MATHEMATISCHES INSTITUT
DER UNIVERSITÄT MÜNCHEN
Prof. Otto Forster

WS 2004/2005
Nov. 4, 2004

Algebraic Number Theory

Problem Sheet #3

Problem 9

Let $d \neq 1$ be a squarefree integer with $d \equiv 1 \bmod 4$.

Show that the ring $\mathbb{Z}[\sqrt{d}]$ is not a UFD.

Problem 10

Let $A = \mathfrak{o}_K$ be the ring of integers in a quadratic number field $K = \mathbb{Q}(\sqrt{d})$, $d \neq 0, 1$ squarefree.

a) Suppose that there exists a unit $\varepsilon \in A^*$ with $N(\varepsilon) = -1$.

Show that every odd prime divisor $p \mid d$ satisfies $p \equiv 1 \bmod 4$.

b) Give examples of units $\varepsilon \in A^*$ with $N(\varepsilon) = -1$ for the cases $d = 5, 10, 13, 41$.

Problem 11

Determine all prime elements $\pi \in \mathbb{Z}[\frac{1+\sqrt{13}}{2}]$ (up to units) with $|N(\pi)| < 50$.

Problem 12

Let $A = \mathfrak{o}_K$ be the ring of integers in a quadratic number field $K = \mathbb{Q}(\sqrt{d})$ and $p \in \mathbb{Z}$ a rational prime which splits in $\mathfrak{o}_K$, i.e. $Ap = \mathfrak{p} \cdot \mathfrak{p}'$ with different prime ideals $\mathfrak{p}, \mathfrak{p}' \subset \mathfrak{o}_K$. Prove:

(1) $\mathfrak{p} \cap \mathfrak{p}' = Ap$,

(2) $\mathfrak{p} + \mathfrak{p}' = A$,

(3) $A/Ap \cong \mathbb{F}_p \times \mathbb{F}_p$.

Due: Tuesday, November 16, 2004, 14:10 h