iff it has the following properties:

i) f is completely multiplicative.

ii) $f(n) = f(n')$ whenever $n \equiv n' \bmod m$.

iii) $f(n) = 0$ for all n with $\gcd(n,m) > 1$.

7.5. Definition (Dirichlet L-series). Let $\chi : \mathbb{N}_1 \to \mathbb{C}$ be a Dirichlet character. The L-series associated to χ is the Dirichlet series

$$L(s,\chi) := \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}.$$

This series converges absolutely for every $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$.

Examples. Let $m = 4$.

i) The principal Dirichlet character modulo 4 has $\chi_{0,4}(n) = 1$ for n odd and $\chi_{0,4}(n) = 0$ for n even. Therefore

$$L(s,\chi_{0,4}) = \sum_{k=0}^{\infty} \frac{1}{(2k+1)^s} = 1 + \frac{1}{3^s} + \frac{1}{5^s} + \frac{1}{7^s} + \frac{1}{9^s} + \ldots$$

Since $2^{-s}\zeta(s) = \sum\limits_{k=1}^{\infty} \dfrac{1}{(2k)^s}$, we have

$$L(s,\chi_{0,4}) = (1 - 2^{-s})\zeta(s),$$

which shows that $L(s,\chi_{0,4})$ can be analytically continued to the whole plane $\mathbb{C}$ as a meromorphic function with a single pole at $s = 1$.

ii) Since $(\mathbb{Z}/4)^* = \{\overline{1}, \overline{3}\}$ has two elements, there is exactly one non-principal Dirichlet character χ_1 modulo 4, namely

$$\chi_1(n) = \begin{cases} 0 & \text{for } n \text{ even,} \\ (-1)^{(n-1)/2} & \text{for } n \text{ odd.} \end{cases}$$

Therefore

$$L(s, \chi_1) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^s} = 1 - \frac{1}{3^s} + \frac{1}{5^s} - \frac{1}{7^s} + \frac{1}{9^s} - + \ldots$$

This Dirichlet series converges to a holomorphic function for $\mathrm{Re}(s) > 0$. For $s = 1$ one gets the well known Leibniz series, hence

$$L(1, \chi_1) = \frac{\pi}{4}.$$

7.6. Theorem. *Let $\chi : \mathbb{N}_1 \to \mathbb{C}$ be a Dirichlet character modulo m. Then*
a) *For $\mathrm{Re}(s) > 1$ one has a product representation*

$$L(s, \chi) = \prod_{p \in \mathbb{P}} \frac{1}{1 - \chi(p) p^{-s}}.$$

b) *If $\chi = \chi_{0m}$ is the principal character, then*

$$L(s, \chi_{0m}) = \Big(\prod_{p|m} (1 - p^{-s}) \Big) \zeta(s),$$

where the product is extended over all prime divisors of m. Hence $L(s, \chi_{0m})$ can be analytically continued to the whole plane $\mathbb{C}$ as a meromorphic function with a single pole at $s = 1$.

c) *If χ is not the principal character, the L-series $L(s, \chi) = \sum_{n=1}^{\infty} \chi(n)/n^s$ has abscissa of convergence $\sigma_c = 0$, hence represents a holomorphic function in the halfplane $H(0)$.*

Proof. a) This follows directly from theorem 6.11 since χ is completely multiplicative.
b) From part a) and the definition of the principal character one gets

$$L(s, \chi_{0m}) = \prod_{p \nmid m} \frac{1}{1 - p^{-s}} = \prod_{p|m} (1 - p^{-s}) \prod_{p \in \mathbb{P}} \frac{1}{1 - p^{-s}}.$$

Since the last product is the Euler product of the zeta function, the assertion follows.
c) By theorem 6.4 it suffices to show that the partial sums $\sum_{n=1}^{N} \chi(n)$ remain bounded as $N \to \infty$. This can be seen as follows: Write $N = qm + r$ with integers q, r, $0 \le r < m$. By theorem 7.3.a) one has $\sum\limits_{n=1}^{qm} \chi(n) = 0$, hence

$$\Big| \sum_{n=1}^{N} \chi(n) \Big| = \Big| \sum_{n=qm+1}^{qm+r} \chi(n) \Big| \le \sum_{n=qm+1}^{qm+r} |\chi(n)| \le \varphi(m), \quad \text{q.e.d.}$$

The next theorem is an analogon of theorem 4.7.

7.7. Theorem. *Let m be an integer ≥ 2 and $\chi : \mathbb{N}_1 \to \mathbb{C}$ a Dirichlet character modulo m. We define the following generalization of the prime zeta function:*

$$P(s,\chi) := \sum_{p\in\mathbb{P}} \frac{\chi(p)}{p^s}.$$

This series converges absolutely in the halfplane $H(1) := \{s \in \mathbb{C} : \mathrm{Re}(s) > 1\}$ and one has

$$P(s,\chi) = \log L(s,\chi) + F_\chi(s),$$

where $F_\chi(s)$ is a bounded function in $H(1)$.

Proof. From the Euler product of the L-function we get for $\mathrm{Re}(s) > 1$

$$\begin{aligned}\log L(s,\chi) &= \sum_{p\in\mathbb{P}} \log \frac{1}{1-\chi(p)p^{-s}} = \sum_{p\in\mathbb{P}}\sum_{k=1}^{\infty} \frac{\chi(p)^k}{kp^{ks}} \\ &= \sum_{p\in\mathbb{P}} \frac{\chi(p)}{p^s} + \sum_{k=2}^{\infty} \frac{1}{k} \sum_{p\in\mathbb{P}} \frac{\chi(p)^k}{p^{ks}}.\end{aligned}$$

The theorem follows with

$$F_\chi(s) = -\sum_{k=2}^{\infty} \frac{1}{k} \sum_{p\in\mathbb{P}} \frac{\chi(p)^k}{p^{ks}},$$

since for $\mathrm{Re}(s) > 1$ we have

$$\Big|\sum_{p\in\mathbb{P}} \frac{\chi(p)^k}{p^{ks}}\Big| \leq \sum_{n=2}^{\infty} \frac{1}{n^k} \leq \frac{1}{k-1},$$

hence

$$|F_\chi(s)| \leq \sum_{k=2}^{\infty} \frac{1}{k(k-1)} = 1.$$