Mathematisches Institut
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Riemann Surfaces

Problem sheet #7

Problem 25

Let X be a Riemann surface and let $\mathcal{O}$ and Ω be the sheaves of holomorphic functions (resp. holomorphic differential 1-forms) on X. Prove that the following sequences of sheaves are exact:

(i) $$0 \longrightarrow \mathbb{C} \longrightarrow \mathcal{O} \xrightarrow{d} \Omega \longrightarrow 0,$$

(ii) $$1 \longrightarrow \mathbb{C}^* \longrightarrow \mathcal{O}^* \xrightarrow{d\log} \Omega \longrightarrow 0.$$

Here $d\log : \mathcal{O}^* \longrightarrow \Omega$ denotes the homomorphism $f \longmapsto \dfrac{df}{f}$.

Problem 26

Let X be a Riemann surface, $a \in X$ and (U, z) a coordinate neighborhood of a with $z(a) = 0$. A holomorphic 1-form $\omega \in \Omega(U \setminus \{a\})$ has a Laurent expansion araound a with respect to this coordinate of the form

$$\omega = f\,dz \quad \text{with} \quad f(z) = \sum_{n=-\infty}^{\infty} c_n z^n.$$

The residue of ω in a is defined by $\mathrm{Res}_a(\omega) := c_{-1}$.

a) Show that the definition of the residue does not depend on the choice of the local coordinate.

b) Let $\mathcal{Q}$ be the sheaf of all meromorphic 1-forms ω on X having residue 0 at every pole of ω. Show that

$$0 \longrightarrow \mathbb{C} \longrightarrow \mathcal{M} \xrightarrow{d} \mathcal{Q} \longrightarrow 0$$

is a short exact sequence of sheaves.

Problem 27

Prove that the following is a short exact sequence of sheaves on a Riemann surface X.

$$0 \longrightarrow \Omega \longrightarrow \mathcal{E}^{1,0} \xrightarrow{d} \mathcal{E}^{(2)} \rightarrow 0.$$

Here $\mathcal{E}^{1,0}$ denotes the sheaf of smooth 1-forms of type $(1,0)$ and $\mathcal{E}^{(2)}$ the sheaf of smooth 2-forms on X.

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p.t.o.

Problem 28

On a Riemann surface X let $\mathcal{E}$ be the sheaf of all smooth (complex valued) functions and $\mathscr{H} \subset \mathcal{E}$ the subsheaf of harmonic functions. Prove that

$$0 \longrightarrow \mathscr{H} \longrightarrow \mathcal{E} \xrightarrow{d''d'} \mathcal{E}^{(2)} \longrightarrow 0$$

is a short exact sequence of sheaves on X.
