**Differential geometry / Differenzierbare Mannigfaltigkeiten**
**WS 2009/10**

D. Kotschick and P. Mayr

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(1) Topological spaces; open and closed sets; Hausdorffness; continuity.
(2) Metric spaces, and their induced topology. This topology is always Hausdorff.
(3) Euclidean space $\mathbb{R}^n$ with its metric topology. By considering balls with rational radii centered at points with rational coordinates, one finds a countable collection of open sets such that all open sets are suitable unions of these. This is abstracted into the notion of a countable basis for the topology.
(4) **Topological manifolds** are locally Euclidean spaces that are Hausdorff and have a countable basis for their topology.
(5) A **differentiable manifold** is a topological manifold together with an atlas whose transition maps are differentiable. Such an atlas is called a differentiable or smooth atlas.
(6) A **differentiable structure** is an equivalence class of atlases, equivalently a maximal atlas.
(7) Every maximal $C^r$ atlas with $r \geq 1$ contains a $C^\infty$ atlas. Because of this fact (which we do not prove), we will restrict ourselves to $C^\infty$ manifolds throughout. So the words differentiable or smooth will usually mean $C^\infty$.

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(8) We define differentiability for maps between differentiable manifolds. A special case concerns real-valued functions.
(9) Diffeomorphisms vs. homeomorphisms.
(10) If we retain from a smooth atlas for a manifold $M$ only the knowledge of the images of charts, together with the identifications that are to be performed according to the transition maps, then we can reconstruct $M$ up to diffeomorphism; see [1, Sections 3.1 and 3.2].

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(11) Dimensions of manifolds and smooth invariance of domain.
(12) Examples of differentiable manifolds and their dimensions: Euclidean spaces $\mathbb{R}^n$, spheres $S^n$, tori $T^n$, $GL(n, \mathbb{R})$,... An open subset of a manifold is a manifold (of the same dimension); products of manifolds are manifolds (and the dimensions add up).
(13) The **tangent bundle** $TM$ of a differentiable manifold $M$ of dimension $n$ is itself a manifold of dimension $2n$. It has a natural projection $\pi\colon TM \longrightarrow M$ for which the preimage $T_xM = \pi^{-1}(x)$ of any point $x \in M$ has a well-defined structure as an $n$-dimensional real vector space. We call this the **tangent space** of $M$ at $x$.

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(14) The projection $\pi\colon TM \longrightarrow M$ is a differentiable map.
(15) For any differentiable map $f\colon M \longrightarrow N$, we define the **derivative** $Df\colon TM \longrightarrow TN$. This restricts to every tangent space $T_xM$ as a linear map $D_xf$ to $T_{f(x)}N$. This is the derivative of $f$ at $x \in M$.

(16) Differentiable **vector bundles** over manifolds; see [1] Section 3.3. Local vs. global triviality; isomorphisms of bundles. The tangent bundle as a vector bundle.

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(17) Every manifold $M$ is paracompact, meaning that every open cover has an open locally finite refinement. We prove the following more precise statement. Given an open covering $\{U_i\}_{i\in I}$ of $M$, there is an atlas $\{(V_k,\varphi_k)\}$ such that the covering by the $V_k$ is a locally finite refinement of the given covering, and such that $\varphi_k(V_k)$ is an open ball $B_3$ of radius 3 for all $k$ and the open sets $W_k=\varphi_k^{-1}(B_1)$ cover $M$.

*Proof.* We prove first that there is a sequence $G_i$, $i=1,2,\ldots$ of open sets with compact closures, such that the $G_i$ cover $M$ and satisfy

$$\overline{G_i}\subset G_{i+1}$$

for all $i$. To this end let $A_i$, $i=1,2,\ldots$ be a countable basis of the topology consisting of open sets with compact closures. (This exists because $M$ is second countable, Hausdorff and locally compact, see the homework assignement.) Set $G_1=A_1$. Suppose inductively that we have defined

$$G_k=A_1\cup\ldots\cup A_{j_k}\ .$$

Then let $j_{k+1}$ be the smallest integer greater than $j_k$ with the property that

$$\overline{G_k}\subset A_1\cup\ldots\cup A_{j_{k+1}}\ ,$$

and define

$$G_{k+1}=A_1\cup\ldots\cup A_{j_{k+1}}\ .$$

This defines the sequence $G_k$ as desired.

Let $\{U_i\}_{i\in I}$ be an arbitrary open covering of $M$. For every $x\in M$ we can find a chart $(V_x,\varphi_x)$ at $x$ with $V_x$ contained in one of the $U_i$ and such that $\varphi_x(V_x)=B_3$. Let $W_x=\varphi_x^{-1}(B_1)$. We can cover each set $\overline{G_k}\setminus G_{k-1}$ by finitely many such $W_{x_j}$ such that at the same time the corresponding $V_{x_j}$ are contained in the open set $G_{k+1}\setminus\overline{G_{k-2}}$. Taking all these $V_{x_j}$ as $i$ ranges over the positive integers we obtain the desired atlas. $\square$

(18) We construct smooth bump functions on $\mathbb{R}^n$ and transfer them to differentiable manifolds via charts. This allows us to construct various kinds of differentiable functions with special properties.

(19) Every open covering of a differentiable manifold admits a subordinate differentiable **partition of unity**. This follows from paracompactness and the existence of smooth bump functions.

(20) The space $\Gamma(E)$ of smooth sections of a vector bundle $\pi\colon E\longrightarrow B$ is a vector space over $\mathbb{R}$ with the operations of addition and scalar multiplication defined point-wise. In the same way, it is also a module over the ring of smooth functions on $B$. Using partitions of unity we see that every (global) section is a sum of (local) sections whose supports are contained in domains of charts.

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(21) A **curve** in a differentiable manifold $M$ is a map $s$ from $\mathbb{R}$ or from some subinterval $I$ of $\mathbb{R}$ to $M$. If such a map is differentiable at $t \in I$, then its derivative $D_t s$ applied to the tangent vector $\frac{\partial}{\partial t}$ of $\mathbb{R}$ gives a vector in $T_{s(t)}M$ denoted by $\dot{s}(t)$ and called the **velocity vector** of $s$ at $t$ (or at $s(t)$). Every tangent vector to $M$ can be realized as a velocity vector of a suitable curve.

(22) Three equivalent definitions of a tangent vector $X_p \in T_pM$ are:
$i)$ the local representative $\tilde{X}^{(\alpha)}_{\varphi_\alpha(p)} \in T_{\varphi_\alpha(p)}\varphi_\alpha(U_\alpha)$ in a chart $(U_\alpha, \varphi_\alpha)$ transforms under a change of chart $(U_\alpha, \varphi_\alpha) \to (U_\beta, \varphi_\beta)$ as $\tilde{X}^{(\beta)}_{\varphi_\beta(p)} = D_{\varphi_\alpha(p)}\psi\,(\tilde{X}^{(\alpha)}_{\varphi_\alpha(p)})$ with $\psi = \varphi_\beta \circ \varphi_\alpha^{-1}$;
$ii)$ as an equivalence class $\langle s \rangle_p$ of curves through $p \in M$ with velocity vector $X_p$, where $s \sim \check{s}$ if $s(0) = p = \check{s}(0)$ and $d\tilde{s}/dt(0) = d\tilde{\check{s}}/dt(0)$ for the local representatives ($\tilde{s} = \varphi \circ s$) in one (and thus any) chart;
$iii)$ as a derivative $D_{\langle s \rangle_p} : C^\infty(U, \mathbb{R}) \to \mathbb{R}$ on smooth, real valued functions evaluated at $p \in U$, in local coordinates: $X_p(f)(p) = \lim_{h\to 0} \frac{1}{h}(\tilde{f}(\tilde{s}(h)) - \tilde{f}(\tilde{s}(0))) = (\tilde{X}^i_{\varphi(p)} \frac{\partial}{\partial x^i})\tilde{f}(x^i)|_{\varphi(p)}$, where $\tilde{f} = f \circ \varphi^{-1}$ and $s(0) = p$.

(23) The definitions $ii)$ and $iii)$ lead to a **natural basis** "$e_i = \frac{\partial}{\partial x_i}$" on the vector space $T_pM$, which links the choice of basis vectors in $T_pM$ to the choice of local coordinates on $M$ in the open set $U \subset M$. The basis vector $e_i$ is *chosen* as the velocity vector of the (pull-back of) the coordinate line $\{x^j = \text{const.}\ \forall j \neq i\}$.

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(24) An **integral curve** $s : (-\epsilon, \epsilon) \to U$ of a vector field $X \in \Gamma(TU)$ through $x \in U \subset \mathbb{R}^n$ is a curve with $s(0) = x$ and velocity vector $\dot{s}(t) = X_{s(t)}\ \forall t \in (-\epsilon, \epsilon)$, $\epsilon > 0$. The existence and uniqueness of $s(t)$ follows from the theory of ODEs (Picard-Lindelöf) for small $\epsilon$.

(25) A **local flow** generated by a vector field $X \in \Gamma(TU)$ is a map $\Phi : (-\epsilon, \epsilon) \times W \to U$, $(t, x) \mapsto \Phi_t(x)$ defined on an open set $W \subseteq U \subset \mathbb{R}^n$. The local flow $\Phi$ exists and is unique on its domain of definition. It defines a local 1-parameter group $\Phi_{t+s} = \Phi_t \circ \Phi_s$ of diffeomorphisms from $W$ to its image.

(26) Patching together compatible local flows in open charts one constructs a local flow on $M$. There is a 1-1 correspondence between (maximal) local flows $\Phi$ on $M$ and vector fields $X \in \Gamma(TM)$ generating the flow. A **global flow** on $M$ is a smooth map $\Phi : \mathbb{R} \times M \to M$, its generator is called a **complete** vector field. A vector field with compact support is complete. On a compact manifold, every vector field is complete and generates a 1-parameter group of diffeomorphisms.

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(27) A **Lie algebra** over $\mathbb{R}$ is a real vector space $V$ with a bilinear map $[.\,,.] : V \times V \to V$, called the **Lie bracket**, satisfying $[X, Y] = -[Y, X]$ and $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$ (Jacobi identity). The space of sections $\Gamma(TM)$ is a Lie algebra with the Lie bracket defined by

$$[X, Y](f) = X(Y(f)) - Y(X(f))\,,$$

for $f \in C^\infty(M, \mathbb{R})$.

(28) The **Lie derivatives** by $X \in \Gamma(TM)$ of a function $f \in C^\infty(M,\mathbb{R})$ and of a vector field $Y \in \Gamma(TM)$ are defined as

$$L_X(f) := \frac{d}{dt}(\Phi_t^* f)|_{t=0}\,, \qquad L_X(Y) := \frac{d}{dt}(\Phi_{-t*}Y)|_{t=0}\,.$$

Here $\Phi_t$ is the flow generated by $X$, $\Phi_t^* f = f \circ \Phi_t$ is the **pull-back** of $f$ and $\Phi_{-t*}Y = D\Phi_{-t}(Y)$ is the **push-forward** of $Y$. The Lie derivatives satisfy $L_X(f) = X(f) \in C^\infty(M,\mathbb{R})$ and $L_X(Y) = [X,Y] \in \Gamma(TM)$.

(29) The flows $\Phi_t^X$ and $\Phi_s^Y$ generated by two vector fields $X$ and $Y$ commute, i.e. $\Phi_t^X \circ \Phi_s^Y = \Phi_s^Y \circ \Phi_t^X$ for all $s$ and $t$, if and only if $[X,Y] = 0$.

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(30) A **differential form** of degree $k$ on a smooth manifold $M$ is a map

$$\omega\colon \mathcal{X}(M) \times \ldots \times \mathcal{X}(M) \longrightarrow C^\infty(M)$$

of $C^\infty(M)$-modules, in other words, it is function-linear in all $k$ arguments. In addition, it is required to satisfy the following condition:

$$\omega(X_{\sigma(1)}, \ldots, X_{\sigma(k)}) = sign(\sigma)\omega(X_1, \ldots, X_k) \tag{1}$$

for all permutations $\sigma \in \Sigma_k$.

(31) We have the following:

**Lemma 1.** *If $\omega$ is a differential form, then the value of the function $\omega(X_1, \ldots, X_k)$ at a point $p \in M$ depends on the vector fields $X_i$ only through their values $X_i(p)$ at the point $p$.*

This means that $\omega$ has a value $\omega(p)$ at $p$, which is a $k$-multilinear map

$$\omega(p)\colon T_pM \times \ldots \times T_pM \longrightarrow \mathbb{R}$$

defined on $(X_1(p), \ldots, X_k(p))$ by extending the $X_i(p)$ to global vector fields, evaluating $\omega$ on these vector fields, and then evaluating the resulting function at $p$. (This multilinear map of course inherits property (1).)

(32) We build a universal model for multilinear maps, first for vector spaces (like $T_pM$), and then for vector bundles (like $TM$). This will allow us to interpret differential forms as sections of suitable vector bundles, so that $\omega(p)$ will be simply the value of the section $\omega$ at $p$. The universal model for bilinear maps on $V \times W$ is given by the **tensor product** $V \times W \longrightarrow V \otimes W$. (See [1] Section 7.1.).

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(33) Iterating the construction of the tensor product we obtain tensor products of $k$ vector spaces which have the universal property for $k$-linear maps. The **tensor algebra** and of a vector space $V$ is the direct sum of the tensor products $T^k(V)$ of $k$ copies of $V$, for $k = 0, 1, 2, \ldots$ endowed with the natural mutiplication given by the tensor product. Here $T^0(V)$ is just the ground field, and $T^1(V)$ is $V$ itself. The tensor algebra is a graded associative algebra. (See [1, Section 7.1].)

(34) For anti-symmetric bilinear maps on $V \times V$ there is a universal object $V \times V \longrightarrow \Lambda^2 V$ obtained as a quotient of $V \otimes V$.

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(35) The **exterior algebra** of a vector space over a field of characteristic $\neq 2$. (See [1, Section 7.2].)
(36) Induced maps on the tensor algebra and on the exterior algebra. The following lemma will be important:

**Lemma 2.** *If $V$ is a vector space of dimension $n$ and $f \colon V \longrightarrow V$ is a linear map, then the induced map $\Lambda^n(f) \colon \Lambda^n(V) \longrightarrow \Lambda^n(V)$ is multiplication by the determinant* $\det(f)$.

See [1, Lemma 7.2.15].

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(37) A vector bundle can be reconstructed, up to bundle isomorphism, from the data given by the transition maps between local trivializations.
(38) Multilinear algebra constructions applied to vector bundles. See [1, Section 7.4].

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(39) **Differential forms** as sections of exterior powers of the cotangent bundle. The wedge product of forms.
(40) **Exterior derivatives**.

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(41) Existence and uniqueness of the exterior derivative.
(42) **Pullback** of differential forms. The pullback commutes with the exterior derivative.
(43) The **Lie derivative** of differential forms.

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(44) **Contraction** of differential forms. Cartan's formula

$$L_X = d \circ i_X + i_X \circ d \, . \tag{2}$$

(45) **Orientability** and **orientations** of vector bundles.

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(46) Orientability of a vector bundle $E$ or rank $k$ is equivalent to the orientability of $E^*$, and also to that of the rank 1 bundle $\Lambda^k E$.
(47) For rank 1 bundles orientability is equivalent to triviality, and to the existence of a no-where zero section.
(48) Orientability and orientations on manifolds via the co-/tangent bundle and its maximal exterior power. Orientability is equivalent to the existence of a volume form.

(49) The **integral** of $n$-forms with compact support on oriented $n$-manifolds. (See [1] Section 8.2.)

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(50) The integral is well-defined.
(51) Motivation for Stokes' Theorem.
(52) **Manifolds with boundary**; their boundaries and their interiors. Every manifold (in the usual sense) is also a "manifold with boundary", but the boundary happens to be empty.

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(53) Orientations of manifolds with boundary and the induced orientation on the boundary.
(54) **Stokes's Theorem** for oriented manifolds with boundary:

$$\int_M d\omega = \int_{\partial M} \omega \ .$$

(See [1] Section 8.2.)

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(55) **Closed** and **exact** $k$-forms; the de Rham complex and its cohomology, called the **de Rham cohomology** $H^k_{dR}(M)$ of a differentiable manifold $M$.
(56) Any differentiable map $f\colon M \longrightarrow N$ induces a map on de Rham cohomology

$$f^*\colon H^k_{dR}(N) \longrightarrow H^k_{dR}(M)$$

defined by pulling back closed forms. (Recall that on forms the pullback commutes with exterior differentiation.)
(57) The **Poincaré lemma**: If $i_{t_0}\colon M \longrightarrow M \times \mathbb{R}$ is the inclusion of $M$ as $M \times \{t_0\}$ and $\pi\colon M \times \mathbb{R} \longrightarrow M$ is the projection, then $i^*_{t_0}$ and $\pi^*$ are inverses of each other on cohomology. Thus $M$ and $M \times \mathbb{R}$ have isomorphic de Rham cohomology.
(58) As consequences of the Poincaré lemma we have in particular a complete calculation of the de Rham cohomology of $\mathbb{R}^n$ by induction on $n$, the statement that on any manifold every closed form is locally exact, and the invariance of de Rham cohomology under smooth homotopies.

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(59) The forms with compact support form a subcomplex of the de Rham complex. Its cohomology is called the (de Rham) cohomology with compact support and denoted $H^k_c(M)$. For compact manifolds this is of course the same as the ordinary de Rham cohomology defined above.
(60) For any oriented $n$-dimensional manifold $M$ without boundary, the integral gives a well-defined surjective linear map:

$$\int_M \colon H^n_c(M) \longrightarrow \mathbb{R}$$
$$[\omega] \longmapsto \int_M \omega \ .$$

(61) By induction on $n$ we find the cohomology of $\mathbb{R}^n$ with compact supports, and, at the same time, the de Rham cohomology of $S^n$.

(62) For any connected, compact, oriented $n$-dimensional manifold $M$ without boundary, we have $H^n_{dR}(M) = \mathbb{R}$.

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(63) We now begin the discussion of **connections** and **curvature** on vector bundles.

Let $E \to M$ be a differentiable vector bundle of rank $k$ over a smooth manifold $M$ of dimension $n$.

**Definition 3.** A **connection** on $E$ is an $\mathbb{R}$-linear map

$$\nabla \colon \Gamma(E) \longrightarrow \Omega^1(E) \tag{3}$$

satisfying the Leibniz rule

$$\nabla(fs) = df \otimes s + f\nabla(s) \tag{4}$$

for all $f \in C^\infty(M)$ and $s \in \Gamma(E)$.

Here $\Omega^1(E) = \Gamma(T^*M \otimes E)$ is the space of 1-forms on $M$ with values in $E$. One can evaluate the 1-form on a vector field $X$ to obtain

$$\nabla_X(s) := \langle \nabla(s), X\rangle \in \Gamma(E)\ . \tag{5}$$

(64) We prove the following fundamental properties of connections:

- A connection $\nabla$ does not increase the support of sections, i. e. if $s \in \Gamma(E)$ vanishes on some open set $U \subset M$, then so does $\nabla(s)$.
- The value of $\nabla(s)$ at a point $p \in B$ depends only on the restriction of $s$ to an orbitrarily small open neighbourhood of $p$. (In other words, $\nabla$ is a differential operator, and $\nabla(s)(p)$ depends only on the germ of $s$ at $p$.)
- If $\nabla_1$ and $\nabla_2$ are connections, then so is $t\nabla_1 + (1-t)\nabla_2$ for all $t \in [0, 1]$.
- If $\nabla_1$ and $\nabla_2$ are connections, then $\nabla_1 - \nabla_2 \in \Omega^1(End(E)) = \Gamma(T^*B \otimes E^* \otimes E)$.

(65) Using these properties and a partition of unity subordinate to a covering of $M$ by open sets over which the restriction of $E$ is trivial, we prove:

**Theorem 4.** *Every vector bundle $E$ admits connections. The space of all connections on $E$ is an affine space for the space $\Omega^1(End(E))$ of 1-forms on $M$ with values in $End(E)$.*

(66) Next we extend the differential operator given by a connection $\nabla$ to bundle-valued forms of higher degree.

**Proposition 5.** *For every connection $\nabla$ on $E \to M$ there is a unique $\mathbb{R}$-linear map*

$$\bar\nabla \colon \Omega^l(E) \longrightarrow \Omega^{l+1}(E)$$

*which satisfies*

$$\bar\nabla(\omega \otimes s) = d\omega \otimes s + (-1)^l \omega \wedge \nabla(s) \tag{6}$$

*for all $\omega \in \Omega^l(M)$ and $s \in \Gamma(E)$. Moreover, this operator satisfies*

$$\bar\nabla(f(\omega \otimes s)) = (df \wedge \omega) \otimes s + f\bar\nabla(\omega \otimes s) \tag{7}$$

*for all smooth functions $f$ on $M$.*

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(67) Consider the operator $\bar{\nabla} \circ \nabla \colon \Omega^0(E) \longrightarrow \Omega^2(E)$ associated with a connection $\nabla$ on $E$. It turns out that this is linear over $C^\infty(M)$, and is therefore given by an element $F^\nabla \in \Omega^2(End(E))$. This is called the **curvature** of $\nabla$.

(68) A (local) **frame** for $E$ is a set of smooth sections $s_1, \ldots, s_k$ defined over some open set $U \subset M$, whose values are linearly independent at every point $p \in U$.

Thus a set of $k$ local smooth sections $s_1, \ldots, s_k$ is a frame if and only if $s_1(p), \ldots, s_k(p)$ is a basis of $E_p = \pi^{-1}(p)$ for every $p \in U$. Therefore a frame defined over $U$ defines a trivialization of $E|_U$, and, conversely, every such trivialization

$$\psi \colon \pi^{-1}(U) \longrightarrow U \times \mathbb{R}^k$$

defines a local frame by setting $s_i(p) = \psi^{-1}(p, e_i)$, where $e_1, \ldots, e_k$ is the standard basis of $\mathbb{R}^k$.

(69) Fix a local frame $s_1, \ldots, s_k$ for the restriction of $E$ to a trivialising open set in $M$. This choice determines a connection $\nabla_0$ defined by the requirement $\nabla_0(s_i) = 0$ for all $i$. Every other connection $\nabla$ differs from $\nabla_0$ by the addition of a 1-form with values in $End(E)$. However, the given trivialization of $E$ induces a trivialization of $End(E)$, and so a 1-form with values in $End(E)$ is nothing but a $k \times k$ matrix of ordinary 1-forms. Thus $\nabla$ can be expressed by the matrix $\omega = (\omega_{ij})$ of 1-forms given by

$$\nabla(s_i) = \sum_{j=1}^{k} \omega_{ij} \otimes s_j \ .$$

(70) From the definition of the curvature we calculate

$$F^\nabla(s_i) = \sum_{j=1}^{k} \Omega_{ij} \otimes s_j$$

with

$$\Omega_{ij} = d\omega_{ij} - \sum_{l=1}^{k} \omega_{il} \wedge \omega_{lj} \ .$$

We can write this briefly as $\Omega = d\omega - \omega \wedge \omega$, where the wedge product on the right-hand-side includes matrix multiplication, and is therefore not necessarily trivial unless $k = 1$.

(71) Similarly we compute $d\Omega = \omega \wedge \Omega - \Omega \wedge \omega$. This is the **Bianchi identity**.

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(72) A choice of a (local) frame is called a **choice of gauge** in physics terminology. The connection and curvature matrices represent $\nabla$ and $F^\nabla$ with respect to this choice. Connections are referred to as gauge fields.

Suppose we have another frame $s'_1, \ldots, s'_k$ on the same domain of definition as the original frame. Let $\omega'$ and $\Omega'$ denote the connection and curvature matrices of $\nabla$ with respect to this new frame. If

$$s'_i = \sum_{i=1}^{k} g_{ij} s_j \ ,$$

we find the following relations: $\omega' = dg\, g^{-1} + g\omega g^{-1}$ and $\Omega' = g\Omega g^{-1}$, where $g = (g_{ij})$. The change of basis $g$ is called a **gauge transformation**, and these formulae show how connection and curvature matrices behave under gauge transformations. The curvature matrix $\Omega$ is more invariant than the connection matrix $\omega$.

(73) Recall that with respect to a frame $s_1, \ldots, s_k$ of $E$ a connection $\nabla$ is expressed by a matrix $(\omega_{ij})$ of one-forms. If we choose a chart for the base manifold $M$ with local coordinates $x_1, \ldots, x_n$, then in the domain of this chart every one-form can be expressed uniquely as a linear combination of the $dx_i$. In particular, there are smooth functions $\omega_{ij}^{\alpha}$ on the domain of the chart such that

$$\omega_{ij} = \sum_{\alpha=1}^{n} \omega_{ij}^{\alpha} dx_{\alpha} \; . \tag{8}$$

Denoting the vector fields $\frac{\partial}{\partial x_\alpha}$ by $\partial_\alpha$, we find the following:

$$\nabla_{\partial_\alpha} s_i = \langle \partial_\alpha, \nabla s_i \rangle = \sum_{j=1}^{k} \langle \partial_\alpha, \omega_{ij} \rangle s_j = \sum_{j=1}^{k} \omega_{ij}^{\alpha} s_j \; .$$

More generally, if

$$s = \sum_{i=1}^{k} f_i s_i \; ,$$

then

$$\nabla_{\partial_\alpha} s = \sum_{j=1}^{k} \left( \frac{\partial f_j}{\partial x_\alpha} + \sum_{i=1}^{k} f_i \omega_{ij}^{\alpha} \right) s_j \; .$$

Writing $A^\alpha$ for the matrix $(\omega_{ij}^{\alpha})$ of functions we see that the covariant derivative $\nabla_{\partial_\alpha}$, which we abbreviate to $\nabla_\alpha$, has the form $\nabla_\alpha = \partial_\alpha + A^\alpha$.

(74) We can now give a first geometric interpretation of the curvature, or at least of its vanishing. A connection $\nabla$ is called **flat** if $F^\nabla = 0$.

**Proposition 6.** $[\nabla_\alpha, \nabla_\beta] s_i = \sum_{j=1}^{k} \Omega_{ij}(\partial_\alpha, \partial_\beta) s_j$

**Corollary 7.** *The connection $\nabla$ is flat if and only if $[\nabla_\alpha, \nabla_\beta] = 0$ for every local coordinate system $x_1, \ldots, x_n$ on the base manifold $M$.*

Thus the curvature quantifies the failure of the commutativity of covariant derivatives.

(75) If $E \to M$ is a vector bundle with a connection $\nabla$, we say that a section $s \in \Gamma(E)$ is **parallel** with respect to $\nabla$ if $\nabla s = 0$. In the special case that $\nabla$ is the connection given by some trivialization, a section is parallel if and only if it is constant in the given trivialization. Thus parallel sections should be thought of as the analogs of constant sections for nontrivial bundles.

(76) We will want to prove the following:

**Proposition 8.** *Let $\pi\colon E \to M$ be a smooth vector bundle with a connection $\nabla$, and $c\colon [0,1] \to M$ a smooth curve in the base. Then for every $v \in \pi^{-1}(c(0))$ there is a unique smooth curve $\tilde{c}\colon [0,1] \to E$ with $\pi \circ \tilde{c} = c$, $\tilde{c}(0) = v$ and $\nabla_{\dot{c}} s = 0$, where $s$ sends $c(t)$ to $\tilde{c}(t)$. Moreover, the map $v \mapsto \tilde{c}(1)$ defines a linear map of vector spaces $\pi^{-1}(c(0)) \to \pi^{-1}(c(1))$.*

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(77) In Proposition 8 the condition $\nabla_{\dot{c}} s = 0$ makes sense although $s$ is not a section over all of $M$ because the covariant derivative is only considered in the direction of $c$, where $s$ is defined.

The Proposition follows from the existence and uniqueness of the solutions of systems of linear ordinary differential equations with given initial conditions, together with the linear dependence of the solutions on the initial values.

**Definition 9.** The linear map

$$\begin{aligned} P_t \colon E_{c(0)} &\longrightarrow E_{c(t)} \\ v &\longmapsto \tilde{c}(t) \end{aligned}$$

is called the **parallel transport** along $c$. It is an isomorphism of vector spaces.

(78) As a consequence of Proposition 8 we have:

**Corollary 10.** *Over a curve every vector bundle with connection admits a framing by parallel sections. Over a one-dimensional base every vector bundle with connection admits local trivializations by parallel frames.*

Here the existence of a parallel frame is over the interval parametrizing the curve. Even if the endpoint of the curve agrees with the starting point, the same may not be true for the initial and ending frames. This is why the second statement is only local.

(79) This corollary fails for base spaces which are not one-dimensional, and this leads to geometric interpretations of the curvature. It will turn out that the corollary encodes the fact that on a one-manifold there is no curvature (as every two-form vanishes identically).

(80) We now prove:

**Theorem 11.** *A vector bundle $E \to M$ with connection $\nabla$ admits local frames consisting of parallel sections if and only if $\nabla$ is flat, i. e. $F^{\nabla} = 0$.*

(81) One of the consequences of this theorem is:

**Corollary 12.** *A vector bundle $E \to M$ admits a flat connection if and only if it has a system of local trivializations for which all transition maps are constant.*

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(82) **Metrics** on vector bundles are smoothly varying fiberwise positive-definite scalar products. Using a partition of unity we prove that every vector bundle admits a metric.

(83) A connection $\nabla$ on a vector bundle $E \to M$ is compatible with a metric $\langle\ ,\ \rangle$ if and only if

$$d\langle s_1, s_2 \rangle = \langle \nabla s_1, s_2 \rangle + \langle s_1, \nabla s_2 \rangle$$

for all pairs of sections $s_1, s_2 \in \Gamma(E)$. Sometimes a connection compatible with some metric is called a metric connection.

**Lemma 13.** *A connection $\nabla$ is compatible with a metric $\langle\ ,\ \rangle$ if and only if its connection matrix $\omega$ with respect to any local frame that is orthonormal with respect to $\langle\ ,\ \rangle$ is skew-symmetric, i. e. $\omega_{ij} = -\omega_{ji}$ for all $i$ and $j$. In this case the curvature matrix $\Omega$ with respect to a local orthonormal frame is also skew-symmetric: $\Omega_{ij} = -\Omega_{ji}$.*

Finally, metric connections always exist. The following is proved by combining the above lemma with the proof of Theorem 4.

**Proposition 14.** *Every vector bundle $E$ with a metric $\langle\ ,\ \rangle$ admits compatible connections. The space of all compatible connections is an affine space for the space $\Omega^1(SkewEnd(E))$ of $1$-forms with values in the endomorphisms of $E$ which are skew-symmetric with respect to $\langle\ ,\ \rangle$.*

Here an endomorphism $A$ is skew-symmetric with respect to $\langle\ ,\ \rangle$ if

$$\langle A(v), w\rangle = -\langle v, A(w)\rangle$$

for all $v$ and $w$.

(84) As an example we consider vector bundles of small rank equipped with metric connections. If the rank is $= 1$, then the skew-symmetry of the connection matrix shows that every metric connection is flat. As every bundle admits a metric and a compatible connection, we conclude that all rank one bundles admit flat connections. In some sense, rank $= 2$ is the first interesting case. Here the curvature is determined by the closed 2-form $\Omega_{12}$ with respect to an orthonormal frame. In the oriented case, this closed form is the same for all oriented orthonormal frames. This leads to the definition of the Euler class, a characteristic class of oriented rank 2 bundles in the de Rham cohomology of the base manifold.

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(85) On a smooth manifold $M$ we now consider connections $\nabla$ on the tangent bundle $TM \to M$. In this case the variables $X$ and $s$ in $\nabla_X s$ are on equal footing, as they are both sections of the tangent bundle. This leads to possible symmetries which make no sense in the more general setting of arbitrary vector bundles.

(86) The **torsion** of a connection $\nabla$ on $TM$ is defined by

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$$

for all $X, Y \in \mathcal{X}(M)$.

**Lemma 15.** *The torsion defines a skew-symmetric map*

$$T\colon \mathcal{X}(M)\times\mathcal{X}(M) \longrightarrow \mathcal{X}(M)$$

*that is bilinear over $C^\infty(M)$.*

A connection $\nabla$ is called **symmetric** if it is torsion-free, i. e. if $T$ vanishes identically[1].

(87) To explain why torsion-freeness is indeed a symmetry condition, we consider the expression of the connection in a local coordinate system $(x_1,\ldots,x_n)$ on $M$. We write $\partial_i$ for the coordinate vector fields $\frac{\partial}{\partial x_i}$, and use the local frame $\partial_1,\ldots,\partial_n$. Then

$$\nabla\partial_i = \sum_{j=1}^n \omega_{ij}\otimes\partial_j\ ,$$

and using (8) we obtain

$$\nabla_{\partial_i}\partial_j = \sum_{k=1}^n \omega^i_{jk}\partial_k\ ,$$

[1]Note that requiring the naive symmetry $\nabla_X Y = \nabla_Y X$ for all $X$ and $Y$ leads to a contradiction.

which is usually written as

$$\nabla_{\partial_i}\partial_j = \sum_{k=1}^{n} \Gamma_{ij}^k \partial_k$$

in classical notation. Therefore, we define the **Christoffel symbols** of the connection $\nabla$ with respect to the coordinate system $(y_1, \ldots, y_n)$ to be $\Gamma_{ij}^k = \omega_{jk}^i$.

Returning to the definition of torsion, we see that

$$T(\partial_i, \partial_j) = \sum_{k=1}^{n} (\omega_{jk}^i - \omega_{ik}^j)\partial_k = \sum_{k=1}^{n} (\Gamma_{ij}^k - \Gamma_{ji}^k)\partial_k \ .$$

As the torsion is linear over the smooth functions, we obtain the following:

**Lemma 16.** *An connection $\nabla$ on the tangent bundle is torsion-free if and only if $\Gamma_{ij}^k = \Gamma_{ji}^k$ for any local coordinate system.*

Thus symmetry of the connection really refers to a symmetry of the Christoffel symbols expressing this connection in local coordinates.

(88) If $E \to M$ is a vector bundle with a connection $\nabla$, then the dual bundle $E^* \to M$ carries a well-defined dual connection $\nabla^*$ characterized by the identity

$$d\langle s, \alpha\rangle = \langle \nabla s, \alpha\rangle + \langle s, \nabla^* \alpha\rangle$$

for all $s \in \Gamma(E)$ and $\alpha \in \Gamma(E^*)$. (The brackets here denote the natural pairing between a bundle and its dual bundle, not a metric.)

In the case of a connection on the tangent bundle, the dual connection $\nabla^*$ on $T^*M$ gives us the following characterization of torsion-freeness:

**Proposition 17.** *An connection $\nabla$ on $TM$ is torsion-free if and only if the exterior derivative on one-forms is given by the composition*

$$\Omega^1(M) = \Gamma(T^*M) \xrightarrow{\nabla^*} \Gamma(T^*M \otimes T^*M) \xrightarrow{\wedge} \Gamma(\Lambda^2 T^*M) = \Omega^2(M) \ .$$

(89) Consider a **Riemannian manifold** $M$, that is a smooth manifold with a metric on its tangent bundle. In this case it turns out that there is a unique symmetric connection that is at the same time compatible with the given metric:

**Proposition 18** (Fundamental Lemma of Riemannian Geometry)**.** *The tangent bundle of a Riemannian manifold admits a unique torsion-free connection compatible with the metric.*

*Proof.* Given vector fields $X$, $Y$ and $Z$, we can use the two requirements, compatibility with the metric $\langle\ ,\ \rangle$ and torsion-freeness, to conclude that the only possible value for $\langle \nabla_X Y, Z\rangle$ is

$$\langle \nabla_X Y, Z\rangle = \frac{1}{2}(\langle [X,Y], Z\rangle + \langle [Z,X], Y\rangle + \langle [Z,Y], X\rangle + L_X\langle Y, Z\rangle + L_Y\langle X, Z\rangle - L_Z\langle X, Y\rangle) \ . \tag{9}$$

This proves uniqueness. To see existence, we use (9) as a definition. As $\langle\ ,\ \rangle$ is non-degenerate, requiring that the equation hold for all $Z$ uniquely defines $\nabla_X Y$. We then check that this $\nabla$ is indeed a connection, is metric-compatible, and torsion-free. □

(90) The curvature of a Riemannian manifold $(M, \langle\ ,\ \rangle)$ is, by definition, the curvature of its **Levi-Civita connection** $\nabla$ given by the Fundamental Lemma of Riemannian Geometry. We write $R$ for the curvature $F^\nabla$ of $\nabla$, and consider this either as

$$\begin{aligned} R\colon \mathcal{X}(M) \times \mathcal{X}(M) \times \mathcal{X}(M) &\longrightarrow \mathcal{X}(M) \\ (X, Y, Z) &\longmapsto R(X, Y)Z\ , \end{aligned}$$

or as

$$\begin{aligned} R\colon \mathcal{X}(M) \times \mathcal{X}(M) \times \mathcal{X}(M) \times \mathcal{X}(M) &\longrightarrow C^\infty(M) \\ (X, Y, Z, T) &\longmapsto \langle R(X, Y)Z, T\rangle\ . \end{aligned}$$

The notation $R(X, Y)Z$ means that the curvature 2-form is evaluated on $X$ and $Y$, and the resulting endomorphism of $TM$ is applied to $Z$. Both of these incarnations of $R$ are called the **Riemann curvature tensor** of $(M, \langle\ ,\ \rangle)$; it is a tensor because it is function-linear in all arguments.

(91) The Riemann curvature tensor of a Riemannian manifold $(M, \langle\ ,\ \rangle)$ is skew-symmetric in $X$ and $Y$, and has the following additional symmetries:

**Lemma 19.** *For all $X, Y, Z, W \in \mathcal{X}(M)$ we have:*
(a) $R(X, Y)Z + R(Y, Z)X + R(Z, X)Y = 0$,
(b) $\langle R(X, Y)Z, W\rangle = -\langle R(X, Y)W, Z\rangle$,
(c) $\langle R(X, Y)Z, W\rangle = \langle R(Z, W)X, Y\rangle$.

(92) Let $X$ and $Y$ be two linearly independent tangent vectors in $T_pM$. Then the expression

$$K(X, Y) = \frac{\langle R(X, Y)Y, X\rangle}{\langle X, X\rangle\langle Y, Y\rangle - \langle X, Y\rangle^2} \tag{10}$$

only depends in the two-dimensional subspace $\sigma = span\{X, Y\} \subset T_pM$, and not on the basis $X$ and $Y$. This is called the **sectional curvature** of $\sigma$.

In the case when $M$ is 2-dimensional, so that $\sigma = T_pM$, the sectional curvature is the same as the Gaussian curvature.

(93) By definition, the sectional curvature is determined by the Riemann tensor. However, the converse is also true:

**Theorem 20.** *If two Riemannian metrics on $M$ have the same sectional curvatures for all tangent 2-planes, then their curvature tensors $R$ agree.*

The proof is just linear algebra, using Lemma 19 and multiple polarisation, arguing only with the trilinear map

$$R\colon T_pM \times T_pM \times T_pM \longrightarrow T_pM$$

at a point.

(94) The proof the Theorem can easily be adapted to prove the following characterization of spaces with constant sectional curvature:

**Proposition 21.** *A Riemannian manifold* $(M, \langle\ ,\ \rangle)$ *has sectional curvature equal to a fixed real number* $K_0 \in \mathbb{R}$ *for all two-planes* $\sigma \subset TM$ *if and only if the following identity holds for all* $X$, $Y$, $Z$ *and* $T \in \mathcal{X}(M)$*:*

$$\langle R(X,Y)Z, T\rangle = -K_0(\langle X, Z\rangle\langle Y, T\rangle - \langle Y, Z\rangle\langle X, T\rangle) \ .$$

The proof is again just linear algebra at a single point. Therefore, one could in theory replace the constant $K_0$ by a function on $M$, and require just that the sectional curvature is constant at every point, meaning that it coincides on any two tangent planes tangent at the same point. For connected manifolds of dimension $\geq 3$ it turns out that constancy at all points implies constancy throughout $M$ (Schur's theorem, which we do not prove); this is of course not true in dimension $2$.

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