Mathematisches Institut
der Universität München
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WS 2012/13
October 26, 2012

Riemann Surfaces

Problem sheet #2

Problem 5 Let $U \subset \mathbb{R}^2 \widehat{=} \mathbb{C}$ be an open subset and $h : U \to \mathbb{R}$ be a harmonic function, i.e. 2-times continuously differentiable and satisfying the differential equation

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} = 0.$$

a) Let $V \subset \mathbb{C}$ be another open subset and $\varphi : V \to U$ be a biholomorphic mapping. Prove that the composite function $h \circ \varphi : V \to \mathbb{R}$ is also harmonic.

Remark. This implies that one can define the notion of harmonic function on a Riemann surface.

b) Let X be a Riemann surface and $f : X \to \mathbb{C}^*$ a holomorphic function without zeros. Prove that the function $x \mapsto \log|f(x)|$ is harmonic on X.

Problem 6

a) Let X be a Riemann surface and $u \colon X \to \mathbb{R}$ a non-constant harmonic function. Prove that u does not attain its maximium.

b) Show that every harmonic function $u : X \to \mathbb{R}$ on a compact Riemann surface X is constant.

Problem 7 Let $p_1, \ldots, p_n \in X$ be points on a compact Riemann surface X and let

$$X' := X \smallsetminus \{p_1, \ldots, p_n\}.$$

(For example $X = \mathbb{P}_1$ and $X' = \mathbb{C}$.) Suppose that $f : X' \to \mathbb{C}$ is a holomorphic function and $W \subset \mathbb{C}$ a non-empty open subset with $f(X') \subset \mathbb{C} \smallsetminus W$. Prove that f is constant.

Problem 8

Let $q \in \mathbb{C}$ with $|q| > 1$ and let G be the multiplicative group $G := \{q^n : n \in \mathbb{Z}\} \subset \mathbb{C}^*$. The quotient $X := \mathbb{C}^*/G$ is defined as the set of all equivalence classes with respect to the equivalence relation

$$z \sim w \quad :\Longleftrightarrow \quad zw^{-1} \in G.$$

a) Prove that there exists a unique structure of a Riemann surface on X such that the canonical projection $\pi : \mathbb{C}^* \to X$ is locally biholomorphic.

b) Show that the Riemann surface X constructed in a) is isomorphic to a torus

$$E_\tau := \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau), \quad \tau \in \mathbb{H} := \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}.$$

Calculate τ.

Hint. Consider the composite map $\mathbb{C} \xrightarrow{\exp} \mathbb{C}^* \xrightarrow{\pi} \mathbb{C}^*/G$.

Due: Wednesday, November 7, 2012, 15 h