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Riemann Surfaces

Problem sheet #10

Problem 37

Let D be a divisor on the Riemann sphere $\mathbb{P}^1$. Prove

(a) $\dim H^0(\mathbb{P}^1, \mathcal{O}_D) = \max(0, 1 + \deg D)$

(b) $\dim H^1(\mathbb{P}^1, \mathcal{O}_D) = \max(0, -1 - \deg D)$

Problem 38

Let $X = \mathbb{C}/\Lambda$ be a torus, $x_0 \in X$ a point and P the divisor with $P(x_0) = 1$ and $P(x) = 0$ for $x \neq x_0$.

a) Prove

$$\dim H^0(X, \mathcal{O}_{nP}) = \begin{cases} 0, & \text{for } n < 0, \\ 1, & \text{for } n = 0, \\ n, & \text{for } n \geqslant 1. \end{cases}$$

Hint. Use the Weierstrass $\wp$-function (Problem 12).

b) Calculate $\dim H^1(X, \mathcal{O}_{nP})$ for all $n \in \mathbb{Z}$.

Problem 39

a) On a Riemann surface X let $\mathfrak{D}$ be the sheaf of divisors, i.e., for $U \subset X$ open, $\mathfrak{D}(U)$ consists of all maps $D : U \to \mathbb{Z}$ such that for every compact set $K \subset U$ there are only finitely many $x \in K$ with $D(x) \neq 0$. Show that $\mathfrak{D}$ together with the natural restriction morphisms is actually a sheaf and that

$$H^1(X, \mathfrak{D}) = 0.$$

Hint. Imitate the proof of $H^1(X, \mathcal{E}) = 0$, using a (discontinuous) integer valued partition of unity.

b) Let $\mathcal{M}^*$ be the sheaf of invertible meromorphic functions, i.e. for $U \subset X$ open, $\mathcal{M}^*(U)$ consists of all meromorphic functions on U which do not vanish identically on any connected component of U. Let $\beta : \mathcal{M}^* \to \mathfrak{D}$ be the map which assigns to every meromorphic function $f \in \mathcal{M}^*(U)$ its divisor $(f) \in \mathfrak{D}(U)$ and let $\alpha : \mathcal{O}^* \to \mathcal{M}^*$ be the natural inclusion map.

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p.t.o.

Show that

$$0 \to \mathcal{O}^* \xrightarrow{\alpha} \mathcal{M}^* \xrightarrow{\beta} \mathfrak{D} \to 0$$

is an exaxt sequence of sheaves and thus there is an exact sequence of groups

$$0 \to H^0(X, \mathcal{O}^*) \longrightarrow H^0(X, \mathcal{M}^*) \longrightarrow \mathrm{Div}(X) \longrightarrow H^1(X, \mathcal{O}^*) \longrightarrow H^1(X, \mathcal{M}^*) \to 0.$$

Problem 40

Let $D^* := \{z \in \mathbb{C} : 0 < |z| < 1\}$ be the punctured unit disk and

$$p_k : D^* \longrightarrow D^*, \quad z \mapsto z^k.$$

Let $\Omega(D^*)$ be the vector space of all holomorphic 1-forms on D^*. A trace mapping

$$\mathrm{Tr} : \Omega(D^*) \longrightarrow \Omega(D^*)$$

with respect to p_k is defined as follows. Since p_k is a covering map, every point $a \in D^*$ has an open neighborhood U such that $p_k^{-1}(U) = V_1 \cup \cdots \cup V_k$, where $V_j \subset D^*$ are disjoint open subsets and $p_k \mid V_j \to U$ is biholomorphic. Let $\varphi_j : U \to V_j$ be the inverse of $p_k \mid V_j$. Now for $\omega \in \Omega(D^*)$ let

$$\mathrm{Tr}(\omega) \mid U := \sum_{j=1}^{k} \varphi_j^* \omega.$$

a) Show that if $\omega \in \Omega(D^*)$ can be continued holomorphically (meromorphically) to the unit disk $D := \{z \in \mathbb{C} : |z| < 1\}$, then $\mathrm{Tr}(\omega)$ can be continued holomorphically (resp. meromorphically) to D.

b) Prove the formula

$$\mathrm{Res}_0(\mathrm{Tr}(\omega)) = \mathrm{Res}_0(\omega) \quad \text{for all } \omega \in \Omega(D^*).$$

c) More generally, for a 1-form

$$\omega = f(z)dz \in \Omega(D^*) \quad \text{with} \quad f(z) = \sum_{n \in \mathbb{Z}} c_n z^n$$

calculate explicitely $\mathrm{Tr}(\omega)$.
