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Riemann Surfaces

Problem sheet #3

Problem 9

Let $f : \mathbb{P}_1 \to \mathbb{P}_1$ be the holomorphic map defined by the rational function

$$f(z) := z + \frac{1}{z}$$

Show that f is a two-sheeted branched covering and determine all its branch points.

Problem 10

a) Prove that the tangent function defines a local homeomorphism $\tan : \mathbb{C} \to \mathbb{P}_1$.

b) Prove that $\tan(\mathbb{C}) = \mathbb{P}_1 \smallsetminus \{\pm i\}$ and that

$$\tan : \mathbb{C} \to \mathbb{P}_1 \smallsetminus \{\pm i\}$$

is a covering map.

Problem 11

Consider the covering maps $\exp\colon \mathbb{C} \to \mathbb{C}^*$ and $\tan\colon \mathbb{C} \to \mathbb{P}_1 \smallsetminus \{\pm i\}$, cf. problem 10.

a) Show that there exists a uniquely defined biholomorphic map $f : \mathbb{C}^* \to \mathbb{P}_1 \smallsetminus \{\pm i\}$ with $f(1) = 0$ and $\lim_{z\to 0} f(z) = -i$.

b) Show that there exists a uniquely defined biholomorphic map $g\colon \mathbb{C} \to \mathbb{C}$ with $g(0) = 0$ which makes the following diagram commutative

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{g} & \mathbb{C} \\ {\scriptstyle\exp}\downarrow & & \downarrow{\scriptstyle\tan} \\ \mathbb{C}^* & \xrightarrow{f} & \mathbb{P}_1 \smallsetminus \{\pm i\} \end{array}$$

and use it to express the function arctan in terms of the logarithm.

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p.t.o.

Problem 12

Let $\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 \subset \mathbb{C}$, ($\omega_1, \omega_2 \in \mathbb{C}$ linearly independent over $\mathbb{R}$), be a lattice. The Weierstrass $\wp$-function with respect to Λ is defined by

$$\wp_\Lambda(z) := \frac{1}{z^2} + \sum_{\omega \in \Lambda \smallsetminus 0} \Big(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \Big).$$

a) Prove that for every compact disc $K_r := \{z \in \mathbb{C} : |z| \leqslant r\}$ there exists a finite subset $\Lambda_0 \subset \Lambda$ such that $\omega \notin K_r$ for all $\omega \in \Lambda \smallsetminus \Lambda_0$ and the series

$$\sum_{\omega \in \Lambda \smallsetminus \Lambda_0} \Big(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \Big).$$

converges uniformly on K_r. This implies that $\wp_\Lambda$ is a meromorphic function on $\mathbb{C}$ with poles of order two exactly at the lattice points $\omega \in \Lambda$.

b) Show that $\wp_\Lambda$ a doubly periodic meromorphic function on $\mathbb{C}$ with respect to Λ, i.e. $\wp_\Lambda(z) = \wp_\Lambda(z + \omega)$ for all $\omega \in \Lambda$ and all $z \in \mathbb{C}$.

Hint. Prove first that the derivative $\wp'_\Lambda(z) = -2 \sum_{\omega \in \Lambda} \frac{1}{(z-\omega)^3}$ is doubly periodic.

c) Since $\wp_\Lambda$ is periodic with respect to Λ, it defines a holomorphic map $\mathbb{C}/\Lambda \to \mathbb{P}_1$. Prove that this map is a two-sheeted branched covering map with exactly 4 branch points at

$$[0], \Big[\frac{\omega_1}{2}\Big], \Big[\frac{\omega_2}{2}\Big], \Big[\frac{\omega_1 + \omega_2}{2}\Big] \in \mathbb{C}/\Lambda.$$

Hint. To determine the zeros of $\wp'_\Lambda$, use that $\wp'_\Lambda$ is an odd function of z, i.e. $\wp'_\Lambda(-z) = -\wp'_\Lambda(z)$ for all $z \in \mathbb{C}$.
