MATHEMATISCHES INSTITUT
DER UNIVERSITÄT MÜNCHEN
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Algebraic Number Theory
Final Written Exam (Klausur)

Problem 1

Let A be the ring of integers in the quadratic number field $\mathbb{Q}(\sqrt{10})$.
Prove that $\sqrt{10}$ is irreducible, but not prime in A.

Problem 2

Let $K = \mathbb{Q}(\sqrt{d})$, $d \neq 0, 1$, squarefree, be a quadratric number field, $\mathfrak{o}_K$ its ring of integers and p a rational prime.

a) For $p = 23$ give examples of $d < 0$ for each of the following cases:
   p is (i) ramified, (ii) inert, (iii) split in $\mathfrak{o}_K$.

b) For $d = 31$ give examples of rational primes p for each of the following cases:
   p is (i) ramified, (ii) inert, (iii) split in $\mathfrak{o}_K$.

Problem 3

Determine the class number of $\mathbb{Q}(\sqrt{-15})$ and give a representative for every ideal class.

Problem 4

Let $K := \mathbb{Q}(\theta)$, where θ is a zero of the irreducible polynomial

$$F(X) = X^3 + X^2 + a \in \mathbb{Q}[X].$$

Calculate the traces $\mathrm{Tr}_{K/\mathbb{Q}}(\theta^k)$ for $k = 1, 2, 3$.

Problem 5

Let $p \neq q$ be two odd rational primes.

a) Show that the property

   $\mathrm{I}(q, p)$: *q is inert (i.e. remains prime) in* $\mathbb{Z}[e^{2\pi i/p}]$

   depends only on the residue class of q mod $2p$.

b) Which rational primes are inert in $\mathbb{Z}[e^{2\pi i/7}]$?
