Theoretical and Numerical Results for Electrorheological Fluids

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CHAPTER 1

Introduction

In this thesis we are working on the mathematical background of the behaviour of electrorheological fluids. Those fluids have a special property: When disposed to an electro–magnetic field their viscosity undergoes a significant change. For example, there exist modern electrorheological fluids which respond to the application of an electric field within 1 ms, their viscosity changing by a factor of 1000.

The first observations of electrorheological fluids were reported by Winslow in 1949 [**Win49**]. While first realizations of electrorheological fluids were quite unstable and had a highly abrasive structure preventing many possibility of application, this drawback has been overcome. Nowadays there exists electrorheological fluids with have the quality and potential for a wide field of applications. These include for example actuators, clutches, shock absorbers, and rehabilition equipment.

The aim of this thesis is to provide insight into the mathematics concerning electrorheological fluids. This includes theoretical results about existence and regularity of solutions as well as numerical stabilizations and their applications to discretization methods. Of course, every investigation of existence and regularity needs a groundwork on the suitable spaces, which are in our case not the classical Sobolev spaces but rather the generalized Orlicz–Sobolev spaces.

There exist several possibilities for modeling the physics of electrorheological fluids. In this thesis we will use a model originally proposed by Rajagopal, Růžička [**RR96**] and further developed by M. Růžička in [**Růž00**]. This model is derived from the general balance laws for mass, linear momentum, angular momentum, energy, the second law of thermodynamics in the form of the Clausius-Duhem inequality and Maxwell's equations in their Minkowskian form. Furthermore the interaction of the electro-magnetic field with the fluid is based on the "dipole current-loop" model (see Grot [**Gro76**] and Pao [**Pao78**]). The full model for an incompressible electrorheological fluid reads

$$\begin{aligned} \operatorname{div}(\mathbf{E}+\mathbf{P}) &= 0, \\ \operatorname{curl}\mathbf{E} &= \mathbf{0}, \\ \rho_0\partial_t\mathbf{u} - \operatorname{div}\mathbf{S} + \rho_0(\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u} + \boldsymbol{\nabla}\pi &= \rho_0\mathbf{f} + [\boldsymbol{\nabla}\mathbf{E}]\mathbf{P}, \\ \operatorname{div}\mathbf{u} &= 0, \end{aligned}$$

where $\mathbf{E}$ is the electric field, $\mathbf{P}$ the polarization, ρ_0 the constant density, $\mathbf{u}$ the velocity, $\mathbf{S}$ the extra stress, π the pressure, and $\mathbf{f}$ the mechanical force with

$$\begin{aligned} \mathbf{S} = \alpha_{21}\big((1+|\mathbf{D}|^2)^{\frac{p-1}{2}} - 1\big)\mathbf{E}\otimes\mathbf{E} + (\alpha_{31}+\alpha_{33}|\mathbf{E}|^2)(1+|\mathbf{D}|^2)^{\frac{p-2}{2}}\mathbf{D} \\ + \alpha_{51}(1+|\mathbf{D}|^2)^{\frac{p-2}{2}}(\mathbf{D}\mathbf{E}\otimes\mathbf{E} + \mathbf{E}\otimes\mathbf{D}\mathbf{E}). \end{aligned}$$

The α_{ij} are material constants and the exponent p depends on the strength of the electric field $|\mathbf{E}|^2$ and satisfies

$$1 < p_\infty \le p(|\mathbf{E}|^2) \le p_0 < \infty.$$

Fortunately the equations for the electro–magnetic field decouple from the equations for $\mathbf{u}$, π, and ρ_0. So we can consider the electric field $\mathbf{E}$ and the polarization $\mathbf{P}$ as given functions and restrict our study on the equations for $\mathbf{u}$, π, and ρ_0. We will further restrict ourselves to the case of constant density neglecting ρ_0. From a mathematical point of view it is of interest to study the simplified system

$$\begin{aligned} \partial_t \mathbf{u} - \operatorname{div}\big(\mathbf{S}(\mathbf{D}\mathbf{u})\big) + (\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u} + \boldsymbol{\nabla}\pi &= \mathbf{f}, \\ \operatorname{div}\mathbf{u} &= 0, \end{aligned} \tag{1.1}$$

with

$$\mathbf{S}(\mathbf{D}\mathbf{u}) = (1 + |\mathbf{D}\mathbf{u}|^2)^{\frac{p-2}{2}}\mathbf{D}\mathbf{u}$$

or an extra stress $\mathbf{S}$ with similar properties. This model is the center of the thesis and all of our investigation are directly connected to it.

From a mathematical point of view one of the first questions arising is the right setting of the used spaces. Let I denote the domain of time and Ω the domain of space, then the natural energy of the model is given by

$$\iint\limits_{I\ \Omega} |\mathbf{D}\mathbf{u}|^{p(x,t)}\,dx\,dt,$$

where $\mathbf{D}\mathbf{u} = \frac{1}{2}(\boldsymbol{\nabla}\mathbf{u} + (\boldsymbol{\nabla}\mathbf{u})^T)$ denotes the symmetric gradient. This energy cannot be expressed in terms of classical Lebesgue and Sobolev spaces and requires the use of generalized Orlicz–Lebesgue and generalized Orlicz–Sobolev spaces. Therefore we give an overview on these spaces in chapter 2. Unfortunately many of the standard results for classical Lebesgue L^q and Sobolev $W^{k,q}$ spaces cannot be transfered to the generalized Orlicz–Lebesgue $L^{p(\cdot)}$ and Orlicz–Sobolev spaces $W^{k,p(\cdot)}$. Some fundamental results do not hold in the generalized case and many questions remain open. To give an example, the translation operator is not continuous in the generalized Orlicz spaces. This is a hard drawback, since most of the standard results about Lebesgue and Sobolev spaces are proved with the help of translations. So we will show that the convolution operator in not continuous on the spaces $L^{p(\cdot)}$ unless we are in a trivial setting still covered by the classical Lebesgue spaces L^q. Nevertheless we will prove that convolution, although based on translations, is still a very useful tool. Indeed we will see that the mollification with an approximation of one is bounded in $L^{p(\cdot)}$ as long as p satisfies a rather weak continuity assumption, namely

$$\omega(R) \le \frac{C}{-\ln R}$$

for all $0 < R < 1$, where ω is the module of continuity of p. We will prove this by providing an even more fundamental result. We will show that the Hardy–Littlewood maximal function operator is bounded on $L^{p(\cdot)}$ under the same continuity assumption on p. Since the maximal function is one of the most important tools in harmonic analysis, this result is a milestone for the theory of generalized Orlicz–Lebesgue and Orlicz–Sobolev spaces. Further results based on the maximal function such as full characterization of Sobolev type embeddings $W^{1,p(\cdot)} \to L^{r(\cdot)}$ and investigations on singular operators are in preparation.

Then in chapter 3 we will discuss in detail the assumptions that we place on the extra stress $\mathbf{S}$ and the exponent p. Rather than restricting ourselves to the case

$$\mathbf{S}(\mathbf{Du}) = (1 + |\mathbf{Du}|^2)^{\frac{p-2}{2}}\mathbf{Du},$$

we will assume that $\mathbf{S}$ is induced by a space and time dependent potential Φ, which satisfies some convexity and growth conditions in term on p. Further we will introduce two stabilizations of the extra stress $\mathbf{S}$, namely the A– and the $\boldsymbol{\lambda}$ approximation $\mathbf{S}^A$ and $\mathbf{S}^{\boldsymbol{\lambda}}$. Roughly spoken, these stabilizations change the extra stress $\mathbf{S}(\mathbf{Du})$ such that it behaves for large $|\mathbf{Du}|$ almost linearly, i.e.

$$\begin{aligned}\sum_{ij}(\tilde{S}_{ij}(\mathbf{A})-\tilde{S}_{ij}(\mathbf{B}))(A_{ij}-B_{ij}) &\geq C(A,\boldsymbol{\lambda})\,|\mathbf{A}-\mathbf{B}|^2\,,\\ |\tilde{\mathbf{S}}(\mathbf{A})-\tilde{\mathbf{S}}(\mathbf{B})| &\leq C(A,\boldsymbol{\lambda})\,|\mathbf{A}-\mathbf{B}|,\end{aligned}$$

where $\tilde{\mathbf{S}}$ is either the A– or the $\boldsymbol{\lambda}$–approximation. Both stabilization behave very similar, but we prefer the A–approximation a little bit, since it only changes the $\mathbf{S}$ for large $|\mathbf{Du}|$. Later in chapter 4 and chapter 6, we will use the A–approximation for questions of regularity and numerical stabilization.

One of the problems when solving system (1.1) numerically, is the constraint $\operatorname{div}\mathbf{u} = 0$, which enforces the use of divergence free test functions or a coupling of the finite element spaces of the velocity and the pressure (BB-condition). One way to overcome this problem is the use of the pressure stabilization, which replaces $\operatorname{div}\mathbf{u} = 0$ by

$$\operatorname{div}\mathbf{u} = \varepsilon\Delta\pi$$

for some $\varepsilon > 0$. In chapter 4 we examine this type of stabilization. Especially we will consider the case of two space dimensions. We will show that there exists a solution $\mathbf{u}$ with Hölder continuous gradients, which is unique in the class of weak solutions. Based on this regularity we show that the error induced by the pressure stabilization is of optimal order ε.

In chapter 5 we will examine system (1.1) in the case of three space dimensions. Under the condition $p_\infty > \frac{7}{5}$ we will show that there exists a strong solution at least for small times. This improves a result of Málek, Nečas, Rokyta, and Růžička [**MNRR96**], who prove short time existence for p constant with $p > \frac{5}{3}$. Furthermore we will improve the regularity result for such short time solutions from

$$\|(\widetilde{D}\mathbf{u})^{\frac{p}{2}}\|_{C(I,L^3(\Omega))}$$

to

$$\|(\widetilde{D}\mathbf{u})^{\frac{p}{2}}\|_{C(I,L^{\frac{12(p_\infty-1)}{p_\infty},\frac{4(p_\infty-1)}{2-p_\infty}})} \leq C \qquad \text{(Lorentz space).}$$

The proof of this is based on an anisotropic interpolation result for parabolic systems, which is proven in the appendix in chapter 8. We will see that this regularity ensures uniqueness, within this class of regularity, exactly up to the bound $p_\infty > \frac{7}{5}$.

Based on the result of chapter 5 we will examine in chapter 6 the fully implicit and the semi implicit Euler time discretization. First results (with p constant) in this direction by A. Prohl and M. Růžička [**PR01**] have indicated that the implicit Euler time discretization without stabilization has only a guaranteed stability up to

$p_\infty > 1.677$. Strong solutions of the time discretized problem, which are needed for a later space discretization, are ensured only for $p_\infty > 1.8$. Since this condition on p_∞ is too restrictive for real fluids, we show how to extend this range up to $p_\infty > 1.588$, even for p non–constant. This improvement is achieved by means of the A–approximation. We will further improve the error estimates in several aspects and show that the results also hold true for the semi implicit Euler discretization.

As in every step of the time discretization the problems regarded are stationary, it is important to investigate the stationary p–Stokes system, i.e.

$$\begin{aligned} -\operatorname{div}\big(\mathbf{S}(\mathbf{Du})\big) + \nabla\pi &= \mathbf{f}, \\ \operatorname{div}\mathbf{u} &= 0. \end{aligned}$$

It is also of general interest to study this system for a better understanding of the interaction of the nonlinear main part (depending on the symmetric gradient only) with the pressure. Therefore we investigate this system in chapter 7. We will derive Meyer–type estimates for weak solutions. That is, a weak solution with energy estimate

$$|\mathbf{Du}|_{p(\cdot)} \le C$$

also satisfies

$$|\mathbf{Du}|_{(1+\delta)p(\cdot)} \le C$$

for some $\delta > 0$.

So far for now, let's go into detail...

CHAPTER 2

Generalized Lebesgue and Sobolev Spaces

When studying the motion of a fluid where the extra stress is induced by a space dependent p-potential (for a definition see chapter 3) one of the main problems is the natural setting of function spaces. The information gathered by the natural energy norm cannot be exactly described within the context of Lebesgue or Sobolev spaces. To be more explicit, let $\mathbf{u}$ be a weak solution of the stationary system

$$-\operatorname{div}(\mathbf{S}(\mathbf{Du})) = \mathbf{f}$$

on a smooth domain with zero boundary values, where $\mathbf{S}$ is induced by a space dependent p–potential. Then the energy norm of this system is naturally given by

$$\int_\Omega \mathbf{S}(\mathbf{Du}) \cdot \mathbf{Du} \, dx.$$

As we will see later, this can be estimated from below by

$$\int_\Omega |\mathbf{Du}|^{p(x)} \, dx,$$

where $p : \Omega \to [1, \infty)$ is a measurable function corresponding to the potential. But this information about $\mathbf{Du}$ cannot be fully qualified by a usual Lebesgue space. On this account we have to make use of generalized Lebesgue and Sobolev spaces. The aim of this chapter is to introduce these spaces, present the known theory, and to derive more fundamental results.

1. The Generalized Lebesgue Spaces $L^{p(\cdot)}(\Omega)$

We start with the definition of the generalized Lebesgue spaces, which have been studied by Hudzik [**Hud80**], Musielak [**Mus83**], Kováčik, Rákosník [**KR91**], Růžička [**Růž00**], and others. Further details and proofs of the statements in this section can be found in their publications.

Let $\Omega \subset \mathbb{R}^d$ be an open, bounded domain and let $p : \Omega \to [1, \infty)$ be measurable. For $I \subset \Omega$ we define

$$p_{0,I} := \operatorname{esssup}_{x \in I} p(x),$$
$$p_{\infty,I} := \operatorname{essinf}_{x \in I} p(x).$$

Further we set $p_0 := p_{0,\Omega}$ and $p_\infty := p_{\infty,\Omega}$. For simplicity we restrict ourselves to the case $p_0 < \infty$.

The notation p_∞ for the smallest value and p_0 for the biggest value is due to historical reasons and was introduced by Růžička. It reflects that in the case of electrorheological fluids the exponent p assumes its maximal value for a vanishing electrical field and its minimal value for an infinitely strong electrical field. Thus the indices represent the strength of the electrical field.

For p as above define $\varphi_p(x,z) := z^{p(x)}$ for all $x \in \Omega$ and all $z \geq 0$. Note that φ_p is of "class Φ" in the sense of Musielak [**Mus83**], i.e.

(a) $\varphi_p(x,z)$ is for every $x \in \Omega$ a nondecreasing, continuous function of $z \geq 0$, such that $\varphi_p(x,0) = 0$, $\varphi_p(x,z) > 0$ for $z > 0$, and $\varphi_p(x,z) \to \infty$ for $z \to \infty$.
(b) The function $\varphi_p(\cdot, z)$ is measurable for all $z \geq 0$.

Let X denote the space of all functions $f : \Omega \to \mathbb{R}$, which are measurable. For $f \in X$ we define the modular $|f|_{p(\cdot)}$ by

$$|f|_{p(\cdot)} := \int_\Omega \varphi_p(x, |f(x)|)\, dx = \int_\Omega |f(x)|^{p(x)}\, dx. \tag{2.1}$$

Then the set

$$L^{\varphi_p}(\Omega) := \{f \in X \,:\, |\lambda f|_{p(\cdot)} \to 0 \text{ for } \lambda \to 0^+\},$$

resp.

$$L_0^{\varphi_p}(\Omega) := \{f \in X \,:\, |f|_{p(\cdot)} < \infty\}.$$

defines the generalized Orlicz space, resp. the generalized Orlicz class. Further let

$$E_{\varphi_p} := \{f \in X \,:\, |\lambda f|_{p(\cdot)} < \infty \text{ for all } \lambda > 0\}.$$

Since $p_0 < \infty$ we know that φ_p satisfies the Δ_2–condition, i.e. there exists an integrable function $h : \Omega \to \mathbb{R}$ and a constant $K > 0$, such that for a.a. $x \in \Omega$ and all $z \geq 0$ there holds

$$\varphi_p(x, 2z) \leq K\, \varphi_p(x,z) + h(x).$$

Indeed $\varphi_p(x,2z) \leq 2^{p_0}\varphi_p(x,z)$. This implies that $L^{\varphi_p}(\Omega) = L_0^{\varphi_p}(\Omega) = E_{\varphi_p}$ (see [**Mus83**] theorem 8.13). So we do not have to distinguish between generalized Orlicz space and generalized Orlicz class. Therefore we introduce the notation $L^{p(\cdot)}(\Omega) := L^{\varphi_p}(\Omega)$. Note that the generalized Orlicz spaces are also called Musielak–Orlicz spaces. The functional defined by

$$\|f\|_{p(\cdot)} := \inf\big\{\lambda > 0 \,:\, |f/\lambda|_{p(\cdot)} \leq 1\big\}$$

is a norm on $L^{p(\cdot)}(\Omega)$, the Luxemburg norm.

It is quite common to use the notation $\|f\|_{p(x)}$ and $L^{p(x)}(\Omega)$ instead of $\|f\|_{p(\cdot)}$ and $L^{p(\cdot)}(\Omega)$. Nevertheless we will use this slightly differing notation in order to exclude the ambiguous case, where $\|f\|_{p(x)}$ denotes the norm of the classical Lebesgue space $L^q(\Omega)$ with $q = p(x)$ for a fixed $x \in \Omega$.

If p is constant, then $\|f\|_{p(\cdot)}$ coincides with the classical Lebesgue norm. Further if $1 \leq r < \infty$, then

$$\|f\|_{rp(\cdot)}^r = \big\||f|^r\big\|_{p(\cdot)}. \tag{2.2}$$

For $p_\infty > 1$ we define the dual exponent p' of p by $1 = \frac{1}{p(x)} + \frac{1}{p'(x)}$ for all $x \in \Omega$. Then the function $p' : \Omega \to (1,\infty)$ is measurable and satisfies $1 < (p')_\infty = (p_0)' \leq (p_\infty)' = (p')_0 < \infty$. Furthermore there holds the following generalization of the Hölder inequality:

$$|\langle f, g\rangle| \leq \big(1 + \tfrac{1}{p_\infty} - \tfrac{1}{p_0}\big)\|f\|_{p(\cdot)}\|g\|_{p'(\cdot)}. \tag{2.3}$$

More precisely, there holds $(L^{p(\cdot)}(\Omega))' = L^{p'(\cdot)}(\Omega)$, so $L^{p(\cdot)}(\Omega)$ is reflexive if $p_\infty > 1$. This enables us to introduce another norm, namely

$$\||f\||_{p(\cdot)} := \sup_{\|g\|_{p'(\cdot)} \le 1} |\langle f, g\rangle|. \tag{2.4}$$

This norm is equivalent to $\|\cdot\|_{p(\cdot)}$ if $p_\infty > 1$.

Closely connected to this Hölder inequality there is the following version of Young's inequality: Let $p : \Omega \to (1,\infty)$ be measurable with $1 < p_\infty$. Further let $f \in L^{p(\cdot)}(\Omega)$, $g \in L^{p'(\cdot)}(\Omega)$, then for all $0 < \delta < 1$ there holds

$$|\langle f, g\rangle| \le \frac{\delta^{p_\infty}}{p_\infty} \left|f\right|_{p(\cdot)} + \frac{\delta^{(p_0)'}}{(p_0)'} \left|g\right|_{p'(\cdot)}. \tag{2.5}$$

This is just a consequence of Young's inequality applied pointwise to $|f(x)g(x)|$ and the fact that $p \mapsto \frac{\delta^p}{p}$ is monotonously decreasing with respect to $p \ge 1$.

For $f \in L^{p(\cdot)}(\Omega)$ we have the following connection between the modular and the norm:

$$\|f\|_{p(\cdot)} \le 1 \quad \Leftrightarrow \quad |f|_{p(\cdot)} \le 1 \quad \Leftrightarrow \quad |f|_{p(\cdot)} \le \|f\|_{p(\cdot)}. \tag{2.6}$$

This shows that norm–convergence, i.e. $\|f_n - f\|_{p(\cdot)} \to 0$, implies modular–convergence, i.e. $|f_n - f|_{p(\cdot)} \to 0$. Moreover, the reverse holds true, so

$$|f_n - f|_{p(\cdot)} \to 0 \qquad \Leftrightarrow \qquad \|f_n - f\|_{p(\cdot)} \to 0. \tag{2.7}$$

Like classical Lebesgue spaces the spaces $L^{p(\cdot)}(\Omega)$ are complete.

Let $q : \Omega \to [1,\infty)$ be measurable with $q \le p$ a.e., then $L^{p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$ continuously and

$$\|f\|_{q(\cdot)} \le (1 + |\Omega|)\, \|f\|_{p(\cdot)}.$$

For $N \in \mathbb{N}$ let f_N be defined by

$$f_N(x) := \begin{cases} f(x), & \text{if } |f(x)| \le N, \\ \operatorname{sgn}(f(x))N, & \text{else.} \end{cases}$$

Then $f_N \to f$ in $L^{p(\cdot)}(\Omega)$. This proves that $L^\infty(\Omega)$ is dense in $L^{p(\cdot)}(\Omega)$. Even more is true: $C_0^\infty(\Omega)$ is a dense subset of $L^{p(\cdot)}(\Omega)$.

If $p_\infty > 1$, then φ_p is uniformly convex in the sense of Musielak (see [**Mus83**] definition 11.3 and remark 11.4), i.e. there exists a function δ mapping the interval $(0,1)$ into itself and a null set $A \in \Omega$, such that all $x \in \Omega \setminus A$, $z > 0$, $0 < a < 1$, and $0 \le b \le a$ there holds

$$\varphi_p\big(x, \tfrac{1+b}{2} z\big) \le (1 - \delta(a)) \frac{\varphi_p(x,z) + \varphi_p(x, bz)}{2}. \tag{2.8}$$

If p is constant then we can choose (see [**Mus83**] remark 11.4)

$$\delta(a) = 1 - 2^{-p+1}(1+a)^p(1+a^p)^{-1}.$$

If p is non–constant, then define

$$\begin{aligned} \delta(a) &:= \operatorname{essinf}_{x\in\Omega} \left(1 - 2^{-p(x)+1}(1+a)^{p(x)}(1+a^{p(x)})^{-1}\right) \\ &= 1 - \operatorname{esssup}_{x\in\Omega} \left(\left(\tfrac{1+a}{2}\right)^{p(x)} \tfrac{2}{1+a^{p(x)}} \right) \\ &= 1 - \left(\tfrac{1+a}{2}\right)^{p_\infty} \tfrac{2}{1+a^{p_\infty}} > 0. \end{aligned}$$

The uniform convexity of φ_p and the Δ_2–condition imply (see [**Mus83**] theorem 11.6) that $L^{p(\cdot)}(\Omega)$ is uniformly convex.

But φ_p has more nice properties: Let $\mathcal{M}(\Omega)$ denote the class of all functions of “class Φ” (see above) of the form

$$\varphi(x,z) = \sum_{i=1}^{n} \varphi_i(z)\chi_{\Omega_i}(x),$$

where χ_{Ω_i} denotes the characteristic function of the pairwise disjoint sets $\Omega_1,\dots,\Omega_n$ with $\Omega = \Omega_1 \cup \dots \cup \Omega_n$ and $\varphi_1,\dots,\varphi_n$ satisfy: $\varphi_i(z)$ is a nondecreasing, continuous function of $z \geq 0$, such that $\varphi_i(0) = 0$, $\varphi_i(z) > 0$ for $z > 0$, and $\varphi_i(z) \to \infty$ for $z \to \infty$.

A function φ belongs to $\mathcal{M}_1$ if and only if there exists a sequence $\varphi_k \in \mathcal{M}$, such that for all $z \geq 0$ and a.e. $x \in \Omega$ there holds $\varphi_k(x,z) \nearrow \varphi(x,z)$ as $k \to \infty$. We will see that $\varphi_p \in \mathcal{M}_1$: Since $p \in L^\infty(\Omega)$ there exist two sequences q_k and r_k of simple functions, i.e. finite linear combinations of indicator functions, such that $q_k \nearrow p$, $r_k \searrow p$, and a.e. there holds $|r_k - q_k| \leq \frac{1}{k}$. By definition of q_k and r_k we have $\varphi_{q_k}, \varphi_{r_k} \in \mathcal{M}$. Define

$$\varphi_{p,k}(x,z) := \min\{\varphi_{q_k}(x,z), \varphi_{r_k}(x,z)\} = \min\left\{z^{q_k(x)}, z^{r_k(x)}\right\}.$$

Since $\mathcal{M}$ is stable with respect to the minimum of pairs, there holds $\varphi_{p,k} \in \mathcal{M}$. Furthermore for all $z \geq 0$ and a.e. $x \in \Omega$ there holds $\varphi_{p,k}(x,z) \nearrow \varphi_p(x,z)$ as $k \to \infty$. Hence $\varphi_p \in \mathcal{M}_1$.

We need one more property of φ_p: A function φ of “class Φ” is an N–function if for a.e. $x \in \Omega$ there holds

$$\lim_{z\to 0^+} \frac{\varphi(x,z)}{z} = 0, \qquad \lim_{z\to\infty} \frac{\varphi(x,z)}{z} = \infty.$$

It is easy to see that if $p_\infty > 1$, then φ_p is an N–function.

Overall we have shown that if $p_\infty > 1$, then φ_p is both an N–function and from $\mathcal{M}_1$. For such functions there exists an interesting interpolation theorem: From theorem 14.16 of [**Mus83**] we immediately deduce

Lemma 2.1. *Let $p,q,r,s : \Omega \to (1,\infty)$ be measurable with $p \leq q$ a.e. Let $1 < p_\infty, q_\infty, r_\infty, s_\infty$, and $p_0, q_0, r_0, s_0 < \infty$. Let T be a linear operator defined on $L^{p(\cdot)}(\Omega) \cap L^{q(\cdot)}(\Omega)$ with values in $L^{r(\cdot)}(\Omega) + L^{s(\cdot)}(\Omega)$, which is continuous as a mapping from $L^{p(\cdot)}(\Omega)$ to $L^{r(\cdot)}(\Omega)$ and from $L^{q(\cdot)}(\Omega)$ to $L^{s(\cdot)}(\Omega)$, i.e.*

$$\|Tf\|_{p(\cdot)} \leq A_0 \|f\|_{r(\cdot)},$$
$$\|Tf\|_{q(\cdot)} \leq A_1 \|f\|_{s(\cdot)}.$$

For $0 < \theta < 1$ define $t,u : \Omega \to [1,\infty)$ by $\frac{1}{t} = \frac{1-\theta}{p} + \frac{\theta}{q}$ and $\frac{1}{u} = \frac{1-\theta}{r} + \frac{\theta}{s}$. Then T is also continuous as a mapping from $L^{t(\cdot)}(\Omega)$ to $L^{u(\cdot)}(\Omega)$ and

$$\|Tf\|_{t(\cdot)} \leq A_0^{1-\theta} A_1^{\theta} \|f\|_{u(\cdot)}.$$

Another interpolation result concerning generalized Lebesgue spaces can be found in [**KR91**].

Unfortunately the spaces $L^{p(\cdot)}(\Omega)$ with non–constant p have also some undesirable properties. Let for example p be continuous but non–constant. Then there exists a function $f \in L^{p(\cdot)}(\Omega)$, such that f is not $p(\cdot)$–mean continuous, i.e. there exists no constant $A > 0$, such that

$$\|\tau_h f\|_{p(\cdot)} \leq A\, \|f\|_{p(\cdot)}$$

for all h small enough, where τ_h is the translation operator defined by $(\tau_h f)(x) = f(x-h)$. Even worse, there exist a function $f \in L^{p(\cdot)}$ and a sequence $h_n \to 0$, such that $\tau_{h_n} f \notin L^{p(\cdot)}(\Omega)$. Since the translation operator plays a fundamental role in the context of Lebesgue and Sobolev spaces, its failure in the context of $L^{p(\cdot)}(\Omega)$ is a strong drawback.

Nevertheless we will see later in this chapter that it is possible to build up tools powerful enough to allow the construction of a quite strong theory with numerous results similar to the ones for standard Lebesgue and Sobolev spaces.

2. The Generalized Sobolev Spaces $W^{k,p(\cdot)}(\Omega)$

Let $k \in \mathbb{N}_0$, then the space $W^{k,p(\cdot)}(\Omega)$ is defined by

$$W^{k,p(\cdot)}(\Omega) := \{f : \Omega \to \mathbb{R} \,|\, f, \dots, f^{(k)} \in L^{p(\cdot)}(\Omega)\},$$

where the derivatives ($f^{(k)}$ is the k-th derivative) are taken in the sense of distributions. These spaces are called generalized Orlicz–Sobolev spaces. They have been studied by Hudzik [**Hud80**], Kováčik, Rákosník [**KR91**], Růžička [**Růž00**], and Edmunds, Rákosník [**ER00**], [**ER92**]. Under special requirements on p some results for the classical Sobolev spaces have been transferred to the generalized Orlicz–Sobolev spaces.

For example it has been shown in [**KR91**] that for uniformly continuous p the Sobolev embeddings hold with an ε–defect:

LEMMA 2.2. *Let $\Omega \subset \mathbb{R}^d$ be an open bounded domain and let $k < d$. Further let $p : \Omega \to [1, \infty)$ be uniformly continuous with $p < d/k$ on $\overline{\Omega}$, then for every ε with $0 < \varepsilon < k/(d-k)$ the embedding*

$$W^{k,p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)-\varepsilon}(\Omega)$$

is continuous, where $q : \Omega \to [1, \infty)$ is given by

$$\frac{1}{q} = \frac{1}{p} - \frac{k}{d}.$$

M. Růžička has proved (see [**Růž00**]) another interesting version of the Sobolev embedding theorem:

LEMMA 2.3. *Let $p : \Omega \to [1, \infty)$ be measurable with $p_0 < d$ and let the level sets $\Omega_q := \{x \in \Omega \,:\, p(x) < q\}$ have Lipschitz boundary. Moreover, let*

$$\int_{p_\infty}^{p_0} c(a)^{q^*} dq < \infty,$$

where $c(q)$ is the continuity constant of the embedding

$$W^{1,q}(\Omega_q) \hookrightarrow L^{q^*}(\Omega_q)$$

and $\frac{1}{q^*} = \frac{1}{q} - \frac{1}{d}$*. Then there exists* $A > 0$*, such that*

$$\int\limits_\Omega \frac{|f(x)|^{p^*(x)}}{\ln(2+|f(x)|)}\,dx \le \Big[1 + \Big(\int\limits_\Omega (|f(x)|^{p(x)} + |\nabla f(x)|^{p(x)})\,dx\Big)^{\frac{p_0^*}{p_0}}\Big]$$

holds for all $f \in W^{1,p(\cdot)}(\Omega)$*, where* $\frac{1}{p^*(x)} = \frac{1}{p(x)} - \frac{1}{d}$.

Later on in [**ER00**], it has been shown that if Ω has Lipschitz boundary and p is uniformly Lipschitz, then the Sobolev embeddings hold true as long as $1 \le p < d$ on $\overline{\Omega}$:

LEMMA 2.4. *Let* $\Omega \subset \mathbb{R}^d$ *be an open bounded set with Lipschitz boundary. Let* $p : \Omega \to [1,\infty)$ *be uniformly Lipschitz continuous with* $p_0 < d$*. Then there exists* $A > 0$*, such that*

$$\|f\|_{p^*(\cdot)} \le A\,\|f\|_{1,p(\cdot)}$$

holds for all $f \in W^{1,p(\cdot)}(\Omega)$*, where* $p^* : \Omega \to [1,\infty)$ *is defined by* $\frac{1}{p*} = \frac{1}{p} - \frac{1}{d}$.

If p satisfies a special cone–growth condition, which ensures that for every $x \in \Omega$ there exists a suitable cone C_x with corner in x, such that $p|_{C_x} \ge p(x)$, then the density of smooth functions in $W^{k,p(\cdot)}(\Omega)$ has been shown in [**ER92**]. Herein the authors used a special mollifying operator, which smoothes the function in the direction of the cones. Please note that if for example p is $C^1(\overline{\Omega})$ and has no stationary point, then p satisfies the cone–growth condition and $C^\infty(\Omega)$ is therefore dense in $W^{k,p(\cdot)}(\Omega)$. Later in this chapter we will show that the cone–growth condition can be replaced by a rather weak uniform continuity condition on p (weaker than uniform Hölder continuity) allowing the presence of stationary points for p, such that $C^\infty(\Omega)$ is still dense in $W^{k,p(\cdot)}(\Omega)$.

3. Discontinuity of Convolution

It is well known that for $1 \le r < \infty$ there holds

$$\|f * \varphi\|_r \le \|f\|_r \|\varphi\|_1$$

as long as $f \in L^r$ and $\varphi \in L^1$. Unfortunately this is not true if we replace L^r by $L^{p(\cdot)}$. Even more, the inequality stays wrong if we insert an arbitrary large, multiplicative constant on the right-hand side. This is the point of the following lemma

LEMMA 2.5. *Let* $\Omega \subset \mathbb{R}^d$ *be open and bounded. Further let* $p, q : \Omega \to (1,\infty)$ *be measurable with* $1 < p_\infty, q_\infty$ *and* $p_0, q_0 < \infty$*, such that there exist open balls* $B_p, B_q \subset \Omega$ *with* $p_{0,B_p} < q_{\infty,B_q}$*. Then there exists no constant* $A > 0$*, such that*

$$\|f * \varphi\|_{q(\cdot)} \le A\,\|f\|_{p(\cdot)} \|\varphi\|_1$$

holds for all $f \in L^{p(\cdot)}(\Omega)$ *and all* $\varphi \in C_0^\infty(\mathbb{R}^d)$.

PROOF. Due to the assumptions on p, q and B_p, B_q, there exist a function $f \in L^{p(\cdot)}(\Omega)$ and a translation $h \in \mathbb{R}^d$, such that $\tau_h f \notin L^{q(x)}(\Omega)$, where

$$(\tau_h f)(x) := \begin{cases} f(x-h) & \text{if } x-h \in \Omega, \\ 0 & \text{else.} \end{cases}$$

Since Ω is bounded, we have $L^{p(\cdot)}(\Omega) \hookrightarrow L^1(\Omega)$, so f and $\tau_h f$ are also bounded in $L^1(\Omega)$. Now let $\varphi \in C_0^\infty(\mathbb{R}^d)$ be a Friedrich's mollifier and define for $\varepsilon > 0$

$$\varphi_\varepsilon(x) := \varepsilon^{-d}\varphi\big(\tfrac{x-h}{\varepsilon}\big),$$

then $f * \varphi_\varepsilon \to \tau_h f$ in $L^1(\Omega)$ as $\varepsilon \to 0^+$. Assume now that there exists a constant $A > 0$ with the desired properties, then

$$\|f * \varphi_\varepsilon\|_{q(\cdot)} \leq A\,\|f\|_{p(\cdot)}\|\varphi_\varepsilon\|_1 \leq A\,\|f\|_{p(\cdot)}.$$

Since $1 < q_\infty \leq q(x) \leq q_0 < \infty$, the space $L^{q(x)}(\Omega)$ is reflexive, so $f * \varphi_\varepsilon$ has a subsequence which converges weakly in $L^{q(\cdot)}(\Omega)$ to a function $g \in L^{q(x)}(\Omega)$. But the weak limit is unique, i.e. the $L^1(\Omega)$ limit and the weak $L^{p(\cdot)}(\Omega)$ limit have to agree, so $g = \tau_h f$. But this is a contradiction to $\tau_h f \notin L^{q(x)}(\Omega)$. □

Let us explain the consequences of this lemma a bit more. Let for example $p : \Omega \to [1,\infty)$ be smooth and bounded, but not locally constant, then we can apply the lemma with $q = p$ to conclude that the convolution is not continuous as a function from $L^{p(\cdot)}(\Omega) \times L^1(\Omega)$ to $L^{p(\cdot)}(\Omega)$. Even more, there is no continuity with an ε-defect as is often found within the context of $L^{p(\cdot)}(\Omega)$ spaces, i.e. there is no continuity as a function from $L^{p(\cdot)}(\Omega) \times L^1(\Omega)$ to $L^{p(\cdot)-\varepsilon}(\Omega)$ for all $\varepsilon > 0$ small enough. To see this, apply lemma 2.5 to $q = p - \varepsilon$ with $\varepsilon > 0$ small enough. Again we retrieve failure of continuity.

All this is a hard drawback and convolution seems not to be useful on $L^{p(\cdot)}(\Omega)$ spaces at all. But this is not true. Later in this chapter we will see that under some uniform continuity condition on p (weaker than uniform Hölder continuity), we still get the convergence of the convolution with a mollifying sequence in the following sense. Let φ be a suitable mollifier (see theorem 2.11 for details). Define $\varphi_\varepsilon(x) := \varepsilon^{-d}\varphi(x/\varepsilon)$ as usual, then $f * \varphi_\varepsilon \to f$ in $L^{p(\cdot)}(\Omega)$ as $\varepsilon \to 0^+$. Let us now state the continuity condition on p which will be needed later. Hereby we use the following notation: For a measurable set $A \subset \mathbb{R}^d$ let $|A|$ denote the Lebesgue measure of A.

4. A Condition on the Exponent

LEMMA 2.6. *Let $\Omega \subset \mathbb{R}^d$ be open and let $p : \Omega \to [1,\infty)$ be uniformly continuous (and thus bounded). Then the following conditions are equivalent:*

(i) There exists a constant C_0, such that for all $x, y \in \Omega$, $|x-y| < \frac{1}{2}$, there holds

$$|p(x) - p(y)| \leq \frac{C_0}{-\ln|x-y|}.$$

(ii) There exists a constant C_1, such that for all open balls $I \subset \mathbb{R}^d$ with $|\Omega \cap I| > 0$, there holds

$$|I|^{p_{\infty,I} - p_{0,I}} \leq C_1.$$

PROOF. Assume that (ii) holds. Let $x, y \in \Omega$ with $|x-y| < \frac{1}{2}$, and let $I \subset \mathbb{R}^d$ denote an open ball with $x, y \in I$ and $\operatorname{diam} I \leq 2|x-y| < 1$. Since Ω is open, we have $|\Omega \cap I| > 0$, so

$$|I|^{p_{\infty,I} - p_{0,I}} \leq C_1.$$

Since $|I| \le \operatorname{diam}(I)^d \le (2|x-y|)^d$, we have

$$\big((2|x-y|)^d\big)^{-|p(x)-p(y)|} \le |I|^{p_{\infty,I}-p_{0,I}} \le C_1$$

and

$$\begin{aligned}|x-y|^{-|p(x)-p(y)|} &\le C_1^{\frac{1}{d}} 2^{|p(x)-p(y)|}\\ &\le C_1^{\frac{1}{d}} 2^{p_0-p_\infty}.\end{aligned}$$

We take the logarithm of this inequality to deduce

$$|p(x)-p(y)| \le \frac{\ln\big(C_1^{\frac{1}{d}} 2^{p_0-p_\infty}\big)}{-\ln|x-y|}.$$

This proves that (ii) implies (i).

Assume now that (i) holds. Let $I \subset \mathbb{R}^d$ be an open ball with $|\Omega \cap I| > 0$, then $1 \le p_\infty \le p_{\infty,I} \le p_{0,I} \le p_0 < \infty$. If $\operatorname{diam}(I) \ge \frac{1}{2}$, then

$$|I|^{p_{\infty,I}-p_{0,I}} = \Big(|B_1(0)|\big(\tfrac{\operatorname{diam}(I)}{2}\big)^d\Big)^{p_{\infty,I}-p_{0,I}} \le \big(|B_1(0)|\tfrac{1}{4^d}\big)^{p_\infty-p_0},$$

therefore we can restrict ourselves to the case $\operatorname{diam}(I) < \frac{1}{2}$. Choose $x_0, x_\infty \in I \cap \Omega$, such that $0 \le \frac{1}{2}(p_{0,I}-p_{\infty,I}) \le p(x_0)-p(x_\infty)$. Since $\operatorname{diam}(I) < \frac{1}{2}$, we have $|x_0-x_\infty| < \frac{1}{2}$, so by assumption on p

$$|p(x_0)-p(x_\infty)| \le \frac{C_0}{-\ln|x_0-x_\infty|},$$

so

$$\exp(C_0) \ge |x_0-x_\infty|^{-|p(x_0)-p(x_\infty)|} \ge |x_0-x_\infty|^{\frac{1}{2}(p_{\infty,I}-p_{0,I})}.$$

Since $|I| \ge |x_0-x_\infty|^d |B_1(0)|$, we get

$$\exp(2C_0) \ge |x_0-x_\infty|^{p_{\infty,I}-p_{0,I}} \ge \Big(\big(\tfrac{|I|}{|B_1(0)|}\big)^{\frac{1}{d}}\Big)^{p_{\infty,I}-p_{0,I}}.$$

Hence

$$\begin{aligned}|I|^{p_{\infty,I}-p_{0,I}} &\le \exp(2dC_0)|B_1(0)|^{p_{\infty,I}-p_{0,I}}\\ &\le \exp(2dC_0)\max\big\{1, |B_1(0)|^{p_\infty-p_0}\big\}.\end{aligned}$$

This proves that (i) implies (ii). □

Corollary 2.7. *Let $\Omega \subset \mathbb{R}^d$ be open and bounded. Further let $p : \Omega \to [1,\infty)$ be uniformly Hölder continuous with index $\alpha > 0$, i.e.*

$$|p(x)-p(y)| \le H\,|x-y|^\alpha$$

for all $|x-y| < \frac{1}{2}$. Then p satisfies the conditions of lemma 2.6.

Proof. Let $\alpha > 0$, then there exists a constant $A > 0$, such that

$$|x-y|^\alpha \le \frac{A}{-\ln|x-y|}$$

for all $|x-y| < \frac{1}{2}$. This shows that uniform Hölder continuity is stronger than the continuity condition (i) of lemma 2.6. □

5. Hardy–Littlewood Maximal Function

Definition 2.8. *Let $\Omega \subset \mathbb{R}^d$ be open. For $f \in L^1(\Omega)$ and $r > 0$ we define $M_{(r)}(f) : \mathbb{R}^d \to \mathbb{R}^{\geq 0}$ and $M : \mathbb{R}^d \to \mathbb{R}^{\geq 0}$ by*

$$M_{(r)}(f)(x) := \frac{1}{|B_r(x)|} \int\limits_{B_r(x)\cap\Omega} |f(y)|\, dy,$$
$$M(f)(x) := \sup_{r>0} M_{(r)}(f)(x).$$

The function $M(f)$ is called the **(Hardy–Littlewood) maximal function** *of f.*

Lemma 2.9. *Let $\Omega \subset \mathbb{R}^d$ be open and let $p : \Omega \to [1, \infty)$ satisfy the conditions of lemma 2.6. Then there exists a constant $C = C(p)$, such that for all $f \in L^{p(\cdot)}(\Omega)$ with $\|f\|_{p(\cdot)} \leq 1$ there holds*

$$\begin{aligned} |M_{(r)}(f)(x)|^{p(x)} &\leq C(p)\Big(M_{(r)}\big(|f(\cdot)|^{p(\cdot)}\big)(x) + 1\Big), \quad \textit{for all } r > 0, \\ |M(f)(x)|^{p(x)} &\leq C(p)\Big(M\big(|f(\cdot)|^{p(\cdot)}\big)(x) + 1\Big). \end{aligned} \tag{2.9}$$

Furthermore all terms involved are finite.

Proof. The proof will be divided into two cases, namely $r \geq \frac{1}{2}$ and $0 < r < \frac{1}{2}$. Let us start with $r \geq \frac{1}{2}$, then

$$\begin{aligned} |M_{(r)}(f)(x)|^{p(x)} &= \Big(\frac{1}{|B_r(x)|} \int\limits_{B_r(x)\cap\Omega} |f(y)|\, dy\Big)^{p(x)} \\ &\leq \Big(\frac{1}{|B_r(x)|} \int\limits_{B_r(x)\cap\Omega} |f(y)|^{p(y)} + 1\, dy\Big)^{p(x)} \\ &\leq \Big(\frac{r^{-d}}{|B_1(0)|} |f|_{p(\cdot)} + 1\Big)^{p(x)} \\ &\leq \Big(\frac{2^d}{|B_1(0)|} + 1\Big)^{p_0}. \end{aligned}$$

Now assume that $0 < r < \frac{1}{2}$.

$$
\begin{aligned}
&|M_{(r)}(f)(x)|^{p(x)} \\
&\quad = \Big(\frac{1}{|B_r(x)|}\int\limits_{B_r(x)\cap\Omega} |f(y)|\,dy\Big)^{p(x)} \\
&\quad \overset{\text{by Jensen}}{\leq} \Big(\frac{1}{|B_r(x)|}\int\limits_{B_r(x)\cap\Omega} |f(y)|^{p_{\infty,B_r(x)}}\,dy\Big)^{\frac{p(x)}{p_{\infty,B_r(x)}}} \\
&\quad \leq \Big(\frac{1}{|B_r(x)|}\int\limits_{B_r(x)\cap\Omega} |f(y)|^{p(y)} + 1\,dy\Big)^{\frac{p(x)}{p_{\infty,B_r(x)}}} \\
&\quad = |B_r(x)|^{-\frac{p(x)}{p_{\infty,B_r(x)}}}\Big(\int\limits_{B_r(x)\cap\Omega} |f(y)|^{p(y)}\,dy + |B_r(x)|\Big)^{\frac{p(x)}{p_{\infty,B_r(x)}}} \\
&\quad \leq |B_r(x)|^{-\frac{p(x)}{p_{\infty,B_r(x)}}}\,2^{\frac{p_0}{p_\infty}}\Big(\tfrac{1}{2}\int\limits_{B_r(x)\cap\Omega} |f(y)|^{p(y)}\,dy + \tfrac{1}{2}|B_r(x)|\Big)^{\frac{p(x)}{p_{\infty,B_r(x)}}}.
\end{aligned}
$$

Since $|f|_{p(\cdot)} \leq 1$ and $0 < r < \frac{1}{2}$, there holds

$$
\tfrac{1}{2}\int\limits_{B_r(x)\cap\Omega} |f(y)|^{p(y)}\,dy + \tfrac{1}{2}|B_r(x)| \leq \tfrac{1}{2}|f|_{p(\cdot)} + \tfrac{1}{2}(2r)^d < \tfrac{1}{2} + \tfrac{1}{2} = 1,
$$

so

$$
\begin{aligned}
&|M_{(r)}(f)(x)|^{p(x)} \\
&\quad \leq |B_r(x)|^{-\frac{p(x)}{p_{\infty,B_r(x)}}}\,2^{\frac{p_0}{p_\infty}}\Big(\tfrac{1}{2}\int\limits_{B_r(x)\cap\Omega} |f(y)|^{p(y)}\,dy + \tfrac{1}{2}|B_r(x)|\Big) \\
&\quad = |B_r(x)|^{\frac{p_{\infty,B_r(x)}-p(x)}{p_{\infty,B_r(x)}}}\,2^{\frac{p_0}{p_\infty}-1}\Big(\frac{1}{|B_r(x)|}\int\limits_{B_r(x)\cap\Omega} |f(y)|^{p(y)}\,dy + 1\Big) \\
&\quad \leq |B_r(x)|^{\frac{p_{\infty,B_r(x)}-p_{0,B_r(x)}}{p_{\infty,B_r(x)}}}\,2^{\frac{p_0}{p_\infty}-1}\Big(\frac{1}{|B_r(x)|}\int\limits_{B_r(x)\cap\Omega} |f(y)|^{p(y)}\,dy + 1\Big).
\end{aligned}
$$

So lemma 2.6 implies

$$
\begin{aligned}
|M_{(r)}(f)(x)|^{p(x)} &\leq C(p)\Big(\frac{1}{|B_r(x)|}\int\limits_{B_r(x)\cap\Omega} |f(y)|^{p(y)}\,dy + 1\Big) \\
&= C(p)\Big(M_{(r)}\big(|f(\cdot)|^{p(\cdot)}\big)(x) + 1\Big).
\end{aligned}
$$

where $C(p)$ does not depend on r. Combining the cases $r \geq \frac{1}{2}$ and $0 < r < \frac{1}{2}$ we have

$$
|M_{(r)}(f)(x)|^{p(x)} \leq C(p)\Big(M_{(r)}\big(|f(\cdot)|^{p(\cdot)}\big)(x) + 1\Big).
$$

Taking the supremum over all $r > 0$ finishes the proof of (2.9). Furthermore for $f \in L^{p(\cdot)}(\Omega)$ we have $|f(\cdot)|^{p(\cdot)} \in L^1(\Omega)$, so the theory of maximal functions on $L^1(\Omega)$

ensures that $M(|f(\cdot)|^{p(\cdot)})(x)$ is finite for a.e. $x \in \Omega$. Hence all the terms in (2.9) are finite. $\square$

THEOREM 2.10. *Let Ω be open and bounded and let $p : \Omega \to [1, \infty)$ be measurable.*

(i) If $f \in L^{p(\cdot)}(\Omega)$ with $1 \le p(x) \le p_0 < \infty$ on Ω, then $M(f)$ is finite almost everywhere on $\mathbb{R}^d$.

(ii) Let p satisfy the conditions of lemma 2.6 and $1 < p_\infty \le p(x) \le p_0 < \infty$ on Ω. Then there exists a constant $C(\Omega, p) > 0$, such that for all $f \in L^{p(\cdot)}(\Omega)$ there holds

$$\|Mf\|_{p(\cdot)} \le C(\Omega, p)\|f\|_{p(\cdot)}. \tag{2.10}$$

Note that the norm $\|\cdot\|_{p(\cdot)}$ measures only the behaviour of $M(f)$ on Ω.

PROOF.

ad (i): Let $f \in L^{p(\cdot)}(\Omega)$, then $f \in L^1(\Omega)$, since Ω is bounded. Therefore the result follows from the theory for constant p, i.e. $p \equiv 1$.

ad (ii): Let $q(x) := p(x)/p_\infty$, so $1 \le q(x) \le p(x) \le p_0 < \infty$. Since Ω is bounded, there exists a constant $A > 0$ such that $\|f\|_{q(\cdot)} \le A\|f\|_{p(\cdot)}$ for all $f \in L^{p(\cdot)}(\Omega)$. Now let $f \in L^{p(\cdot)}(\Omega)$ with $\|f\|_{p(\cdot)} \le 1/A$ be arbitrary, then $\|f\|_{q(x)} \le 1$. We will show that $|M(f)|_{q(x)}$ is bounded independently of the choice of f. Since q satisfies the conditions of lemma 2.6 and $\|f\|_{q(x)} \le 1$, we can apply lemma 2.9 to get

$$\begin{aligned}
|M(f)|_{p(\cdot)} &= \big\|(M(f))^q\big\|^{p_\infty}_{L^{p_\infty}(\Omega)} \\
&\le \big\|C(p)\big(M(|f(\cdot)^{q(\cdot)}|) + 1\big)\big\|^{p_\infty}_{L^{p_\infty}(\Omega)} \\
&\le C(p)^{p_\infty}\Big(\big\|M(|f(\cdot)^{q(\cdot)}|)\big\|_{L^{p_\infty}(\Omega)} + \|1\|_{L^{p_\infty}(\Omega)}\Big)^{p_\infty}.
\end{aligned}$$

The theory of the maximal function with exponent $p_\infty > 1$ ensures

$$\begin{aligned}
|M(f)|_{p(\cdot)} &\le C(p)^{p_\infty}\Big(C(p_\infty)\big\||f(\cdot)^{q(\cdot)}|\big\|_{L^{p_\infty}(\Omega)} + \|1\|_{L^{p_\infty}(\Omega)}\Big)^{p_\infty} \\
&= C(p)^{p_\infty}\Big(C(p_\infty)\big|f\big|^{\frac{1}{p_\infty}}_{p(\cdot)} + \|1\|_{L^{p_\infty}(\Omega)}\Big)^{p_\infty} \\
&\le C(\Omega, p).
\end{aligned}$$

So $|M(f)|_{p(\cdot)}$, and thus $\|M(f)\|_{p(\cdot)}$, are bounded independently of f with $\|f\|_{p(\cdot)} \le 1/A$. Since $M(\cdot)$ and $\|\cdot\|_{p(\cdot)}$ are homogeneous with respect to positive scalars, i.e. $M(\lambda f) = |\lambda| M(f)$ and $\|\lambda f\|_{p(\cdot)} = |\lambda|\|f\|_{p(\cdot)}$, we see that

$$\|M(f)\|_{p(\cdot)} = A\|f\|_{p(\cdot)}\Big\|M\Big(\frac{f}{A\|f\|_{p(\cdot)}}\Big)\Big\|_{p(\cdot)} \le A\|f\|_{p(\cdot)}C(\Omega, p).$$

This proves the desired result.

$\square$

6. Convolution

We have already seen in section 3 that the convolution, although continuous as a function from $L^q(\Omega) \times L^1(\Omega) \to L^q(\Omega)$ for all constants $1 \le q < \infty$, is not continuous if $L^q(\Omega)$ is replaced by $L^{p(\cdot)}(\Omega)$. As we have seen in lemma 2.5 the situation is even worse. So the content of the following theorem is rather surprising.

THEOREM 2.11. *Let $\Omega \subset \mathbb{R}^d$ be open and bounded. Further let $p : \Omega \to [1, \infty)$ satisfy the conditions of lemma 2.6. Let $\varphi : \mathbb{R}^d \to \mathbb{R}$ be an integrable function and set $\varphi_\varepsilon(x) = \varepsilon^{-n}\varphi(x/\varepsilon)$ for all $\varepsilon > 0$. Suppose that the least decreasing radial majorant of φ is integrable, i.e. let $\psi(x) = \sup_{|y|\geq|x|}|\varphi(y)|$ then $\int_{\mathbb{R}^d} \psi(x)\,dx = A < \infty$. Then with the same A*

(i) $\sup_{\varepsilon>0}|(f * \varphi_\varepsilon)(x)| \leq A\,M(f)(x)$ *for all* $f \in L^{p(\cdot)}(\Omega)$.
(ii) If in addition $\int_{\mathbb{R}^d} \varphi(x)\,dx = 1$, *then* $\lim_{\varepsilon\to 0^+}(f*\varphi_\varepsilon)(x) = f(x)$ *almost everywhere in* Ω *for all* $f \in L^{p(\cdot)}(\Omega)$.
(iii) For all $f \in L^{p(\cdot)}(\Omega)$ *there holds* $f * \varphi_\varepsilon \to f$ *in* $L^{p(\cdot)}(\Omega)$ *as* $\varepsilon \to 0^+$.
(iv) For all $f \in L^{p(\cdot)}(\Omega)$ *there holds (independently of* $\varepsilon > 0$*)*

$$\|f * \varphi_\varepsilon\|_{p(\cdot)} \leq C(A, \Omega, p)\|M(f)\|_{p(\cdot)} \leq C(A, \Omega, p)\|f\|_{p(\cdot)}.$$

PROOF. Since Ω is bounded, we have $L^{p(\cdot)}(\Omega) \hookrightarrow L^1(\Omega)$. So (i) and (ii) follow immediately from theorem 2 page 62 of [**Ste70**]. To prove (iii) let $f \in L^{p(\cdot)}(\Omega)$. Using (i) we estimate for $x \in \Omega$

$$\begin{aligned} |(f * \varphi_\varepsilon)(x) - f(x)|^{p(x)} &\leq C(p)\big(|(f * \varphi_\varepsilon)(x)| + |f(x)|\big)^{p(x)} \\ &\leq C(p)\big(AM(f)(x) + |f(x)|\big)^{p(x)}, \end{aligned} \tag{2.11}$$

where the right-hand side is due to theorem 2.10 a $L^1(\Omega)$ function. Hence with (ii) and the theorem of dominated convergence we get

$$\begin{aligned} \lim_{\varepsilon\to 0^+} |f * \varphi_\varepsilon - f|_{p(\cdot)} &= \lim_{\varepsilon\to 0^+} \int_\Omega |(f * \varphi_\varepsilon)(x) - f(x)|^{p(x)}\,dx \\ &= \int_\Omega \lim_{\varepsilon\to 0^+} |(f * \varphi_\varepsilon)(x) - f(x)|^{p(x)}\,dx = 0. \end{aligned}$$

So we have convergence in the modular, which implies convergence in the norm. This proves $\|f * \varphi_\varepsilon - f\|_{p(\cdot)} \to 0$ as $\varepsilon \to 0^+$. The remaining property (iv) is an immediate consequence of (i), theorem 2.10, and the fact that $|f| \leq |g|$ implies $\|f\|_{p(\cdot)} \leq \|g\|_{p(\cdot)}$. □

CHAPTER 3

The Potential and the Extra Stress

Earlier we have stated that the extra stress tensor $\mathbf{S}$ is induced by a potential. Thus properties of the potential can be transferred to properties of the extra stress. In this section we will give the exact requirements for the potentials we are looking at. This is done either for the case where the potential does only depend on the absolute value of the symmetric gradient, and for the case where the potential is additionally time and space dependent. Afterwards we will give some examples of potentials satisfying these requirements. We continue by deriving useful properties of the extra stress and other partial derivatives of the potential. In this context we will introduce the dual viscosity, which is connected to the error of the extra stress and appears quite naturally in the dual problem of the error equation.

Since we are dealing with functions from $\Omega \times \mathbb{R}^{n\times n}$ to $\mathbb{R}$, we will distinguish the partial derivatives by ∂_i and ∂_{jk}. The single index means a partial derivative with respect to the i-th space coordinate. The double index represents a partial derivative with respect to the (j,k)-component of the underlying space of $n \times n$-matrices. By $\boldsymbol{\nabla}$ we denote the space gradient, while $\boldsymbol{\nabla}_{n\times n}$ denotes the matrix consisting of the partial derivatives with respect to the space of matrices. In a few cases we use d_i instead of ∂_i to indicate a total derivative. Note that by $\mathbf{B}^{\mathrm{sym}}$ we denote the symmetric part of a matrix $\mathbf{B} \in \mathbb{R}^{n\times n}$, i.e. $\mathbf{B}^{\mathrm{sym}} = \frac{1}{2}(\mathbf{B} + \mathbf{B}^T)$. Further let $\mathbb{R}^{n\times n}_{\mathrm{sym}}$ be the subspace of $\mathbb{R}^{n\times n}$ consisting of the symmetric matrices. Moreover we use C as a constant which is generic but does not depend on the ellipticity constants.

1. The Potential

Let us first consider the case where the extra stress only depends on the absolute value of the symmetric gradient, so we have no further space dependency.

Definition 3.1. *Let $1 < p \leq 2$ and let $F : \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$ be a convex function, which is C^2 on $\mathbb{R}^{\geq 0}$, such that $F(0) = 0$, $F'(0) = 0$, and the induced function $\Phi : \mathbb{R}^{n\times n} \to \mathbb{R}^{\geq 0}$, defined through $\Phi(\mathbf{B}) = F(|\mathbf{B}^{\mathrm{sym}}|)$, satisfies*

$$\sum_{jklm} (\partial_{jk}\partial_{lm}\Phi)(\mathbf{B}) C_{jk} C_{lm} \geq \gamma_1 (1 + |\mathbf{B}^{\mathrm{sym}}|^2)^{\frac{p-2}{2}} |\mathbf{C}^{\mathrm{sym}}|^2, \tag{3.1}$$

$$\left|(\boldsymbol{\nabla}^2_{n\times n}\Phi)(\mathbf{B})\right| \leq \gamma_2 (1 + |\mathbf{B}^{\mathrm{sym}}|^2)^{\frac{p-2}{2}} \tag{3.2}$$

for all $\mathbf{B}, \mathbf{C} \in \mathbb{R}^{n\times n}$ with constants $\gamma_1, \gamma_2 > 0$. Such a function F, resp. Φ, is called a p**-potential** *and the corresponding constants γ_1, γ_2 are called the ellipticity constants of F, resp. Φ.*

Remark 3.2. *Observe that for all $\mathbf{B} \in \mathbb{R}^{n\times n} \setminus \{\mathbf{0}\}$*

$$(\partial_{jk}\Phi)(\mathbf{B}) = F'(|\mathbf{B}^{\mathrm{sym}}|) \frac{B^{\mathrm{sym}}_{jk}}{|\mathbf{B}^{\mathrm{sym}}|},$$

$$(\partial_{jk}\partial_{lm}\Phi)(\mathbf{B}) = F'(|\mathbf{B}^{\mathrm{sym}}|) \Big(\frac{\delta^{\mathrm{sym}}_{jk,lm}}{|\mathbf{B}^{\mathrm{sym}}|} - \frac{B^{\mathrm{sym}}_{jk} B^{\mathrm{sym}}_{lm}}{|\mathbf{B}^{\mathrm{sym}}|^3} \Big) + F''(|\mathbf{B}^{\mathrm{sym}}|) \frac{B^{\mathrm{sym}}_{jk}}{|\mathbf{B}^{\mathrm{sym}}|} \frac{B^{\mathrm{sym}}_{lm}}{|\mathbf{B}^{\mathrm{sym}}|},$$

where $\delta^{\text{sym}}_{jk,lm} := \frac{1}{2}(\delta_{jl}\delta_{km} + \delta_{jm}\delta_{kl})$. *Hence*

$$\sum_{jklm}(\partial_{jk}\partial_{lm}\Phi)(\mathbf{B})B_{jk}B_{lm} = F''(|\mathbf{B}^{\text{sym}}|)|\mathbf{B}^{\text{sym}}|^2.$$

So by (3.1) *and* (3.2) *we conclude that for all* $\mathbf{B} \in \mathbb{R}^{n\times n} \setminus \{\mathbf{0}\}$

$$\gamma_1(1 + |\mathbf{B}^{\text{sym}}|^2)^{\frac{p-2}{2}} \leq F''(|\mathbf{B}^{\text{sym}}|) \leq \gamma_2(1 + |\mathbf{B}^{\text{sym}}|^2)^{\frac{p-2}{2}}. \tag{3.3}$$

Since $F'' \in C^2(\mathbb{R}^{\geq 0})$, *this estimate also holds for* $\mathbf{B} = \mathbf{0}$. *From the formula above for* $\partial_{jk}\Phi(\mathbf{B})$, *the continuity of* F' *at zero with* $F'(0) = 0$, *and the boundedness of* $B^{\text{sym}}_{jk}/|\mathbf{B}^{\text{sym}}|$ *in* $\mathbb{R}^{n\times n} \setminus \{\mathbf{0}\}$, *we deduce*

$$(\partial_{jk}\Phi)(\mathbf{0}) = \mathbf{0}.$$

Remark 3.3. *Let* $\mathbf{B}, \mathbf{C} \in \mathbb{R}^{n\times n}$. *Due to* $\Phi(\mathbf{B}) = F(|\mathbf{B}^{\text{sym}}|)$, *we have* $\Phi(\mathbf{B}) = \Phi(\mathbf{B}^{\text{sym}})$, *thus the* $\partial_{jk}\partial_{lm}\Phi$ *are symmetric in* j, k *and* l, m *and* $(j, k), (l, m)$. *This implies that*

$$\sum_{jklm}(\partial_{jk}\partial_{lm}\Phi)(\mathbf{B})C_{jk}C_{lm} = \sum_{jklm}(\partial_{jk}\partial_{lm}\Phi)(\mathbf{B}^{\text{sym}})C^{\text{sym}}_{jk}C^{\text{sym}}_{lm}, \tag{3.4}$$

$$\boldsymbol{\nabla}_{n\times n}\Phi(\mathbf{B}) = \boldsymbol{\nabla}_{n\times n}\Phi(\mathbf{B}^{\text{sym}}), \tag{3.5}$$

$$(\boldsymbol{\nabla}^2_{n\times n}\Phi)(\mathbf{B}) = (\boldsymbol{\nabla}^2_{n\times n}\Phi)(\mathbf{B}^{\text{sym}}). \tag{3.6}$$

Thus it suffices to verify (3.1) (3.2) *for all symmetric matrices. Since later we will mostly deal with symmetric matrices, we will in some cases leave out the symmetrization of the matrices, i.e. we will use* $\mathbf{B}$ *instead of* $\mathbf{B}^{\text{sym}}$ *and restrict the admitted matrices to the symmetric ones.*

Definition 3.4. *We define the* dual viscosity $\boldsymbol{\sigma} : \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n} \to (\mathbb{R}^{n\times n})^2$ *of a potential* Φ *for all* $\mathbf{B}, \mathbf{C} \in \mathbb{R}^{n\times n}$ *by*

$$\sigma_{jklm}(\mathbf{B}, \mathbf{C}) = \int_0^1 (\partial_{jk}\partial_{lm}\Phi)([\mathbf{C}, \mathbf{B}]_s)\, ds,$$

where $[\mathbf{C}, \mathbf{B}]_s := \mathbf{C} + s(\mathbf{B} - \mathbf{C})$.

The reason for introducing the dual viscosity is that it appears quite naturally when examining the difference of the extra stress $\mathbf{S} := \boldsymbol{\nabla}_{n\times n}\Phi$, which appears for example in the error equation and its dual problem. The dual problem is explained in chapter 4 section 5 in more detail. Let us only mention so far that the dual problem is important for deriving optimal error estimates of the pressure stabilization. We chose the name *dual viscosity* for $\boldsymbol{\sigma}$, since it is just the generalized viscosity of the dual problem. For all $\mathbf{B} \in \mathbb{R}^{n\times n}$

$$\mathbf{S}(\mathbf{B}) = \boldsymbol{\nabla}_{n\times n}\Phi(\mathbf{B}) = F'(|\mathbf{B}^{\text{sym}}|)\tfrac{\mathbf{B}^{\text{sym}}}{|\mathbf{B}^{\text{sym}}|},$$

so $\mathbf{S}$ only depends on the symmetric part of $\mathbf{B}$. Let us now examine the difference of the extra stresses. For all $\mathbf{B}, \mathbf{C} \in \mathbb{R}^{n\times n}$

$$\begin{aligned} S_{jk}(\mathbf{B}) - S_{jk}(\mathbf{C}) &= (\partial_{jk}\Phi)(\mathbf{B}) - (\partial_{jk}\Phi)(\mathbf{C}) \\ &= \int_0^1 \tfrac{d}{ds}\Big((\partial_{jk}\Phi)([\mathbf{C},\mathbf{B}]_s)\Big)\,ds \\ &= \sum_{lm}\int_0^1 (\partial_{jk}\partial_{lm}\Phi)([\mathbf{C},\mathbf{B}]_s)(B_{lm}-C_{lm})\,ds \\ &= \sum_{lm}\sigma_{jklm}(\mathbf{B},\mathbf{C})(B_{lm}-C_{lm}), \end{aligned}$$

or shortly $\mathbf{S}(\mathbf{B}) - \mathbf{S}(\mathbf{C}) = \boldsymbol{\sigma}(\mathbf{B},\mathbf{C})(\mathbf{B}-\mathbf{C})$. From (3.1) and (3.2) we conclude that for all $\mathbf{B}, \mathbf{C}, \mathbf{Q} \in \mathbb{R}^{n\times n}$

$$\sum_{jklm}\sigma_{jklm}(\mathbf{B},\mathbf{C})Q_{jk}Q_{lm} \geq \gamma_1 \int_0^1 (1+|[\mathbf{C},\mathbf{B}]_s^{\mathrm{sym}}|^2)^{\frac{p-2}{2}}\,ds\,|\mathbf{Q}^{\mathrm{sym}}|^2,$$

$$|\boldsymbol{\sigma}(\mathbf{B},\mathbf{C})| \leq \gamma_2 \int_0^1 (1+|[\mathbf{C},\mathbf{B}]_s^{\mathrm{sym}}|^2)^{\frac{p-2}{2}} ds.$$

As in [**PR01**], we deduce from this the existence of constants c_1, c_2 independent of γ_1, γ_2, such that

$$\sum_{jklm}\sigma_{jklm}(\mathbf{B},\mathbf{C})Q_{jk}Q_{lm} \geq c_1\,\gamma_1(1+|\mathbf{B}^{\mathrm{sym}}|^2+|\mathbf{C}^{\mathrm{sym}}|^2)^{\frac{p-2}{2}}|\mathbf{Q}^{\mathrm{sym}}|^2, \tag{3.7}$$

$$|\boldsymbol{\sigma}(\mathbf{B},\mathbf{C})| \leq c_2\,\gamma_2(1+|\mathbf{B}^{\mathrm{sym}}|^2+|\mathbf{C}^{\mathrm{sym}}|^2)^{\frac{p-2}{2}}, \tag{3.8}$$

where we have implicitly used $0 \geq p-2 > -1$ (see also [**MNRR96**]). With these inequalities and $\mathbf{S}(\mathbf{B}) - \mathbf{S}(\mathbf{C}) = \boldsymbol{\sigma}(\mathbf{B},\mathbf{C})(\mathbf{B}-\mathbf{C})$, it is easy to check that $\mathbf{S}$ fulfills the following properties:

THEOREM 3.5. *For all $\mathbf{B}, \mathbf{C} \in \mathbb{R}^{n\times n}_{\mathrm{sym}}$ there holds*

$$\mathbf{S}(\mathbf{0}) = \mathbf{0}, \tag{3.9}$$

$$\begin{aligned} \sum_{ij}(S_{ij}(\mathbf{B})-S_{ij}(\mathbf{C}))(B_{ij}-C_{ij}) &\geq c_1\gamma_1(1+|\mathbf{B}|^2+|\mathbf{C}|^2)^{\frac{p-2}{2}}|\mathbf{B}-\mathbf{C}|^2, \\ \sum_{ij}S_{ij}(\mathbf{B})B_{ij} &\geq c_1\,\gamma_1(1+|\mathbf{B}|^2)^{\frac{p-2}{2}}|\mathbf{B}|^2, \\ &\geq c_1\gamma_1\,(2^{-\frac{1}{2}}|\mathbf{B}|^p-1). \end{aligned} \tag{3.10}$$

$$\begin{aligned} |\mathbf{S}(\mathbf{B})-\mathbf{S}(\mathbf{C})| &\leq c_2\,\gamma_2(1+|\mathbf{B}|^2+|\mathbf{C}|^2)^{\frac{p-2}{2}}\,|\mathbf{B}-\mathbf{C}|, \\ |\mathbf{S}(\mathbf{B})| &\leq c_2\,\gamma_2(1+|\mathbf{B}|^2)^{\frac{p-2}{2}}\,|\mathbf{B}|. \end{aligned} \tag{3.11}$$

The same inequalities hold true for all $\mathbf{B}, \mathbf{C} \in \mathbb{R}^{n\times n}$ if $|\mathbf{B}|$ and $|\mathbf{C}|$ is replaced on the right-hand side by $|\mathbf{B}^{\mathrm{sym}}|$ and $|\mathbf{C}^{\mathrm{sym}}|$ (compare lemma 3.3).

The inequalities involving $\mathbf{S}(\mathbf{B})-\mathbf{S}(\mathbf{C})$ follow from $\mathbf{S}(\mathbf{B})-\mathbf{S}(\mathbf{C}) = \boldsymbol{\sigma}(\mathbf{B},\mathbf{C})(\mathbf{B}-\mathbf{C})$ and the estimates for $\boldsymbol{\sigma}$. Then choose $\mathbf{C} = \mathbf{0}$ for the other inequalities. In the last inequality of (3.10) we have used

$$(1+|\mathbf{B}|^2)^{\frac{p-2}{2}}|\mathbf{B}|^2 \geq (2|\mathbf{B}|^2)^{\frac{p-2}{2}}|\mathbf{B}|^2 - 1 = 2^{\frac{p-2}{2}}|\mathbf{B}|^p - 1 \geq 2^{-\frac{1}{2}}|\mathbf{B}|^p - 1.$$

From remark 3.2, (3.10), and (3.11) there follows for $\mathbf{B} \in \mathbb{R}^{n\times n}$

$$c_1\gamma_1(1+|\mathbf{B}^{\mathrm{sym}}|^2)^{\frac{p-2}{2}}|\mathbf{B}^{\mathrm{sym}}| \leq F'(|\mathbf{B}^{\mathrm{sym}}|) \leq c_2\gamma_2(1+|\mathbf{B}^{\mathrm{sym}}|^2)^{\frac{p-2}{2}}|\mathbf{B}^{\mathrm{sym}}|. \tag{3.12}$$

2. Time and Space Dependent Potentials

We will now define the properties of a time and space dependent potential.

Definition 3.6. *Let $p : I\times\Omega \to (1,2]$ be a $W^{1,\infty}(I\times\Omega)$ function and let there exist $p_\infty, p_0 \in (1,2]$ such that $1 < p_\infty \leq p(t,x) \leq p_0 \leq 2$ for all $t\in I$, $x \in \Omega$. Let $F : I\times\Omega\times\mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$ be such that for a.e. $(t,x) \in I\times\Omega$ the function $F(t,x,\cdot)$ is a $p(t,x)$–potential (see definition 3.1) and the ellipticity constants do not depend on (t,x), i.e. the function $\Phi : \mathbb{R}^{n\times n} \to \mathbb{R}^{\geq 0}$ defined through $\Phi(t,x,\mathbf{B}) = F(t,x,|\mathbf{B}^{\mathrm{sym}}|)$ satisfies*

$$\sum_{jklm}(\partial_{jk}\partial_{lm}\Phi)(x,\mathbf{B})C_{jk}C_{lm} \geq \gamma_1(1+|\mathbf{B}^{\mathrm{sym}}|^2)^{\frac{p(t,x)-2}{2}}|\mathbf{C}^{\mathrm{sym}}|^2, \tag{3.13}$$

$$\big|(\boldsymbol{\nabla}^2_{n\times n}\Phi)(x,\mathbf{B})\big| \leq \gamma_2(1+|\mathbf{B}^{\mathrm{sym}}|^2)^{\frac{p(t,x)-2}{2}} \tag{3.14}$$

for all $\mathbf{B},\mathbf{C} \in \mathbb{R}^{n\times n}$ with constants $\gamma_1,\gamma_2 > 0$. Further we assume that F is continuously differentiable with respect to t and x and that $(\partial_t F)(t,x,\cdot) : \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$ and $(\partial_j F)(t,x,\cdot) : \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$ are C^1–functions on $\mathbb{R}^{\geq 0}$ and C^2–functions on $\mathbb{R}^{>0}$ for all $t \in I$, $x \in \Omega$. Moreover, assume that for $j = 1,\dots,d$

$$\begin{aligned}
(\partial_t F)(t,x,0) &= 0,\\
(\partial_t F)(t,x,R) &> 0 \qquad \text{for all } R > 0,\\
(\partial_j F)(t,x,0) &= 0,\\
(\partial_j F)(t,x,R) &> 0 \qquad \text{for all } R > 0,\\
\big|(\partial_t\boldsymbol{\nabla}_{n\times n}\Phi)(t,x,\mathbf{B})\big| &\leq \gamma_3(1+|\mathbf{B}^{\mathrm{sym}}|^2)^{\frac{p(t,x)-1}{2}}\ln(1+|\mathbf{B}^{\mathrm{sym}}|),\\
\big|(\boldsymbol{\nabla}\boldsymbol{\nabla}_{n\times n}\Phi)(t,x,\mathbf{B})\big| &\leq \gamma_3(1+|\mathbf{B}^{\mathrm{sym}}|^2)^{\frac{p(t,x)-1}{2}}\ln(1+|\mathbf{B}^{\mathrm{sym}}|).
\end{aligned} \tag{3.15}$$

with $\gamma_3 > 0$. Such a function F, resp. Φ, is called a **time and space dependent p-potential** *and the corresponding constants γ_1,γ_2 and γ_3 are called the ellipticity and growth constants of F, resp. Φ. The function p is called the* **exponent** *of the potential. In the absence of time we use the term* **space dependent p-potential**.

First of all let us mention that all the estimates that we have derived for $\mathbf{S}$ and Φ so far (see theorem 3.5) do also hold in the time and space dependent case: For fixed t and x we have just the case of the previous section.

Although F now also depends on the time and space, we will still write $F'(t,x,s)$ to indicate the partial derivative with respect to s. We have the following useful formula for $\partial_t\boldsymbol{\nabla}_{n\times n}$ and $\boldsymbol{\nabla}\boldsymbol{\nabla}_{n\times n}$: For all $\mathbf{B} \in \mathbb{R}^{n\times n}\setminus\{\mathbf{0}\}$ there holds

$$\begin{aligned}
(\partial_t\partial_{jk}\Phi)(x,\mathbf{B}) &= (\partial_t F)'(x,|\mathbf{B}^{\mathrm{sym}}|)\tfrac{B^{\mathrm{sym}}_{jk}}{|\mathbf{B}^{\mathrm{sym}}|},\\
(\partial_i\partial_{jk}\Phi)(x,\mathbf{B}) &= (\partial_i F)'(x,|\mathbf{B}^{\mathrm{sym}}|)\tfrac{B^{\mathrm{sym}}_{jk}}{|\mathbf{B}^{\mathrm{sym}}|}.
\end{aligned} \tag{3.16}$$

If $\mathbf{B} = \mathbf{0}$, then the regularity of F and the boundedness of $\frac{B^{\mathrm{sym}}_{jk}}{|\mathbf{B}^{\mathrm{sym}}|}$ imply

$$(\partial_t \partial_{jk} \Phi)(x, \mathbf{0}) = \mathbf{0},$$
$$(\partial_i \partial_{jk} \Phi)(x, \mathbf{0}) = \mathbf{0}.$$

Note that due to $\Phi(t, x, \mathbf{B}) = \Phi(t, x, \mathbf{B}^{\mathrm{sym}})$ it suffices to verify (3.13), (3.14) and (3.15) for symmetric matrices (see also remark 3.3). Further we remark that we will sometimes omit the time and space dependency and write $\Phi(\mathbf{B})$ instead of $\Phi(t, x, \mathbf{B})$. But we will only do so when it is clear that we are in a time and space dependent context.

3. Examples of Potentials

Certainly there arises the question of the existence of a p-potential and especially of a space dependent p-potential. In fact there do exist such potentials. The two standard examples are

$$F_1(t, x, s) = \int_0^s (1 + a^2)^{\frac{p(t,x)-2}{2}} a \, da \qquad \text{and} \quad F_2(t, x, s) = \int_0^s (1 + a)^{p(t,x)-2} a \, da,$$

where $p \in W^{1,\infty}(I \times \Omega)$ and $1 < p_\infty \le p(t, x) \le p_0 \le 2$ for all $t \in I$, $x \in \Omega$. For the partial derivatives with respect to s we conclude

$$\begin{aligned} F_1'(t, x, s) &= (1 + s^2)^{\frac{p(t,x)-2}{2}} s, \\ F_1''(t, x, s) &= (1 + s^2)^{\frac{p(t,x)-4}{2}} \big((p(t, x) - 1)s^2 + 1\big), \\ F_2'(t, x, s) &= (1 + s)^{p(t,x)-2} s, \\ F_2''(t, x, s) &= (p(t, x) - 1)\,(1 + s)^{p(t,x)-2}. \end{aligned} \tag{3.17}$$

The partial derivatives $\partial_t F_1'$, $\partial_j F_1'$, $\partial_t F_2'$ and $\partial_j F_2'$ are also of interest:

$$\begin{aligned} (\partial_t F_1)'(t, x, s) &= (1 + s^2)^{\frac{p(t,x)-2}{2}} s \ln(1 + s^2)(\partial_t p)(t, x), \\ (\partial_j F_1)'(t, x, s) &= (1 + s^2)^{\frac{p(t,x)-2}{2}} s \ln(1 + s^2)(\partial_j p)(t, x), \\ (\partial_t F_2)'(t, x, s) &= (1 + s)^{p(t,x)-2} s \ln(1 + s)(\partial_t p)(t, x), \\ (\partial_j F_2)'(t, x, s) &= (1 + s)^{p(t,x)-2} s \ln(1 + s)(\partial_j p)(t, x). \end{aligned} \tag{3.18}$$

We will now verify that F_1 and F_2 are time and space dependent p-potentials. Let $\Phi_1(t, x, \mathbf{B}) := F_1(t, x, |\mathbf{B}^{\mathrm{sym}}|)$ and $\Phi_2(t, x, \mathbf{B}) := F_2(t, x, |\mathbf{B}^{\mathrm{sym}}|)$ for all $\mathbf{B} \in \mathbb{R}^{n \times n}$. Due to remark 3.3 we will restrict ourselves to symmetric matrices, i.e. let $\mathbf{B}, \mathbf{C} \in \mathbb{R}^{n \times n}_{\mathrm{sym}}$, then by remark 3.2 and (3.17)

$$
\begin{aligned}
&\sum_{jklm}(\partial_{jk}\partial_{lm}\Phi_1)(t,x,\mathbf{B})C_{jk}C_{lm}\\
&\qquad= F_1'(|\mathbf{B}|)\big(\tfrac{|\mathbf{C}|^2}{|\mathbf{B}|}-\tfrac{|\mathbf{C}\cdot\mathbf{B}|^2}{|\mathbf{B}|^3}\big)+F_1''(|\mathbf{B}|)\tfrac{|\mathbf{C}\cdot\mathbf{B}|^2}{|\mathbf{B}|^2}\\
&\qquad= (1+|\mathbf{B}|^2)^{\frac{p(t,x)-2}{2}}|\mathbf{C}|^2\\
&\qquad\quad+(1+|\mathbf{B}|^2)^{\frac{p(t,x)-4}{2}}\tfrac{|\mathbf{C}\cdot\mathbf{B}|^2}{|\mathbf{B}|^2}\big(-(1+|\mathbf{B}|^2)+(p(t,x)-1)|\mathbf{B}|^2+1\big)\\
&\qquad= (1+|\mathbf{B}|^2)^{\frac{p(t,x)-2}{2}}|\mathbf{C}|^2+(1+|\mathbf{B}|^2)^{\frac{p(t,x)-2}{2}}\tfrac{|\mathbf{C}\cdot\mathbf{B}|^2}{1+|\mathbf{B}|^2}(p(t,x)-2)\\
&\qquad\geq (p(t,x)-1)(1+|\mathbf{B}|^2)^{\frac{p(t,x)-2}{2}}|\mathbf{C}|^2,
\end{aligned}
\tag{3.19}
$$

$$
\begin{aligned}
&\sum_{jklm}(\partial_{jk}\partial_{lm}\Phi_2)(t,x,\mathbf{B})C_{jk}C_{lm}\\
&\qquad= F_2'(|\mathbf{B}|)\big(\tfrac{|\mathbf{C}|^2}{|\mathbf{B}|}-\tfrac{|\mathbf{C}\cdot\mathbf{B}|^2}{|\mathbf{B}|^3}\big)+F_2''(|\mathbf{B}|)\tfrac{|\mathbf{C}\cdot\mathbf{B}|^2}{|\mathbf{B}|^2}\\
&\qquad= (1+|\mathbf{B}|)^{p(t,x)-2}|\mathbf{C}|^2\\
&\qquad\quad+(p(t,x)-2)(1+|\mathbf{B}|)^{p(t,x)-2}\tfrac{|\mathbf{C}\cdot\mathbf{B}|^2}{|\mathbf{B}|^2}\\
&\qquad\geq (p(t,x)-1)(1+|\mathbf{B}|)^{p(t,x)-2}|\mathbf{C}|^2\\
&\qquad\geq (p(t,x)-1)\big(\sqrt{2}\big)^{p(t,x)-2}(1+|\mathbf{B}|^2)^{\frac{p(t,x)-2}{2}}|\mathbf{C}|^2\\
&\qquad\geq (p(t,x)-1)2^{-\frac{1}{2}}(1+|\mathbf{B}|^2)^{\frac{p(t,x)-2}{2}}|\mathbf{C}|^2,
\end{aligned}
\tag{3.20}
$$

and

$$
\begin{aligned}
&|(\boldsymbol{\nabla}^2_{n\times n}\Phi_1)(t,x,\mathbf{B})|\\
&\qquad\leq F_1'(|\mathbf{B}|)\tfrac{\sqrt{d^2+1}}{|\mathbf{B}|}+F_1''(|\mathbf{B}|)\\
&\qquad= (1+|\mathbf{B}|^2)^{\frac{p(t,x)-4}{2}}\big(\sqrt{d^2+1}(1+|\mathbf{B}|^2)+(p(t,x)-1)|\mathbf{B}|^2+1\big)\\
&\qquad\leq (\sqrt{d^2+1}+1)\,(1+|\mathbf{B}|^2)^{\frac{p(t,x)-2}{2}},\\
&|(\boldsymbol{\nabla}^2_{n\times n}\Phi_2)(t,x,\mathbf{B})|\\
&\qquad\leq F_2'(|\mathbf{B}|)\tfrac{\sqrt{d^2+1}}{|\mathbf{B}|}+F_2''(|\mathbf{B}|)\\
&\qquad= (1+|\mathbf{B}|)^{p(t,x)-2}\Big(\sqrt{d^2+1}+(p(t,x)-2)\tfrac{|\mathbf{B}|}{1+|\mathbf{B}|}+1\Big)\\
&\qquad\leq (\sqrt{d^2+1}+1)\,(1+|\mathbf{B}|^2)^{\frac{p(t,x)-2}{2}},
\end{aligned}
$$

where we have repeatedly used $p \le 2$. By equations (3.16) and (3.18) we get

$$\begin{aligned}
|(\boldsymbol{\nabla}\boldsymbol{\nabla}_{n\times n}\Phi_1)(t,x,\mathbf{B})| &\le (1+|\mathbf{B}|^2)^{\frac{p(t,x)-2}{2}}|\mathbf{B}|\ln(1+|\mathbf{B}|^2)\,|(\boldsymbol{\nabla}p)(t,x)| \\
&\le 2\,(1+|\mathbf{B}|^2)^{\frac{p(t,x)-1}{2}}\ln(1+|\mathbf{B}|)\,|(\boldsymbol{\nabla}p)(t,x)| \\
|(\partial_t\boldsymbol{\nabla}_{n\times n}\Phi_1)(t,x,\mathbf{B})| &\le (1+|\mathbf{B}|^2)^{\frac{p(t,x)-2}{2}}|\mathbf{B}|\ln(1+|\mathbf{B}|^2)\,|(\partial_t p)(t,x)| \\
&\le 2\,(1+|\mathbf{B}|^2)^{\frac{p(t,x)-1}{2}}\ln(1+|\mathbf{B}|)\,|(\partial_t p)(t,x)| \\
|(\boldsymbol{\nabla}\boldsymbol{\nabla}_{n\times n}\Phi_2)(t,x,\mathbf{B})| &\le (1+|\mathbf{B}|)^{p(t,x)-2}|\mathbf{B}|\ln(1+|\mathbf{B}|)\,|(\boldsymbol{\nabla}p)(t,x)| \\
&\le \sqrt{2}\,(1+|\mathbf{B}|^2)^{\frac{p(t,x)-1}{2}}\ln(1+|\mathbf{B}|)\,|(\boldsymbol{\nabla}p)(t,x)| \\
|(\partial_t\boldsymbol{\nabla}_{n\times n}\Phi_2)(t,x,\mathbf{B})| &\le (1+|\mathbf{B}|)^{p(t,x)-2}|\mathbf{B}|\ln(1+|\mathbf{B}|)\,|(\partial_t p)(t,x)| \\
&\le \sqrt{2}\,(1+|\mathbf{B}|^2)^{\frac{p(t,x)-1}{2}}\ln(1+|\mathbf{B}|)\,|(\partial_t p)(t,x)|.
\end{aligned}$$

So if $p \in W^{1,\infty}(I\times\Omega)$, then Φ_1 and Φ_2 are space and time dependent p-potentials.

4. The A–Approximation

In the study of an incompressible fluid with the extra stress given by a p-potential Φ, one has to overcome lots of difficulties which arise mostly due to the non–quadratic growth. This becomes especially crucial for small values of p. To get past these problems it is useful to approximate Φ by a 2–potential (i.e. quadratic growth), which differs only slightly from Φ. Let us introduce two such potentials, namely the A–approximation and the $\boldsymbol{\lambda}$–approximation.

Let $A \ge 1$. The A–approximation $F^A : \mathbb{R}^{\ge 0} \to \mathbb{R}^{\ge 0}$ of the potential F is defined by

$$F^A(R) := \begin{cases} F(R) & \text{for } R \le A, \\ F(A) + (R-A)F'(A) + \frac{1}{2}(R-A)^2 F''(A) & \text{for } R > A. \end{cases}$$

Then F^A is convex. As usual we define $\Phi^A(\mathbf{B}) := F^A(|\mathbf{B}^{\mathrm{sym}}|)$, $\mathbf{S}^A(\mathbf{B}) := (\boldsymbol{\nabla}_{n\times n}\Phi^A)(\mathbf{B})$. Note that for $\mathbf{B} \in \mathbb{R}^{n\times n}$ with $|\mathbf{B}^{\mathrm{sym}}| > A$ there holds (see remark 3.2)

$$(\partial_{jk}\Phi^A)(\mathbf{B}) = \big(F'(A) + (|\mathbf{B}^{\mathrm{sym}}| - A)F''(A)\big)\tfrac{B^{\mathrm{sym}}_{jk}}{|\mathbf{B}^{\mathrm{sym}}|}. \tag{3.21}$$

This implies for $|\mathbf{B}^{\mathrm{sym}}| > A$

$$\begin{aligned}
(\partial_{jk}\partial_{lm}\Phi^A)(\mathbf{B})) = \quad & F'(A)\Big(\tfrac{\delta^{\mathrm{sym}}_{jk,lm}}{|\mathbf{B}^{\mathrm{sym}}|} - \tfrac{B^{\mathrm{sym}}_{jk}B^{\mathrm{sym}}_{lm}}{|\mathbf{B}^{\mathrm{sym}}|^3}\Big) + F''(A)\tfrac{B^{\mathrm{sym}}_{jk}}{|\mathbf{B}^{\mathrm{sym}}|}\tfrac{B^{\mathrm{sym}}_{lm}}{|\mathbf{B}^{\mathrm{sym}}|} \\
&+ (|\mathbf{B}^{\mathrm{sym}}| - A)F''(A)\Big(\tfrac{\delta^{\mathrm{sym}}_{jk,lm}}{|\mathbf{B}^{\mathrm{sym}}|} - \tfrac{B^{\mathrm{sym}}_{jk}B^{\mathrm{sym}}_{lm}}{|\mathbf{B}^{\mathrm{sym}}|^3}\Big).
\end{aligned}$$

Let $\kappa := \frac{A}{|\mathbf{B}^{\mathrm{sym}}|}$, then $0 < \kappa < 1$ and $|\kappa\mathbf{B}| = A$. Using remark 3.2 it follows that

$$(\partial_{jk}\partial_{lm}\Phi)(\kappa\mathbf{B}) = \tfrac{1}{\kappa}F'(A)\Big(\tfrac{\delta^{\mathrm{sym}}_{jk,lm}}{|\mathbf{B}^{\mathrm{sym}}|} - \tfrac{B^{\mathrm{sym}}_{jk}B^{\mathrm{sym}}_{lm}}{|\mathbf{B}^{\mathrm{sym}}|^3}\Big) + F''(A)\tfrac{B^{\mathrm{sym}}_{jk}}{|\mathbf{B}^{\mathrm{sym}}|}\tfrac{B^{\mathrm{sym}}_{lm}}{|\mathbf{B}^{\mathrm{sym}}|}.$$

Thus for $|\mathbf{B}^{\rm sym}| > A$

$$\begin{aligned}(\partial_{jk}\partial_{lm}\Phi^A)(\mathbf{B}) &= \kappa(\partial_{jk}\partial_{lm}\Phi)(\kappa\mathbf{B}) + (1-\kappa)F''(A)\tfrac{B^{\rm sym}_{jk}}{|\mathbf{B}^{\rm sym}|}\tfrac{B^{\rm sym}_{lm}}{|\mathbf{B}^{\rm sym}|}\\ &\quad + (|\mathbf{B}^{\rm sym}|-A)F''(A)\Big(\tfrac{\delta^{\rm sym}_{jk,lm}}{|\mathbf{B}^{\rm sym}|} - \tfrac{B^{\rm sym}_{jk}B^{\rm sym}_{lm}}{|\mathbf{B}^{\rm sym}|^3}\Big)\\ &= \kappa(\partial_{jk}\partial_{lm}\Phi)(\kappa\mathbf{B}) + (1-\kappa)F''(A)\delta^{\rm sym}_{jk,lm}.\end{aligned}$$

Hence

$$(\partial_{jk}\partial_{lm}\Phi^A)(\mathbf{B}) = \begin{cases}(\partial_{jk}\partial_{lm}\Phi)(\mathbf{B}) & \text{for } |\mathbf{B}^{\rm sym}|\le A,\\ \frac{A}{|\mathbf{B}^{\rm sym}|}(\partial_{jk}\partial_{lm}\Phi)\big(\frac{A}{|\mathbf{B}^{\rm sym}|}\mathbf{B}^{\rm sym}\big) & \\ +\big(1-\frac{A}{|\mathbf{B}^{\rm sym}|}\big)F''(A)\delta^{\rm sym}_{jk,lm} & \text{for } |\mathbf{B}^{\rm sym}|> A.\end{cases} \tag{3.22}$$

Let $\mathbf{C}\in\mathbb{R}^{n\times n}$. From (3.1), (3.2), (3.3) and remark 3.3 we conclude that

$$\begin{aligned}&\sum_{jklm}(\partial_{jk}\partial_{lm}\Phi^A)(\mathbf{B})C_{jk}C_{lm}\\ &\qquad\ge\begin{cases}\gamma_1\,(1+|\mathbf{B}^{\rm sym}|^2)^{\frac{p-2}{2}}|\mathbf{C}^{\rm sym}|^2 & \text{for } |\mathbf{B}^{\rm sym}|\le A,\\ \gamma_1\,(1+A^2)^{\frac{p-2}{2}}|\mathbf{C}^{\rm sym}|^2 & \text{for } |\mathbf{B}^{\rm sym}|> A,\end{cases}\end{aligned}$$

and

$$\big|(\boldsymbol{\nabla}^2_{n\times n}\Phi^A)(\mathbf{B})\big|\le\begin{cases}\gamma_2\,(1+|\mathbf{B}^{\rm sym}|^2)^{\frac{p-2}{2}} & \text{for } |\mathbf{B}^{\rm sym}|\le A,\\ \gamma_2\,(1+A^2)^{\frac{p-2}{2}} & \text{for } |\mathbf{B}^{\rm sym}|> A.\end{cases}$$

Since $A\ge 1$ and $1<p\le 2$ there follows

$$\sum_{jklm}(\partial_{jk}\partial_{lm}\Phi^A)(\mathbf{B})C_{jk}C_{lm}\ge\begin{cases}2^{\frac{p-2}{2}}\,\gamma_1\,A^{p-2}|\mathbf{C}^{\rm sym}|^2,\\ \gamma_1\,(1+|\mathbf{B}^{\rm sym}|^2)^{\frac{p-2}{2}}|\mathbf{C}^{\rm sym}|^2,\end{cases} \tag{3.23}$$

$$\big|(\boldsymbol{\nabla}^2_{n\times n}\Phi^A)(\mathbf{B})\big|\le d\,\gamma_2. \tag{3.24}$$

Further we derive from the definition of Φ^A and (3.21) that

$$(\partial_{jk}\Phi^A)(\mathbf{B}) = \begin{cases}(\partial_{jk}\Phi)(\mathbf{B}) & \text{for } |\mathbf{B}^{\rm sym}|\le A,\\ (\partial_{jk}\Phi)\big(\frac{A}{|\mathbf{B}^{\rm sym}|}\mathbf{B}^{\rm sym}\big) & \\ +\big(1-\frac{A}{|\mathbf{B}^{\rm sym}|}\big)F''(A)\mathbf{B}^{\rm sym} & \text{for } |\mathbf{B}^{\rm sym}|> A.\end{cases} \tag{3.25}$$

For the difference $\mathbf{S}^A-\mathbf{S}=\boldsymbol{\nabla}_{n\times n}\Phi^A-\boldsymbol{\nabla}_{n\times n}\Phi$ we deduce

$$S^A_{jk}(\mathbf{B})-S_{jk}(\mathbf{B}) = \begin{cases}0 & \text{for } |\mathbf{B}^{\rm sym}|\le A,\\ S_{jk}\big(\frac{A}{|\mathbf{B}^{\rm sym}|}\mathbf{B}^{\rm sym}\big)-S_{jk}(\mathbf{B}^{\rm sym}) & \\ +\big(1-\frac{A}{|\mathbf{B}^{\rm sym}|}\big)F''(A)\mathbf{B}^{\rm sym} & \text{for } |\mathbf{B}^{\rm sym}|> A.\end{cases}$$

For $|\mathbf{B}^{\rm sym}|>A$ there follows from (3.3) and (3.11) that

$$\begin{aligned}|\mathbf{S}^A(\mathbf{B})-\mathbf{S}(\mathbf{B})|\le\ & c_2\,\gamma_2(1+A^2+|\mathbf{B}^{\rm sym}|^2)^{\frac{p-2}{2}}\big(1-\tfrac{A}{|\mathbf{B}^{\rm sym}|}\big)|\mathbf{B}^{\rm sym}|\\ &+|\mathbf{B}^{\rm sym}|\gamma_2(1+A^2)^{\frac{p-2}{2}}.\end{aligned}$$

This implies for all $\mathbf{B} \in \mathbb{R}^{n\times n}$ and $\rho \geq 0$

$$|\mathbf{S}^A(\mathbf{B}) - \mathbf{S}(\mathbf{B})| \leq \begin{cases} 0 & \text{for } |\mathbf{B}^{\text{sym}}| \leq A, \\ 2\,c_2\,\gamma_2 A^{p-2}|\mathbf{B}^{\text{sym}}| & \text{for } |\mathbf{B}^{\text{sym}}| > A, \end{cases} \\ \leq 2\,c_2\,\gamma_2 A^{p-2-\rho}|\mathbf{B}^{\text{sym}}|^{1+\rho}. \tag{3.26}$$

LEMMA 3.7. *If the p–potential F, Φ additionally satisfies*

$$\begin{aligned} \big|\partial_r F''(t,x,s)\big| &\leq C\,(1+s)^{p(t,x)-2}\ln(1+s), \\ \big|\partial_t F''(t,x,s)\big| &\leq C\,(1+s)^{p(t,x)-2}\ln(1+s), \end{aligned} \tag{3.27}$$

for almost all $t \in I$, $x \in \Omega$ and all $s \geq 0$, then for all $\mathbf{B} \in \mathbb{R}^{d\times d}$

$$\begin{aligned} \big|(\partial_t \boldsymbol{\nabla}_{n\times n}\Phi^A)(t,x,\mathbf{B})\big| &\leq C\,(1+|\mathbf{B}^{\text{sym}}|^2)^{\frac{p(t,x)-1}{2}}\ln(1+|\mathbf{B}^{\text{sym}}|), \\ \big|(\boldsymbol{\nabla}\boldsymbol{\nabla}_{n\times n}\Phi^A)(t,x,\mathbf{B})\big| &\leq C\,(1+|\mathbf{B}^{\text{sym}}|^2)^{\frac{p(t,x)-1}{2}}\ln(1+|\mathbf{B}^{\text{sym}}|). \end{aligned} \tag{3.28}$$

PROOF. Due to (3.21) there holds

$$\begin{aligned} (\partial_t\partial_{jk}\Phi^A)(\mathbf{B}) &= \big(\partial_t F'(A) + (|\mathbf{B}^{\text{sym}}| - A)\partial_t F''(A)\big)\tfrac{B^{\text{sym}}_{jk}}{|\mathbf{B}^{\text{sym}}|} \\ &= (\partial_t\partial_{jk}\Phi)(\mathbf{B}) + (|\mathbf{B}^{\text{sym}}| - A)\partial_t F''(A)\tfrac{B^{\text{sym}}_{jk}}{|\mathbf{B}^{\text{sym}}|}, \\ (\partial_r\partial_{jk}\Phi^A)(\mathbf{B}) &= \big(\partial_r F'(A) + (|\mathbf{B}^{\text{sym}}| - A)\partial_t F''(A)\big)\tfrac{B^{\text{sym}}_{jk}}{|\mathbf{B}^{\text{sym}}|} \\ &= (\partial_r\partial_{jk}\Phi)(\mathbf{B}) + (|\mathbf{B}^{\text{sym}}| - A)\partial_r F''(A)\tfrac{B^{\text{sym}}_{jk}}{|\mathbf{B}^{\text{sym}}|}, \end{aligned}$$

This and (3.27) proves the lemma. □

Let us remark, that both example potentials F_1, Φ_1 and F_2, Φ_2 from section 3 satisfy the additional requirement (3.28) as long as $p \in W^{1,\infty}(I \times \Omega)$. This is indeed a direct consequence of (3.18).

5. The λ–Approximation

DEFINITION 3.8. *Let $\boldsymbol{\lambda} = (\lambda_1, \lambda_2) \in (0,1] \times \mathbb{R}^{>0}$ and $\lambda_2 \geq 2-p$. Then the $\boldsymbol{\lambda}$–approximation is defined by $F^{\boldsymbol{\lambda}} : \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$*

$$F^{\boldsymbol{\lambda}}(R) := \int_0^R (1+\lambda_1 s^{\lambda_2})^{\frac{2-p}{\lambda_2}} F'(s)\,ds.$$

Further we set $\Phi^{\boldsymbol{\lambda}}(\mathbf{B}) := F^{\boldsymbol{\lambda}}(|\mathbf{B}^{\text{sym}}|)$ and $\mathbf{S}^{\boldsymbol{\lambda}}(\mathbf{B}) := \boldsymbol{\nabla}_{n\times n}\Phi^{\boldsymbol{\lambda}}(\mathbf{B})$ for all $\mathbf{B} \in \mathbb{R}^{n\times n}$.

We calculate

$$\begin{aligned} (F^{\boldsymbol{\lambda}})'(R) &= (1+\lambda_1 R^{\lambda_2})^{\frac{2-p}{\lambda_2}} F'(R), \\ \partial_{jk}\Phi^{\boldsymbol{\lambda}}(\mathbf{B}) &= (1+\lambda_1|\mathbf{B}^{\text{sym}}|^{\lambda_2})^{\frac{2-p}{\lambda_2}} F'(|\mathbf{B}^{\text{sym}}|)\tfrac{B^{\text{sym}}_{jk}}{|\mathbf{B}^{\text{sym}}|}, \\ (\partial_{jk}\partial_{lm}\Phi^{\boldsymbol{\lambda}})(\mathbf{B}) &= \quad (1+\lambda_1|\mathbf{B}^{\text{sym}}|^{\lambda_2})^{\frac{2-p}{\lambda_2}}(\partial_{jk}\partial_{lm}\Phi)(\mathbf{B}) \\ &\quad + \lambda_1(2-p)(1+\lambda_1|\mathbf{B}^{\text{sym}}|^{\lambda_2})^{\frac{2-p-\lambda_2}{\lambda_2}}|\mathbf{B}^{\text{sym}}|^{\lambda_2-1}F'(|\mathbf{B}^{\text{sym}}|)\tfrac{B^{\text{sym}}_{jk}B^{\text{sym}}_{lm}}{|\mathbf{B}^{\text{sym}}|^2}. \end{aligned}$$

Let $\mathbf{C} \in \mathbb{R}^{n\times n}$, then

$$\begin{aligned}&\sum_{jklm}(\partial_{jk}\partial_{lm}\Phi^{\boldsymbol{\lambda}})(\mathbf{B})C_{jk}C_{lm}\\ &\quad= (1+\lambda_1|\mathbf{B}^{\mathrm{sym}}|^{\lambda_2})^{\frac{2-p}{\lambda_2}}\sum_{jklm}(\partial_{jk}\partial_{lm}\Phi)(\mathbf{B})C_{jk}C_{lm}\\ &\qquad+\lambda_1(2-p)(1+\lambda_1|\mathbf{B}^{\mathrm{sym}}|^{\lambda_2})^{\frac{2-p-\lambda_2}{\lambda_2}}|\mathbf{B}^{\mathrm{sym}}|^{\lambda_2-1}F'(|\mathbf{B}^{\mathrm{sym}}|)\tfrac{|\mathbf{B}^{\mathrm{sym}}\cdot\mathbf{C}^{\mathrm{sym}}|^2}{|\mathbf{B}^{\mathrm{sym}}|^2}.\end{aligned}$$

Since $1 < p \le 2$, the last term is non–negative, so due to (3.1) there holds

$$\begin{aligned}&\sum_{jklm}(\partial_{jk}\partial_{lm}\Phi^{\boldsymbol{\lambda}})(\mathbf{B})C_{jk}C_{lm}\\ &\quad\ge \gamma_1\,(1+\lambda_1|\mathbf{B}^{\mathrm{sym}}|^{\lambda_2})^{\frac{2-p}{\lambda_2}}(1+|\mathbf{B}^{\mathrm{sym}}|^2)^{\frac{p-2}{2}}|\mathbf{C}^{\mathrm{sym}}|^2\\ &\quad\overset{p\le 2}{\ge}\begin{cases}\gamma_1\,(1+|\mathbf{B}^{\mathrm{sym}}|^2)^{\frac{p-2}{2}}|\mathbf{C}^{\mathrm{sym}}|^2,\\ \frac{1}{2}\,\gamma_1\,\lambda_1^{\frac{2-p}{\lambda_2}}|\mathbf{C}^{\mathrm{sym}}|^2.\end{cases}\end{aligned}\tag{3.29}$$

Furthermore, by (3.2) and (3.12) we have

$$\begin{aligned}\big|(\boldsymbol{\nabla}^2_{n\times n}\Phi^{\boldsymbol{\lambda}})(\mathbf{B})\big| \le\ & \gamma_2\left(\frac{(1+\lambda_1|\mathbf{B}^{\mathrm{sym}}|^{\lambda_2})^{\frac{1}{\lambda_2}}}{(1+|\mathbf{B}^{\mathrm{sym}}|^2)^{\frac12}}\right)^{2-p}\\ &+c_2\,\gamma_2\left(\frac{(1+\lambda_1|\mathbf{B}^{\mathrm{sym}}|^{\lambda_2})^{\frac{1}{\lambda_2}}}{(1+|\mathbf{B}^{\mathrm{sym}}|^2)^{\frac12}}\right)^{2-p}\frac{\lambda_1|\mathbf{B}^{\mathrm{sym}}|^{\lambda_2}}{1+\lambda_1|\mathbf{B}^{\mathrm{sym}}|^{\lambda_2}}\\ \le\ & 2(1+c_2)\,\gamma_2.\end{aligned}\tag{3.30}$$

For the difference $\mathbf{S}^{\boldsymbol{\lambda}} - \mathbf{S} = \boldsymbol{\nabla}_{n\times n}\Phi^{\boldsymbol{\lambda}} - \boldsymbol{\nabla}_{n\times n}\Phi$ we deduce

$$\mathbf{S}^{\boldsymbol{\lambda}}(\mathbf{B}) - \mathbf{S}(\mathbf{B}) = \Big((1+\lambda_1|\mathbf{B}^{\mathrm{sym}}|^{\lambda_2})^{\frac{2-p}{\lambda_2}} - 1\Big)F'(|\mathbf{B}^{\mathrm{sym}}|)\tfrac{\mathbf{B}^{\mathrm{sym}}}{|\mathbf{B}^{\mathrm{sym}}|}.$$

Note that $0 \le (1+t)^q - 1 \le |q|t$ for all $q \le 1$ and $t \ge 0$, so with (3.12):

$$\begin{aligned}|\mathbf{S}^{\boldsymbol{\lambda}}(\mathbf{B}) - \mathbf{S}(\mathbf{B})| &\le c_2\,\lambda_1\tfrac{2-p}{\lambda_2}\gamma_2|\mathbf{B}^{\mathrm{sym}}|^{\lambda_2}(1+|\mathbf{B}^{\mathrm{sym}}|^2)^{\frac{p-2}{2}}|\mathbf{B}^{\mathrm{sym}}|\\ &\le c_2\,\gamma_2\,\lambda_1|\mathbf{B}^{\mathrm{sym}}|^{p-1+\lambda_2}.\end{aligned}\tag{3.31}$$

6. Approximated Potential

Let us summarize the common properties of $\mathbf{S}^A$ and $\mathbf{S}^{\boldsymbol{\lambda}}$. Let $\widetilde{F}$ be either F^A or $F^{\boldsymbol{\lambda}}$ and define $\widetilde{\Phi}$ and $\widetilde{\mathbf{S}}$ analogously. Then for all $\mathbf{B}, \mathbf{C} \in \mathbb{R}^{n\times n}$ there holds

$$\sum_{jklm}(\partial_{jk}\partial_{lm}\Phi^{A})(\mathbf{B})C_{jk}C_{lm} \ge \begin{cases}2^{\frac{p-2}{2}}\,\gamma_1 A^{p-2}\,|\mathbf{C}^{\mathrm{sym}}|^2,\\ C\,\gamma_1\,(1+|\mathbf{B}^{\mathrm{sym}}|^2)^{\frac{p-2}{2}}|\mathbf{C}^{\mathrm{sym}}|^2,\end{cases}\tag{3.32}$$

$$\sum_{jklm}(\partial_{jk}\partial_{lm}\Phi^{\boldsymbol{\lambda}})(\mathbf{B})C_{jk}C_{lm} \ge \begin{cases}\frac12\,\gamma_1\lambda_1^{\frac{2-p}{\lambda_2}}\,|\mathbf{C}^{\mathrm{sym}}|^2,\\ \gamma_1\,(1+|\mathbf{B}^{\mathrm{sym}}|^2)^{\frac{p-2}{2}}|\mathbf{C}^{\mathrm{sym}}|^2,\end{cases}\tag{3.33}$$

$$\big|(\boldsymbol{\nabla}^2\Phi^A)(\mathbf{B})\big| \le d\,\gamma_2,\tag{3.34}$$

$$\big|(\boldsymbol{\nabla}^2\Phi^{\boldsymbol{\lambda}})(\mathbf{B})\big| \le 2(1+c_2)\,\gamma_2.\tag{3.35}$$

If we further introduce $\rho \geq 0$, then for all $\mathbf{B} \in \mathbb{R}^{n\times n}_{\mathrm{sym}}$ we deduce from (3.26) and (3.31).

$$|\mathbf{S}^A(\mathbf{B}) - \mathbf{S}(\mathbf{B})| \leq 2\, c_2\, \gamma_2\, A^{p-2-\rho}|\mathbf{B}^{\mathrm{sym}}|^{1+\rho}, \tag{3.36}$$

$$|\mathbf{S}^{\boldsymbol{\lambda}}(\mathbf{B}) - \mathbf{S}(\mathbf{B})| \leq c_2\, \gamma_2\, \lambda_1 |\mathbf{B}^{\mathrm{sym}}|^{p-1+\lambda_2}. \tag{3.37}$$

Remark 3.9. *It is quite remarkable that these estimates for the A–approximation and the $\boldsymbol{\lambda}$–approximation merge into one another if we choose $\rho = p - 2 + \lambda_2$ and $A^{-\lambda_2} = \lambda_1$. In this case*

$$\gamma_1 A^{p-2} = \gamma_1 \lambda_1^{\frac{2-p}{\lambda_2}} \qquad \text{and} \qquad \gamma_2 A^{p-2-\rho}|\mathbf{B}^{\mathrm{sym}}|^{1+\rho} = \gamma_2 \lambda_1 |\mathbf{B}^{\mathrm{sym}}|^{p-1+\lambda_2},$$

for all $\mathbf{B} \in \mathbb{R}^{n\times n}$, so (3.32)–(3.37) get unified. In addition, the conditions $A \geq 1$ and $\rho \geq 0$ correspond exactly to $\lambda_1 \in (0,1]$ and $\lambda_2 \geq 2-p$. From this point of view it is sufficient to investigate just one of the approximated potentials, namely F^A or $F^{\boldsymbol{\lambda}}$. We will do so by restricting our study of the numerics to the A–approximation. With a suitable choice of $\boldsymbol{\lambda}$, all the results will also hold for the $\boldsymbol{\lambda}$–approximation.

From (3.32)–(3.37) it follows in analogy to (3.7) and (3.8) that

$$\sum_{jklm} \sigma^A_{jklm}(\mathbf{B},\mathbf{C}) Q_{jk} Q_{lm} \geq \begin{cases} C\, \gamma_1\, A^{p-2}|\mathbf{Q}^{\mathrm{sym}}|^2, \\ C\, \gamma_1\, (1 + |\mathbf{B}^{\mathrm{sym}}|^2 + |\mathbf{C}^{\mathrm{sym}}|^2)^{\frac{p-2}{2}}\, |\mathbf{Q}^{\mathrm{sym}}|^2, \end{cases} \tag{3.38}$$

$$|\boldsymbol{\sigma}(\mathbf{B},\mathbf{C})| \leq C\, \gamma_2. \tag{3.39}$$

for all $\mathbf{B},\mathbf{C},\mathbf{Q} \in \mathbb{R}^{n\times n}$. Thus we deduce (compare theorem 3.5):

Theorem 3.10. *For all $\mathbf{B},\mathbf{C} \in \mathbb{R}^{n\times n}_{\mathrm{sym}}$ there holds*

$$\mathbf{S}^A(\mathbf{0}) = \mathbf{0}, \tag{3.40}$$

$$\begin{aligned} \sum_{ij}(S^A_{ij}(\mathbf{B}) - S^A_{ij}(\mathbf{C}))(B_{ij} - C_{ij}) &\geq \begin{cases} C\, \gamma_1\, A^{p-2}|\mathbf{B}-\mathbf{C}|^2\,, \\ C\, \gamma_1\, (1 + |\mathbf{B}|^2 + |\mathbf{C}|^2)^{\frac{p-2}{2}}\, |\mathbf{B}-\mathbf{C}|^2\,, \end{cases} \\ \sum_{ij} S^A_{ij}(\mathbf{B}) B_{ij} &\geq \begin{cases} C\, \gamma_1\, A^{p-2}|\mathbf{B}|^2\,, \\ C\, \gamma_1\, (1 + |\mathbf{B}|^2)^{\frac{p-2}{2}}\, |\mathbf{B}|^2\,, \\ C\, \gamma_1\, (2^{-\frac{1}{2}}|\mathbf{B}|^p - 1)\,, \end{cases} \end{aligned} \tag{3.41}$$

$$\begin{aligned} |\mathbf{S}^A(\mathbf{B}) - \mathbf{S}^A(\mathbf{C})| &\leq C\, \gamma_2 |\mathbf{B} - \mathbf{C}|, \\ |\mathbf{S}^A(\mathbf{B})| &\leq C\, \gamma_2\, |\mathbf{B}|. \end{aligned} \tag{3.42}$$

7. Assumption on the Exponent p

In the following sections we want to derive more useful properties of space and time dependent p-potentials using the properties of generalized Lebesgue and Sobolev spaces. Among others we will make use of the Sobolev embeddings on $W^{k,p(\cdot)}(\Omega)$. Since it is not yet known (see section 2 in chapter 2) which minimal requirements on p are necessary in order to ensure the validity of the Sobolev embeddings, we will restrict ourselves to the case where p is uniformly Lipschitz on the time space cylinder $I \times \Omega$. In this case lemma 2.4 ensures that the Sobolev embeddings are valid.

ASSUMPTION 3.11. *For all the following sections we assume that the exponent p of the space (and time) dependent potential Φ is uniformly Lipschitz on the space domain (time cylinder), i.e.*

$$p \in W^{1,\infty}(\Omega), \qquad \textit{resp.} \qquad p \in W^{1,\infty}(I \times \Omega),$$

and $1 < p_\infty \leq p_0 \leq 2$.

8. Special Energies

Later in our examinations of the stationary and instationary p-Navier–Stokes problem we will encounter the following two important expressions

$$\mathcal{I}_\Phi(t, \mathbf{u}) := \Big\langle \sum_r \sum_{jk\alpha\beta} (\partial_{\alpha\beta}\partial_{jk}\Phi)(t, x, \mathbf{Du})\partial_r D_{\alpha\beta}\mathbf{u}, \partial_r D_{jk}\mathbf{u} \Big\rangle, \tag{3.43}$$

$$\mathcal{J}_\Phi(t, \mathbf{u}) := \Big\langle \sum_{jk\alpha\beta} (\partial_{\alpha\beta}\partial_{jk}\Phi)(t, x, \mathbf{Du})\partial_t D_{\alpha\beta}\mathbf{u}, \partial_t D_{jk}\mathbf{u} \Big\rangle, \tag{3.44}$$

where $\mathbf{u}$ denotes a sufficiently smooth velocity function over the space time cylinder. The brackets $\langle \cdot, \cdot \rangle$ stand for integration over the space domain Ω. These two expressions will arise when we are going to test the equation of motions with $-\Delta\mathbf{u}$ and $\partial_t^2\mathbf{u}$. Since $\mathcal{I}_\Phi$ and $\mathcal{J}_\Phi$ are very similar, it is useful to introduce another functor $\mathcal{G}_\Phi$ by

$$\mathcal{G}_\Phi(t, \mathbf{w}, \mathbf{v}) := \Big\langle \sum_{jk\alpha\beta} (\partial_{\alpha\beta}\partial_{jk}\Phi)(t, x, \mathbf{Dw}) D_{\alpha\beta}\mathbf{v}, D_{jk}\mathbf{v} \Big\rangle, \tag{3.45}$$

where $\mathbf{w} : \Omega \to \mathbb{R}^d$ and $\mathbf{v} : \Omega \to \mathbb{R}^d$ (or $\mathbf{v} : \Omega \to \mathbb{R}^{d\times d}$) are sufficiently smooth functions. In most cases we will simply write $\mathcal{I}_\Phi(\mathbf{u})$, $\mathcal{J}_\Phi(\mathbf{u})$, and $\mathcal{G}_\Phi(\mathbf{w}, \mathbf{v})$ instead $\mathcal{I}_\Phi(t, \mathbf{u})$, $\mathcal{J}_\Phi(t, \mathbf{u})$, and $\mathcal{G}_\Phi(t, \mathbf{w}, \mathbf{v})$. With this convention we have

$$\mathcal{I}_\Phi(\mathbf{u}) = \mathcal{G}_\Phi(\mathbf{Du}, \boldsymbol{\nabla}\mathbf{u}), \qquad \mathcal{J}_\Phi(\mathbf{u}) = \mathcal{G}_\Phi(\mathbf{Du}, \partial_t\mathbf{u}). \tag{3.46}$$

Let Φ be a space (and time) dependent p-potential, then due to the properties of Φ we estimate

$$\mathcal{G}_\Phi(\mathbf{w}, \mathbf{v}) \geq \gamma_1 \int_\Omega (1 + |\mathbf{Dw}|^2)^{\frac{p-2}{2}} |\mathbf{Dv}|^2 \, dx.$$

The expression $(1 + |\mathbf{Dw}|^2)^{\frac{1}{2}}$ appears quite often in all the chapters, so it is very useful to introduce the shortcut

$$\widetilde{D}\mathbf{w} := (1 + |\mathbf{Dw}|^2)^{\frac{1}{2}}. \tag{3.47}$$

This gives

$$\mathcal{G}_\Phi(\mathbf{w}, \mathbf{v}) \geq \gamma_1 \int_\Omega (\widetilde{D}\mathbf{w})^{p-2} |\mathbf{Dv}|^2 \, dx. \tag{3.48}$$

As a consequence

$$\mathcal{I}_\Phi(\mathbf{u}) \geq C\,\gamma_1 \int_\Omega (\widetilde{D}\mathbf{u})^{p-2} |\boldsymbol{\nabla}\mathbf{Du}|^2 \, dx, \tag{3.49}$$

$$\mathcal{J}_\Phi(\mathbf{u}) \geq C\,\gamma_1 \int_\Omega (\widetilde{D}\mathbf{u})^{p-2} |\partial_t\mathbf{Du}|^2 \, dx. \tag{3.50}$$

Since in general $|\boldsymbol{\nabla}^2\mathbf{u}| \le 2\,|\boldsymbol{\nabla}\mathbf{Du}|$ (see Appendix), $|\boldsymbol{\nabla}\mathbf{Du}|$ can always be replaced by $|\boldsymbol{\nabla}^2\mathbf{u}|$ by increasing the multiplicative constant.

Closely connected to the quantities $\mathcal{I}_\Phi(\mathbf{u})$ and $\mathcal{J}_\Phi(\mathbf{u})$ is the function $(\widetilde{D}\mathbf{u})^{\frac{p}{2}}$, which will be very important when examining the regularity of solutions. This is the content of the following lemma:

Lemma 3.12. *Let Φ and p be as in assumption 3.11. Then there exists a constant $C > 0$, such that for all (sufficiently smooth) $\mathbf{u}$ and almost all times $t \in I$ there holds*

$$\gamma_1\|\boldsymbol{\nabla}\big((\widetilde{D}\mathbf{u})^{\frac{p}{2}}\big)\|_2^2 \le C\left(\mathcal{I}_\Phi(\mathbf{u}) + \|\boldsymbol{\nabla}p\|_\infty^2 \int_\Omega (\widetilde{D}\mathbf{u})^p \ln^2(\widetilde{D}\mathbf{u})\,dx\right), \tag{3.51}$$

$$\gamma_1\|\partial_t\big((\widetilde{D}\mathbf{u})^{\frac{p}{2}}\big)\|_2^2 \le C\left(\mathcal{J}_\Phi(\mathbf{u}) + \|\partial_t p\|_\infty^2 \int_\Omega (\widetilde{D}\mathbf{u})^p \ln^2(\widetilde{D}\mathbf{u})\,dx\right). \tag{3.52}$$

Proof. Observe that

$$\begin{aligned}\boldsymbol{\nabla}\big((\widetilde{D}\mathbf{u})^{\frac{p}{2}}\big) &= \sum_{jk} \tfrac{p}{2}(\widetilde{D}\mathbf{u})^{\frac{p-4}{2}}(D_{jk}\mathbf{u})\,(\boldsymbol{\nabla}D_{jk}\mathbf{u}) \\ &\quad + (\widetilde{D}\mathbf{u})^{\frac{p}{2}}\ln(\widetilde{D}\mathbf{u})\tfrac{1}{2}(\boldsymbol{\nabla}p).\end{aligned} \tag{3.53}$$

Raising this to the power of two and integrating over Ω proves the first inequality. If we replace $\boldsymbol{\nabla}$ in the calculations above by ∂_t, we get the result for $\mathcal{J}_\Phi(\mathbf{u})$. ☐

In the following we will derive more useful estimates for $\mathcal{G}_\Phi(\mathbf{w},\mathbf{v})$, $\mathcal{I}_\Phi(\mathbf{u})$ and $\mathcal{J}_\Phi(\mathbf{u})$:

Lemma 3.13. *Let Φ and p be as in assumption 3.11. Then for all (sufficiently smooth) $\mathbf{u}$ and almost all times $t \in I$ there holds*

$$\gamma_1|\boldsymbol{\nabla}^2\mathbf{u}|_{p(\cdot)} \le C\,\mathcal{I}_\Phi(\mathbf{u}) + \gamma_1|\widetilde{D}\mathbf{u}|_{p(\cdot)}, \tag{3.54}$$

$$\gamma_1|\partial_t\mathbf{Du}|_{p(\cdot)} \le C\,\mathcal{J}_\Phi(\mathbf{u}) + \gamma_1|\widetilde{D}\mathbf{u}|_{p(\cdot)}. \tag{3.55}$$

Proof. Note that for all $q \in [1,2]$, $a \ge 0$, $b \ge 1$ there holds

$$a^q \le a^2 b^{q-2} + b^q. \tag{3.56}$$

Indeed, there is nothing to prove if $q = 2$, so let $1 \le q < 2$. In this case $1 < \frac{2}{q} < \infty$, and Young's inequality gives

$$a^q = (a^2b^{q-2})^{\frac{q}{2}}(b^{\frac{(2-q)q}{2}}) \overset{\text{Young}}{\le} a^2b^{q-2} + b^q.$$

Now (3.56) implies

$$|\boldsymbol{\nabla}^2\mathbf{u}|^p \le (\widetilde{D}\mathbf{u})^{p-2}|\boldsymbol{\nabla}^2\mathbf{u}|^2 + (\widetilde{D}\mathbf{u})^p.$$

almost everywhere. Since in general $|\boldsymbol{\nabla}^2\mathbf{u}| \le 2|\boldsymbol{\nabla}\mathbf{D}\mathbf{u}|$ (see appendix) we deduce

$$\begin{aligned}|\boldsymbol{\nabla}^2\mathbf{u}|_{p(\cdot,t)} &\le \int_\Omega (\widetilde{D}\mathbf{u})^{p-2}|\boldsymbol{\nabla}^2\mathbf{u}|^2\,dx + |\widetilde{D}\mathbf{u}|_{p(\cdot,t)}\\ &\le 4\int_\Omega (\widetilde{D}\mathbf{u})^{p-2}|\boldsymbol{\nabla}\mathbf{D}\mathbf{u}|^2\,dx + |\widetilde{D}\mathbf{u}|_{p(\cdot,t)}\\ &\overset{(3.49)}{\le} \frac{C}{\gamma_1}\mathcal{I}_\Phi(\mathbf{u}) + |\widetilde{D}\mathbf{u}|_{p(\cdot,t)}.\end{aligned}$$

The estimate for $\partial_t\mathbf{D}\mathbf{u}$ follows analogously. □

LEMMA 3.14. *Let Φ and p be as in assumption 3.11. Then for all (sufficiently smooth) $\mathbf{u}$ and $\mathbf{v}$ and for all $1 \le q \le 2$ there holds:*

$$\|\mathbf{v}\|_q \le \tfrac{C}{\gamma_1}\left(\tfrac{1}{\gamma_1}\mathcal{G}_\Phi(\mathbf{w},\mathbf{v})\right)^{\frac12}\|(\widetilde{D}\mathbf{w})^{\frac{2-p}{2}}\|_{\frac{2q}{2-q}},$$

where $\frac{2q}{2-q} = \infty$ for $q = 2$.

PROOF. Observe that $1 \le \frac{2}{q} < \infty$ and $1 < (\frac{2}{q})' = \frac{2}{2-q} \le \infty$. Further for $1 \le q < 2$

$$\begin{aligned}\|\mathbf{v}\|_q^q &= \int_\Omega \left((\widetilde{D}\mathbf{w})^{p-2}|\mathbf{v}|^2\right)^{\frac{q}{2}}(\widetilde{D}\mathbf{w})^{\frac{(2-p)q}{2}}dx\\ &\le \left(\int_\Omega (\widetilde{D}\mathbf{w})^{p-2}|\mathbf{v}|^2dx\right)^{\frac{q}{2}}\|(\widetilde{D}\mathbf{w})^{\frac{(2-p)q}{2}}\|_{\frac{2}{2-q}}\\ &= \left(\int_\Omega (\widetilde{D}\mathbf{w})^{p-2}|\mathbf{v}|^2dx\right)^{\frac{q}{2}}\|(\widetilde{D}\mathbf{w})^{\frac{(2-p)}{2}}\|_{\frac{2q}{2-q}}^q.\end{aligned}$$

This and (3.48) prove the lemma for $q < 2$. The case $q = 2$ is similar. □

Note that this lemma is applicable to $\mathcal{I}_\Phi(\mathbf{u}) = \mathcal{G}_\Phi(\mathbf{D}\mathbf{u}, \boldsymbol{\nabla}\mathbf{u})$ and as well to $\mathcal{J}_\Phi(\mathbf{u}) = \mathcal{G}_\Phi(\mathbf{D}\mathbf{u}, \partial_t\mathbf{u})$. Analogously we have

LEMMA 3.15. *Let Φ and p be as in assumption 3.11. Then for all (sufficiently smooth) $\mathbf{u}$ and $\mathbf{v}$ and for all $1 \le q \le 2$ there holds:*

$$\|\mathbf{D}(\mathbf{u}-\mathbf{v})\|_q \le C\left\langle \mathbf{S}(\mathbf{D}\mathbf{u})-\mathbf{S}(\mathbf{D}\mathbf{v}), \mathbf{D}(\mathbf{u}-\mathbf{v})\right\rangle^{\frac12}\|(\widetilde{D}\mathbf{u})^{\frac{2-p}{2}} + (\widetilde{D}\mathbf{v})^{\frac{2-p}{2}}\|_{\frac{2q}{2-q}},$$

where $\frac{2q}{2-q} = \infty$ for $q = 2$.

PROOF. Analogously to the proof of lemma 3.14

$$\|\mathbf{D}(\mathbf{u}-\mathbf{v})\|_q^q = \int_\Omega \Big((\widetilde{D}\mathbf{u}+\widetilde{D}\mathbf{v})^{p-2}|\mathbf{D}(\mathbf{u}-\mathbf{v})|^2\Big)^{\frac{q}{2}} (\widetilde{D}\mathbf{u}+\widetilde{D}\mathbf{v})^{\frac{(2-p)q}{2}}\,dx$$

$$\le \Bigg(\int_\Omega (\widetilde{D}\mathbf{u}+\widetilde{D}\mathbf{v})^{p-2}|\mathbf{D}(\mathbf{u}-\mathbf{v})|^2 dx\Bigg)^{\frac{q}{2}} \Big\|(\widetilde{D}\mathbf{u}+\widetilde{D}\mathbf{v})^{\frac{(2-p)q}{2}}\Big\|_{\frac{2}{2-q}}$$

$$\overset{(3.10)}{\le} \frac{C}{\gamma_1}\big\langle \mathbf{S}(\mathbf{D}\mathbf{u})-\mathbf{S}(\mathbf{D}\mathbf{v}),\mathbf{D}(\mathbf{u}-\mathbf{v})\big\rangle^{\frac{q}{2}} \Big\|(\widetilde{D}\mathbf{u})^{\frac{(2-p)}{2}}+(\widetilde{D}\mathbf{v})^{\frac{(2-p)}{2}}\Big\|_{\frac{2q}{2-q}}^q.$$

This proves the lemma for $q<2$. The case $q=2$ is similar. □

In view of the A- and $\boldsymbol{\lambda}$-approximation, we define

$$\mathcal{G}_\Phi{}^A := \mathcal{G}_{\Phi^A}, \qquad \mathcal{I}_\Phi{}^A := \mathcal{I}_{\Phi^A}, \qquad \mathcal{J}_\Phi{}^A := \mathcal{J}_{\Phi^A},$$
$$\mathcal{G}_\Phi{}^{\boldsymbol{\lambda}} := \mathcal{G}_{\Phi^{\boldsymbol{\lambda}}}, \qquad \mathcal{I}_\Phi{}^{\boldsymbol{\lambda}} := \mathcal{I}_{\Phi^{\boldsymbol{\lambda}}}, \qquad \mathcal{J}_\Phi{}^{\boldsymbol{\lambda}} := \mathcal{J}_{\Phi^{\boldsymbol{\lambda}}}.$$

The estimates for Φ^A and $\Phi^{\boldsymbol{\lambda}}$ imply (note: $A\ge 1$, $\lambda_1\le 1$):

$$\mathcal{G}_\Phi{}^A(\mathbf{w},\mathbf{v}) \ge \begin{cases} C\,\gamma_1 A^{p_\infty-2}\int_\Omega |\mathbf{D}\mathbf{v}|^2\,dx, \\ C\,\gamma_1 \int_\Omega (\widetilde{D}\mathbf{w})^{p-2}|\mathbf{D}\mathbf{v}|^2\,dx, \end{cases} \tag{3.57}$$

$$\mathcal{G}_\Phi{}^{\boldsymbol{\lambda}}(\mathbf{w},\mathbf{v}) \ge \begin{cases} C\,\gamma_1 \lambda_1^{\frac{2-p_\infty}{\lambda_2}}\int_\Omega |\mathbf{D}\mathbf{v}|^2\,dx, \\ C\,\gamma_1 \int_\Omega (\widetilde{D}\mathbf{w})^{p-2}|\mathbf{D}\mathbf{v}|^2\,dx. \end{cases} \tag{3.58}$$

The consequences for $\mathcal{I}_\Phi{}^A$, $\mathcal{J}_\Phi{}^A$, $\mathcal{I}_\Phi{}^{\boldsymbol{\lambda}}$, $\mathcal{J}_\Phi{}^{\boldsymbol{\lambda}}$ are evident. Nevertheless for the sake of completeness we state them for Φ^A.

$$\mathcal{I}_\Phi{}^A(\mathbf{u}) \ge \begin{cases} C\,\gamma_1 A^{p_\infty-2}\int_\Omega |\boldsymbol{\nabla}\mathbf{D}\mathbf{u}|^2\,dx, \\ C\,\gamma_1 \int_\Omega (\widetilde{D}\mathbf{u})^{p-2}|\boldsymbol{\nabla}\mathbf{D}\mathbf{u}|^2\,dx, \end{cases} \tag{3.59}$$

$$\mathcal{J}_\Phi{}^A(\mathbf{u}) \ge \begin{cases} C\,\gamma_1 A^{p_\infty-2}\int_\Omega |\partial_t\mathbf{D}\mathbf{u}|^2\,dx, \\ C\,\gamma_1 \int_\Omega (\widetilde{D}\mathbf{u})^{p-2}|\partial_t\mathbf{D}\mathbf{u}|^2\,dx. \end{cases} \tag{3.60}$$

Note that in the derivation of lemma 3.12, 3.13, and 3.14 we have only used (3.49) and (3.50). Since we have just shown that $\mathcal{I}_\Phi{}^A(\mathbf{u})$ fulfills the same estimate with constants independent of A, both lemmas 3.12 and 3.13 remain valid if we replace $\mathcal{I}_\Phi(\mathbf{u})$ by $\mathcal{I}_\Phi{}^A(\mathbf{u})$. Certainly the same holds true for the $\boldsymbol{\lambda}$-approximation.

Later we will derive more interesting estimates connected to $\mathcal{I}_\Phi$, $\mathcal{J}_\Phi$ and their approximated versions, but most of them will depend on the dimension of the underlying space. Therefore we will postpone these estimates to the appropriate chapters.

CHAPTER 4

2D Flow – Pressure Stabilization

1. Introduction

In this chapter we will study solutions of the following instationary shear dependent flow problem in two space dimensions

$$
\begin{aligned}
\partial_t \mathbf{u} - \operatorname{div}(\mathbf{S}(\mathbf{D}\mathbf{u})) + (\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u} + \boldsymbol{\nabla}\pi &= \mathbf{f} && \text{on } I\times\Omega,\\
\operatorname{div}\mathbf{u} &= 0 && \text{on } I\times\Omega,\\
\mathbf{u}(0) &= \mathbf{u}_0 && \text{on } \Omega.
\end{aligned}
\tag{4.1}
$$

Here Ω denotes the two–dimensional torus, $I=[0,T]$ denotes a finite time interval. Furthermore $\mathbf{u}$ is the velocity, π the pressure, and $\mathbf{f}$ represents the data. We assume that the extra stress $\mathbf{S}$ is induced by a time and space dependent p–potential F and Φ with $p\in W^{1,\infty}(I\times\Omega)$, as we have defined in chapter 3, which additionally satisfies (3.28). Due to the space–periodic setting we will only consider data and solutions with zero mean value. This ensures the validity of the Poincaré inequality and the wellposedness of the inverse Laplacian equation.

These flows appear for example in the study of electrorheological fluids, where the behaviour of the fluids strongly depends on the applied electrical field. An important model for electrorheological fluids, which is closely connected to the system above, has been developed by K. R. Rajagopal and M. Růžička in [**RR96**]. Therein the exponent p is a function of $|\mathbf{E}|$, where $\mathbf{E}$ is the applied electrical field. See [**Růž00**] for a comprehensive work on such fluids.

In the view of numerics it is also quite of interest to study approximations of this system, which do not require the use of divergence free functions. One widely spread strategy is the use of pressure stabilization, that is $\operatorname{div}\mathbf{u}=0$ is replaced by $\operatorname{div}\mathbf{u}=\varepsilon\Delta\pi$ for some $\varepsilon>0$. This method is for example used in the Van–Kahn, the Chorin, and the Chorin–Uzawa scheme. For a description of these methods we refer the reader to the book [**Pro97**] by A. Prohl. The important observation for this approximation is that in the linear Stokes case $-\Delta\mathbf{u}+\boldsymbol{\nabla}\pi=\mathbf{f}$ the error due to the pressure stabilization is of the order ε. This fact is also valid for the linear Navier–Stokes problem if the convective term $(\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u}$ is adjusted accordingly. There are two canonical ways, each with its own advantages, how to adapt the convective part to the stabilization. Both of the systems have in common that the trilinear form b induced by the (adapted) convective term satisfies $b(\mathbf{u},\mathbf{u},\mathbf{u})=0$, which is of absolute importance for all analysis. Note that if $\operatorname{div}\mathbf{v}=0$, there holds $\langle(\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{w},\mathbf{w}\rangle=0$ for arbitrary $\mathbf{w}$. But since $\operatorname{div}\mathbf{u}=\varepsilon\Delta\pi\not\equiv 0$ we have $\langle(\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u},\mathbf{u}\rangle\not\equiv 0$. The first way to compensate this is the anti–symmetrization of the trilinear form with respect to the last component, i.e.

$$\begin{aligned} b(\mathbf{u},\mathbf{v},\mathbf{w}) &:= \tfrac{1}{2}\big\langle(\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{v},\mathbf{w}\big\rangle - \langle(\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{w},\mathbf{v}\rangle \\ &= \big\langle(\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{v} + \tfrac{1}{2}(\operatorname{div}\mathbf{u})\mathbf{v},\mathbf{w}\big\rangle. \end{aligned}$$

The other possibility is to use

$$b(\mathbf{u},\mathbf{v},\mathbf{w}) := \langle((\mathbf{u}-\varepsilon\boldsymbol{\nabla}\pi)\cdot\boldsymbol{\nabla})\mathbf{v},\mathbf{w}\rangle,$$

which corresponds to the projection of the first component in the trilinear form $\langle(\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{v},\mathbf{w}\rangle$ to the space of divergence free functions. Note that both adapted versions of the convective term are used within the area of numerics. Therefore in addition to (4.1) we will study the systems

$$\begin{aligned} \partial_t\mathbf{u} - \operatorname{div}(\mathbf{S}(\mathbf{D}\mathbf{u})) + (\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u} + \tfrac{1}{2}(\operatorname{div}\mathbf{u})\mathbf{u} + \boldsymbol{\nabla}\pi &= \mathbf{f} && \text{on } I\times\Omega, \\ \operatorname{div}\mathbf{u} &= \varepsilon\Delta\pi && \text{on } I\times\Omega, \\ \mathbf{u}(0) &= \mathbf{u}_0 && \text{on } \Omega, \end{aligned} \tag{4.2}$$

and

$$\begin{aligned} \partial_t\mathbf{u} - \operatorname{div}(\mathbf{S}(\mathbf{D}\mathbf{u})) + \big((\mathbf{u}-\varepsilon\boldsymbol{\nabla}\pi)\cdot\boldsymbol{\nabla}\big)\mathbf{u} + \boldsymbol{\nabla}\pi &= \mathbf{f} && \text{on } I\times\Omega, \\ \operatorname{div}\mathbf{u} &= \varepsilon\Delta\pi && \text{on } I\times\Omega, \\ \mathbf{u}(0) &= \mathbf{u}_0 && \text{on } \Omega. \end{aligned} \tag{4.3}$$

In particular we will address the problem of existence, uniqueness, and $C^{1,\alpha}$–regularity of strong solutions for large times and data. Let us mention once again that these systems unify with (4.1) if $\varepsilon = 0$.

In the case of no stabilization, i.e. $\varepsilon = 0$, and p constant these questions have been examined by P. Kaplický, J. Málek and J. Stará. In [**KMS97b**] they show that for zero initial data, i.e. $\mathbf{u}_0 = \mathbf{0}$, and $p > \frac{4}{3}$ there exists a unique, strong solution of (4.1), which satisfies $\boldsymbol{\nabla}\mathbf{u} \in C^{0,\alpha}(I\times\Omega)$ for some $\alpha > 0$. Note that for the stationary system P. Kaplický, J. Málek and J. Stará have also studied the case of zero Dirichlet boundary conditions (see [**KMS99**] and [**KMS97a**]).

We will see that by refining the ideas of [**KMS97b**], it is possible to extend these results to the case of p non constant, i.e. $p \in W^{1,\infty}(I\times\Omega)$, as long as $\frac{3}{2} < p_\infty \le p_0 \le 2$. Particularly we will prove that the systems above, i.e. (4.1), (4.2), and (4.3), have a unique, strong solution, which satisfies $\boldsymbol{\nabla}\mathbf{u} \in C^{0,\alpha}(I\times\Omega)$ for some $\alpha > 0$, where α does not depend on ε.

Furthermore in contrast to [**KMS97b**] we have included the possibility of non–zero initial data, since this is quite crucial for the application to numerics. Moreover our result above also covers the pressure stabilized setting. Thereby we carefully pay attention that the norms involved do not depend on ε as long as $0 \le \varepsilon \le \varepsilon_0$, where ε_0 does only depend on the exponent p, the data f, and the initial data $\mathbf{u}_0$. So the theorem below provides all necessary information regarding regularity to establish a numerical error analysis for a pressure stabilized time discretization.

Provided with the existence, uniqueness, and regularity of the stabilized systems we will investigate the order of perturbation with respect to ε. We will see that, as in the linear Navier–Stokes case, the error, measured in the $L^{p(\cdot)}(I\times\Omega)$ norm, is of order ε. The proof of this is related very closely to the existence of strong solutions to the dual problem of the error. While for the linear Stokes problem the equation of the error and its dual problem is just another Stokes problem, this is not the case in our non–linear setting. More precisely, the dual problem to the error equation has the form

$$\begin{aligned} \partial_t \mathbf{w} + \operatorname{div}(\boldsymbol{\sigma}(\mathbf{Du}, \mathbf{Dv})\mathbf{Dw}) + \boldsymbol{\nabla}\eta + \mathbf{H}(\mathbf{w}) &= \mathbf{g} && \text{on } I \times \Omega, \\ \operatorname{div} \mathbf{w} &= 0 && \text{on } I \times \Omega, \\ \mathbf{w}(T) &= 0 && \text{on } \Omega, \end{aligned}$$

for some function $\mathbf{g}$ and a linear operator $\mathbf{H}$, where $\mathbf{u}$, resp. $\mathbf{v}$, are the solutions to the non–stabilized, resp. the stabilized, p–Stokes system. Unfortunately it is not clear if this system has a strong solution under the sole condition $\mathbf{u}, \mathbf{v} \in C(I, W^{2,p(\cdot)}(\Omega))$. In order to get strong solutions of the dual problem we need $C^{0,\alpha}(I \times \Omega)$ regularity $(\alpha > 0)$ of $\boldsymbol{\nabla}\mathbf{u}$ and $\boldsymbol{\nabla}\mathbf{v}$. Luckily, the results above show that this is true for two space dimensions. But for higher space dimensions (three and above) this is indeed a grave problem, as it is not known if gradients of strong solutions to the p–Stokes system are in $C^{0,\alpha}(I \times \Omega)$.

2. Stokes Flow — Weak Solutions

In this section we will examine the existence and regularity of weak solutions to the pressure stabilized, instationary, generalized Stokes problem

$$\begin{aligned} \partial_t \mathbf{u} - \operatorname{div}(\mathbf{TDu}) + \boldsymbol{\nabla}\pi &= \operatorname{div} \mathbf{G} && \text{on } I \times \Omega, \\ \operatorname{div} \mathbf{u} &= \varepsilon \Delta \pi && \text{on } I \times \Omega, \\ \mathbf{u}(0) &= \mathbf{u}_0 && \text{on } \Omega, \end{aligned} \tag{4.4}$$

where Ω denotes the d–dimensional torus, $I = [0, T]$ a finite time interval, and $\varepsilon \geq 0$. Moreover, $\mathbf{T} : I \times \Omega \to \mathbb{R}^{d\times d\times d\times d}$ satisfies $\mathbf{T} \in L^\infty(I \times \Omega)$,

$$T_{ij,kl} = T_{ji,kl} = T_{ij,lk} = T_{kl,ij},$$

and for all $\mathbf{B} \in \mathbb{R}^{d\times d}$

$$\gamma_1 |\mathbf{B}^{\text{sym}}|^2 \leq \sum_{jklm} T_{jk,lm} B_{jk} B_{lm} \leq \gamma_2 |\mathbf{B}^{\text{sym}}|^2.$$

We use $\mathbf{TDu}$ as the abbreviated form of $(\mathbf{TDu})_{lm} = \sum_{jk} T_{jk,lm} D_{lm}\mathbf{u}$. Note that the symmetry of $\mathbf{T}$ implies $\mathbf{TDu} = \mathbf{T}\boldsymbol{\nabla}\mathbf{u}$. Nevertheless we chose to use $\mathbf{TDu}$ in order to point out the fundamental structure of the problem. Further note that section 2 is valid for any space dimension $d \geq 2$. But throughout the other sections we will assume $d = 2$.

We are looking for solutions with space mean value zero, i.e.

$$\langle u_i(t), 1\rangle = \langle \pi(t), 1\rangle = 0 \qquad \text{for all times } t \in I = [0, T]$$

and $i = 1, \ldots, d$. So additionally to the Lebesgue spaces $L^q(\Omega)$ and the Sobolev spaces $W^{k,q}(\Omega)$ we need the following versions with mean value zero:

$$\begin{aligned} L^q_0(\Omega) &:= \{f \in L^q(\Omega) \,:\, \langle f, 1\rangle = 0\}, \\ W^{k,q}_0(\Omega) &:= \{f \in W^{k,q}(\Omega) \,:\, \langle f, 1\rangle = 0\} \text{ for } k \geq 0. \end{aligned}$$

Analogously we define the generalized versions

$$\begin{aligned} L^{q(\cdot)}_0(\Omega) &:= \{f \in L^{q(\cdot)}(\Omega) \,:\, \langle f, 1\rangle = 0\}, \\ W^{k,q(\cdot)}_0(\Omega) &:= \{f \in W^{k,q(\cdot)}(\Omega) \,:\, \langle f, 1\rangle = 0\} \text{ for } k \geq 0. \end{aligned}$$

With these spaces $\Delta : W_0^{s+2,2}(\Omega) \to W_0^{s,2}(\Omega)$ is an isomorphism for all $s \in \mathbb{R}$, which is a standard result from Fourier analysis on the torus. Further we need the spaces:

$$\begin{aligned}
\mathcal{D}(\Omega) &:= C_0^\infty(\Omega) \qquad \text{(smooth functions with compact support)},\\
\mathcal{D}_0(\Omega) &:= \{\boldsymbol{\varphi} \in \mathcal{D}(\Omega) \,:\, \langle \varphi_i, 1\rangle = 0 \text{ for } i = 1,\dots,d\},\\
\mathcal{V}_0 &:= \{\boldsymbol{\varphi} \,:\, \boldsymbol{\varphi} \in \mathcal{D}_0(\Omega), \operatorname{div}\boldsymbol{\varphi} = 0\},\\
W^{k,q}_{\mathrm{div},0}(\Omega) &:= \{\mathbf{u} \in W_0^{k,q}(\Omega) \,:\, \operatorname{div}\mathbf{u} = 0\},\\
L^q_{\mathrm{div},0}(\Omega) &:= \{\mathbf{u} \in L^q_0(\Omega) \,:\, \operatorname{div}\mathbf{u} = 0 \text{ (as a distribution)}\},
\end{aligned}$$

where $k \geq 0$, $1 \leq q \leq \infty$. Moreover, to handle the traces of $\mathbf{u}$ with respect to the time we need the Besov spaces

$$B^s_{p,q}(\Omega) \qquad \text{and} \qquad B^s_p(\Omega) := B^s_{p,p}(\Omega)$$

with $p, q \in (1,\infty)$ and $s \in \mathbb{R}$, which can be defined by real interpolation of the spaces $W^{k,p}(\Omega)$, i.e.

$$B^s_{p,q}(\Omega) = [W^{k_1,p}(\Omega), W^{k_2,p}(\Omega)]_{\theta,q}$$

for $0 < \theta < 1$, $1 < q < \infty$, $k_1 \neq k_2$, and $s = (1-\theta)k_1 + \theta k_2$. For more details regarding Besov spaces see Bergh, Löfström [**BL76**] and Triebel [**Tri78**]. We write $B^s_{p,\mathrm{div}}(\Omega)$ for the set of functions $\mathbf{f} \in B^s_p(\Omega)$ with $\operatorname{div}\mathbf{f} = 0$ (in the sense of distributions).

We say $\mathbf{u}, \pi$ is a weak solution of (4.4) if and only if $\mathbf{u} \in C(I, L^2_0(\Omega))$, $\mathbf{u} \in L^2(I, W_0^{1,2}(\Omega))$, $\pi \in L^2(I, L^2(\Omega))$, and (4.4) is satisfied in the sense of distributions. The following lemma proves existence of weak solutions.

LEMMA 4.1. *Let $\mathbf{u}_0 \in L^2_{\mathrm{div},0}(\Omega)$, $\mathbf{G} \in L^2(I, L^2(\Omega; \mathbb{R}^{d\times d}))$, and let $\mathbf{T}$ be as described above with ellipticity constants γ_1, γ_2. Let $\mathbf{G}^{\mathrm{sym}}$ denote the symmetric part, i.e. $\mathbf{G}^{\mathrm{sym}} = \frac{1}{2}(\mathbf{G} + \mathbf{G}^T)$, and $\mathbf{G}^{\mathrm{anti}}$ the antisymmetric part of $\mathbf{G}$, i.e. , i.e. $\mathbf{G}^{\mathrm{anti}} = \frac{1}{2}(\mathbf{G} - \mathbf{G}^T)$. Then for all $\varepsilon \geq 0$ there exists a unique weak solution $\mathbf{u}, \pi$ of (4.4) with*

$$\begin{aligned}
&\|\mathbf{u}\|^2_{C(I,L^2(\Omega))} + (1-\mu)\gamma_1 \|\mathbf{D}\mathbf{u}\|^2_{L^2(I,L^2(\Omega))}\\
&\qquad \leq \tfrac{1}{\gamma_1}\|\mathbf{G}^{\mathrm{sym}}\|^2_{L^2(I,L^2(\Omega))} + \tfrac{C}{\gamma_1\mu}\|\mathbf{G}^{\mathrm{anti}}\|^2_{L^2(I,L^2(\Omega))} + C\,\|\mathbf{u}_0\|_2^2, \qquad (4.5)\\
&\gamma_1\|\nabla\mathbf{u}\|^2_{L^2(I,L^2(\Omega))} \leq \tfrac{C}{\gamma_1}\|\mathbf{G}\|^2_{L^2(I,L^2(\Omega))} + C\,\|\mathbf{u}_0\|_2^2,
\end{aligned}$$

for all $\mu > 0$ where C does not depend on ε, μ nor T. Further there holds $\mathbf{u} \in C(I, L^2_0(\Omega))$.

Moreover, if $\varepsilon = 0$ or $\mathbf{T} = \gamma_2\,\mathrm{Id}^{\mathrm{sym}}$, i.e. $T_{jk,lm} = \frac{1}{2}(\delta_{jk,lm} + \delta_{jk,ml})$, then

$$\|\pi\|_{L^2(I,L^2(\Omega))} \leq C\left(\tfrac{\gamma_2}{\gamma_1}\|\mathbf{G}\|_{L^2(I,L^2(\Omega))} + \tfrac{\gamma_2}{\sqrt{\gamma_1}}\|\mathbf{u}_0\|_2\right) \qquad (4.6)$$

with constants independent of ε and T. (if $\varepsilon \neq 0$ and $\mathbf{T} \neq \gamma_2\,\mathrm{Id}^{\mathrm{sym}}$ we only prove $\pi \in L^2(I, L^2_0(\Omega))$ without a norm estimate that is dependent on ε.)

PROOF. Let us first assume that $\gamma_2 = 1$. Later we will see that the general case follows from the case $\gamma_2 = 1$ by a scaling argument in time.

Case $\gamma_2 = 1, \varepsilon = 0$: Let $\mathbb{S}$ denote the Stokes operator, then there exists (see [**MNRR96**]) a countable set $\{\boldsymbol{\omega}^r\}$ of eigenfunctions with corresponding eigenvalues λ_r, which is orthonormal with respect to the $L^2(\Omega)$ scalar product. Define X_N by $X_N := \operatorname{span}\{\boldsymbol{\omega}^1, \dots, \boldsymbol{\omega}^N\}$. Let us recall that we are only looking for (space periodic) solutions with mean value zero. So the $\boldsymbol{\omega}^r$ all fulfill $\langle\boldsymbol{\omega}^r, 1\rangle = 0$. Define $P^N\mathbf{u} := \sum_{r=1}^N \langle\mathbf{u}, \boldsymbol{\omega}^r\rangle\boldsymbol{\omega}^r$, then

$$\lambda_r\langle\mathbf{u}^N, \boldsymbol{\omega}^r\rangle = \langle\mathbf{u}^N, \mathbb{S}\boldsymbol{\omega}^r\rangle = \langle\boldsymbol{\nabla}\mathbf{u}^N, \boldsymbol{\nabla}\boldsymbol{\omega}^r\rangle$$

and the $P^N : W_0^{s,2} \to (X_N, \|\cdot\|_{s,2})$ are uniformly continuous for all $0 \le s \le 2$. (See [**MNRR96**] for a proof.)

Let us define $\mathbf{u}^N(t,x) = \sum_{r=1}^N c_r^N(t)\boldsymbol{\omega}^r(x)$, where the coefficients $c_r^N(t)$ solve the Galerkin system (for all $1 \le r \le N$)

$$\begin{aligned} \big\langle\partial_t\mathbf{u}^N, \boldsymbol{\omega}^r\big\rangle + \langle\mathbf{T}\mathbf{D}\mathbf{u}^N, \mathbf{D}\boldsymbol{\omega}^r\rangle &= \langle\mathbf{G}, \boldsymbol{\nabla}\boldsymbol{\omega}^r\rangle, \\ \mathbf{u}^N(0) &= P^N\mathbf{u}_0. \end{aligned} \tag{4.7}$$

Since the matrix $\langle\boldsymbol{\omega}_j, \boldsymbol{\omega}_k\rangle$ with $j, k = 1, \dots, N$ is positive definite, this can be rewritten as a system of ordinary differential equations. This in turn fulfills the Carathéodory conditions and is therefore solvable locally in time, i.e. on a small time interval $I^* = [0, T^*)$. Since $\mathbf{G} \in L^2(I^*, L_0^2(\Omega))$ we have $c_r^N, \partial_t c_r^N \in L^2(I^*)$ (norms may depend on N). This implies $\mathbf{u}^N, \partial_t\mathbf{u}^N \in L^2(I^*, X_N)$. To ensure solvability for large times at least for this finite dimensional problem we have to establish a first a priori estimate.

Since $\mathbf{u}^N, \partial_t\mathbf{u}^N \in L^2(I^*, X_N)$, we can test the Galerkin system (4.7) with $\mathbf{u}^N$ and get

$$\tfrac{1}{2}d_t\|\mathbf{u}^N\|_2^2 + \langle\mathbf{T}\mathbf{D}\mathbf{u}^N, \mathbf{D}\mathbf{u}^N\rangle = \langle\mathbf{G}, \boldsymbol{\nabla}\mathbf{u}^N\rangle$$

Thus

$$\begin{aligned} \tfrac{1}{2}d_t\|\mathbf{u}^N\|_2^2 + \langle\mathbf{T}\mathbf{D}\mathbf{u}^N, \mathbf{D}\mathbf{u}^N\rangle &= \langle\mathbf{G}^{\mathrm{sym}}, \boldsymbol{\nabla}\mathbf{u}^N\rangle + \langle\mathbf{G}^{\mathrm{anti}}, \boldsymbol{\nabla}\mathbf{u}^N\rangle. \\ &= \langle\mathbf{G}^{\mathrm{sym}}, \mathbf{D}\mathbf{u}^N\rangle + \langle\mathbf{G}^{\mathrm{anti}}, \boldsymbol{\nabla}\mathbf{u}^N\rangle. \end{aligned} \tag{4.8}$$

The coercivity of $\mathbf{T}$ and Young's inequality implies

$$\begin{aligned} \tfrac{1}{2}d_t\|\mathbf{u}^N\|_2^2 + \gamma_1\|\mathbf{D}\mathbf{u}^N\|_2^2 \le \tfrac{1}{2\gamma_1}\|\mathbf{G}^{\mathrm{sym}}\|_2^2 &+ \tfrac{\gamma_1}{2}\|\mathbf{D}\mathbf{u}^N\|_2^2 \\ &+ \tfrac{1}{2\gamma_1\delta}\|\mathbf{G}^{\mathrm{anti}}\|_2^2 + \tfrac{\delta\gamma_1}{2}\|\boldsymbol{\nabla}\mathbf{u}^N\|_2^2 \end{aligned}$$

for all $\delta > 0$. Due to Korn's inequality there holds $\|\boldsymbol{\nabla}\mathbf{u}^N\|_2 \le C\,\|\mathbf{D}\mathbf{u}^N\|_2$. Thus for all $\mu > 0$ there holds

$$\begin{aligned} \tfrac{1}{2}d_t\|\mathbf{u}^N\|_2^2 + \gamma_1\|\mathbf{D}\mathbf{u}^N\|_2^2 \le \tfrac{1}{2\gamma_1}\|\mathbf{G}^{\mathrm{sym}}\|_2^2 &+ \tfrac{\gamma_1}{2}\|\mathbf{D}\mathbf{u}^N\|_2^2 \\ &+ \tfrac{C}{2\gamma_1\mu}\|\mathbf{G}^{\mathrm{anti}}\|_2^2 + \tfrac{\mu\gamma_1}{2}\|\mathbf{D}\mathbf{u}^N\|_2^2. \end{aligned}$$

This implies

$$d_t\|\mathbf{u}^N\|_2^2 + \gamma_1(1-\mu)\|\mathbf{D}\mathbf{u}^N\|_2^2 \le \tfrac{1}{\gamma_1}\|\mathbf{G}^{\mathrm{sym}}\|_2^2 + \tfrac{C}{\gamma_1\mu}\|\mathbf{G}^{\mathrm{anti}}\|_2^2.$$

Integration over $(0,t)$ with varying $t \in I^*$ implies

$$\begin{aligned} &\sup_{I^*}\|\mathbf{u}^N\|_2^2 + \gamma_1(1-\mu)\|\mathbf{D}\mathbf{u}^N\|_{L^2(I^*,L^2(\Omega))}^2 \\ &\quad\le \tfrac{1}{\gamma_1}\|\mathbf{G}^{\mathrm{sym}}\|_{L^2(I^*,L^2(\Omega))}^2 + \tfrac{C}{\gamma_1\mu}\|\mathbf{G}^{\mathrm{anti}}\|_{L^2(I^*,L^2(\Omega))}^2 + \|\mathbf{u}_0^N\|_2^2 \\ &\quad\le \tfrac{1}{\gamma_1}\|\mathbf{G}^{\mathrm{sym}}\|_{L^2(I^*,L^2(\Omega))}^2 + \tfrac{C}{\gamma_1\mu}\|\mathbf{G}^{\mathrm{anti}}\|_{L^2(I^*,L^2(\Omega))}^2 + \|P^N\mathbf{u}_0\|_2^2 \\ &\quad\le \tfrac{1}{\gamma_1}\|\mathbf{G}^{\mathrm{sym}}\|_{L^2(I,L^2(\Omega))}^2 + \tfrac{C}{\gamma_1\mu}\|\mathbf{G}^{\mathrm{anti}}\|_{L^2(I,L^2(\Omega))}^2 + C\,\|\mathbf{u}_0\|_2^2. \end{aligned} \tag{4.9}$$

Note that the constants do not depend on N and T^*. Since $\langle \mathbf{u}^N, \boldsymbol{\omega}^r\rangle = c_r^N$ and $\|\boldsymbol{\omega}^r\|_2 = 1$, this implies (independent of N, T^*)

$$\|c_r^N\|_{L^\infty(I^*)} \le C, \qquad r = 1, \dots, N. \tag{4.10}$$

Let $t_0 \in I^* \cap I$, then (4.10) implies that there exists an $\tau > 0$ (independent of T^*), such that (4.7) has a solution on $[t_0, t_0+\tau)$ with initial value $c_r^N(t_0)$. Iteration of this argument implies that (4.7) is solvable on the whole interval I. As a consequence we can replace I^* by I before intgrating over the time interval. This proves that the a priori estimates (4.9) remain valid if I^* is replaced by I, i.e.

$$\begin{aligned}\sup_I \|\mathbf{u}^N\|_2^2 &+ \gamma_1(1-\mu)\|\mathbf{D}\mathbf{u}^N\|^2_{L^2(I,L^2(\Omega))} \\ &\le \tfrac{1}{\gamma_1}\|\mathbf{G}^{\mathrm{sym}}\|^2_{L^2(I,L^2(\Omega))} + \tfrac{C}{\gamma_1\mu}\|\mathbf{G}^{\mathrm{anti}}\|^2_{L^2(I,L^2(\Omega))} + C\|\mathbf{u}_0\|_2^2\end{aligned} \tag{4.11}$$

with constants independent of T. With Korn's inequality and the special choice $\mu = \frac{1}{2}$ there follows

$$\gamma_1\|\boldsymbol{\nabla}\mathbf{u}^N\|^2_{L^2(I,L^2(\Omega))} \le \tfrac{C}{\gamma_1}\|\mathbf{G}\|^2_{L^2(I,L^2(\Omega))} + C\|\mathbf{u}_0\|_2^2. \tag{4.12}$$

Thus there exists a subsequence (still denoted by $\mathbf{u}^N$) and a function $\mathbf{u}$, such that

$$\mathbf{u}^N \overset{*}{\rightharpoonup} \mathbf{u} \qquad \text{in } L^\infty(I, L^2(\Omega)), \tag{4.13}$$

$$\boldsymbol{\nabla}\mathbf{u}^N \rightharpoonup \boldsymbol{\nabla}\mathbf{u} \qquad \text{in } L^2(I, L^2_0(\Omega)). \tag{4.14}$$

Since the eigenfunctions $\{\boldsymbol{\omega}^r\}$ are orthonormal, the operator P^N is an orthogonal, self-adjoint projection with respect to the $L^2(\Omega)$ scalar product. Let $\boldsymbol{\varphi} \in L^2(I, W^{1,2}_0(\Omega))$ with $\|\boldsymbol{\varphi}\|_{L^2(I,W_0^{1,2}(\Omega))} \le 1$, then

$$\begin{aligned}\int_I \langle \partial_t \mathbf{u}^N, \boldsymbol{\varphi}\rangle\, dt &= \int_I \langle P^N\partial_t \mathbf{u}^N, \boldsymbol{\varphi}\rangle\, dt = \int_I \langle \partial_t \mathbf{u}^N, P^N\boldsymbol{\varphi}\rangle\, dt \\ &\overset{(4.7)}{=} \int_I -\langle \mathbf{T}\mathbf{D}\mathbf{u}^N, \mathbf{D}P^N\boldsymbol{\varphi}\rangle + \langle \mathbf{G}, \boldsymbol{\nabla}P^N\boldsymbol{\varphi}\rangle\, dt.\end{aligned}$$

Thus (recall $\gamma_2 = 1$)

$$\begin{aligned}\Big|\int_I \langle \partial_t \mathbf{u}^N, \boldsymbol{\varphi}\rangle\, dt\Big| &\le \big(\|\boldsymbol{\nabla}\mathbf{u}^N\|_{L^2(I,L^2(\Omega))} + \|\mathbf{G}\|_{L^2(I,L^2(\Omega))}\big)\|P^N\boldsymbol{\varphi}\|_{L^2(I,W_0^{1,2}(\Omega))} \\ &\le \big(\|\boldsymbol{\nabla}\mathbf{u}^N\|_{L^2(I,L^2(\Omega))} + \|\mathbf{G}\|_{L^2(I,L^2(\Omega))}\big) C\,\|\boldsymbol{\varphi}\|_{L^2(I,W_0^{1,2}(\Omega))} \\ &\le C\,\big(\|\boldsymbol{\nabla}\mathbf{u}^N\|_{L^2(I,L^2(\Omega))} + \|\mathbf{G}\|_{L^2(I,L^2(\Omega))}\big) \\ &\overset{(4.12)}{\le} C\left(\tfrac{1}{\gamma_1}\|\mathbf{G}\|_{L^2(I,L^2(\Omega))} + \tfrac{1}{\sqrt{\gamma_1}}\|\mathbf{u}_0\|_2\right).\end{aligned}$$

This proves

$$\begin{aligned}\|\partial_t\mathbf{u}^N\|_{L^2(I,(W_0^{1,2}(\Omega))')} &= \|\partial_t\mathbf{u}^N\|_{(L^2(I,W_0^{1,2}(\Omega)))'} \\ &\le C\left(\tfrac{1}{\gamma_1}\|\mathbf{G}\|_{L^2(I,L^2(\Omega))} + \tfrac{1}{\sqrt{\gamma_1}}\|\mathbf{u}_0\|_2\right).\end{aligned} \tag{4.15}$$

Thus passing to a subsequence (still denoted by $\mathbf{u}^N$) we get

$$\partial_t\mathbf{u}^N \rightharpoonup \partial_t\mathbf{u} \qquad \text{in } L^2(I, (W_0^{1,2}(\Omega))'), \tag{4.16}$$

where we have used that the limit in $\mathcal{D}(I\times\Omega)'$ is unique. Now (4.14) and (4.16) imply

$$\mathbf{u} \in L^2(I, W_0^{1,2}(\Omega)),$$
$$\partial_t \mathbf{u} \in L^2(I, (W_0^{1,2}(\Omega))').$$

Thus by parabolic interpolation

$$\mathbf{u} \in C(I, L_0^2(\Omega)).$$

Moreover due to (4.14), (4.16), and the lemma of Aubin–Lions (see lemma 8.1) there exists a subsequence (still denoted by $\mathbf{u}^N$), such that

$$\mathbf{u}^N \to \mathbf{u} \qquad \text{in } L^2(I, (W_0^{1,2}(\Omega))'). \tag{4.17}$$

This and (4.16) imply

$$\mathbf{u}^N \to \mathbf{u} \qquad \text{in } C(I, (W_0^{1,2}(\Omega))'). \tag{4.18}$$

Since $\mathbf{u}^N(0) = P^N\mathbf{u}_0 \to \mathbf{u}_0$ in $L^2(\Omega)$, we deduce $\mathbf{u}(0) = \mathbf{u}_0$, so $\mathbf{u}$ has the correct initial value. Since system (4.7) is linear we further deduce from (4.13), (4.14), (4.16), (4.17), and (4.18) that

$$\big\langle \partial_t \mathbf{u}, \boldsymbol{\omega}^r \big\rangle + \langle \mathbf{TDu}, \mathbf{D}\boldsymbol{\omega}^r\rangle = \langle \mathbf{G}, \boldsymbol{\nabla}\boldsymbol{\omega}^r\rangle \qquad \text{in } \mathcal{D}(I)' \tag{4.19}$$

for all $\boldsymbol{\omega}^r$. Since $\operatorname{div}\mathbf{u}^N = 0$ we also get $\operatorname{div}\mathbf{u} = 0$. Furthermore there holds by (4.11), (4.12), and (4.15)

$$\begin{aligned}\sup_I \|\mathbf{u}\|_2^2 &+ \gamma_1(1-\mu)\|\mathbf{Du}\|^2_{L^2(I,L^2(\Omega))} \\ &\le \tfrac{1}{\gamma_1}\|\mathbf{G}^{\text{sym}}\|^2_{L^2(I,L^2(\Omega))} + \tfrac{C}{\gamma_1\mu}\|\mathbf{G}^{\text{anti}}\|^2_{L^2(I,L^2(\Omega))} + C\|\mathbf{u}_0\|_2^2,\end{aligned} \tag{4.20}$$

$$\|\partial_t \mathbf{u}\|_{L^2(I,(W_0^{1,2}(\Omega))')} \le C\left(\tfrac{1}{\gamma_1}\|\mathbf{G}\|_{L^2(I,L^2(\Omega))} + \tfrac{1}{\sqrt{\gamma_1}}\|\mathbf{u}_0\|_2\right), \tag{4.21}$$

$$\gamma_1\|\boldsymbol{\nabla}\mathbf{u}\|^2_{L^2(I,L^2(\Omega))} \le \tfrac{C}{\gamma_1}\|\mathbf{G}\|^2_{L^2(I,L^2(\Omega))} + C\|\mathbf{u}_0\|_2^2. \tag{4.22}$$

Since $\bigcup_N \mathcal{D}(I, X_N)$ is dense in $\mathcal{D}(I, W^{1,2}_{\text{div},0}(\Omega))$ (see [**MNRR96**]) and $\mathbf{u}$ satisfies (4.21) and (4.22) there holds

$$\partial_t \mathbf{u} - \operatorname{div}(\mathbf{TDu}) = \operatorname{div}\mathbf{G} \qquad \text{in } \big(\mathcal{D}(I, W^{1,2}_{\text{div},0}(\Omega))\big)'. \tag{4.23}$$

Furthermore from (4.21) and (4.22) we deduce

$$\|\partial_t \mathbf{u} - \operatorname{div}(\mathbf{TDu})\|_{L^2(I,(W_0^{1,2}(\Omega))')} \le C\left(\tfrac{1}{\gamma_1}\|\mathbf{G}\|_{L^2(I,L^2(\Omega))} + \tfrac{1}{\sqrt{\gamma_1}}\|\mathbf{u}_0\|_2\right).$$

Moreover,

$$\|\operatorname{div}\mathbf{G}\|_{L^2(I,(W_0^{1,2}(\Omega))')} \le C\,\|\mathbf{G}\|_{L^2(I,L^2(\Omega))} \le \tfrac{C}{\gamma_1}\,\|\mathbf{G}\|_{L^2(I,L^2(\Omega))}.$$

Thus by the theorem of De Rham and the theorem on negative norms there exists a pressure $\pi \in L^2(I, L_0^2(\Omega))$ with

$$\partial_t \mathbf{u} - \operatorname{div}(\mathbf{TDu}) + \boldsymbol{\nabla}\pi = \operatorname{div}\mathbf{G} \qquad \text{in } \big(\mathcal{D}(I, W_0^{1,2}(\Omega))\big)'$$

and

$$\|\pi\|_{L^2(I,L_0^2(\Omega))} \le C\left(\tfrac{1}{\gamma_1}\|\mathbf{G}\|_{L^2(I,L^2(\Omega))} + \tfrac{1}{\sqrt{\gamma_1}}\|\mathbf{u}_0\|_2\right).$$

Overall we have shown for $\gamma_2 = 1$ and $\varepsilon = 0$ that there exists a weak solution $\mathbf{u}, \pi$ with the desired properties. It remains to show uniqueness. Let us assume that $\mathbf{v}, q$

is another solution to the same problem, then $\mathbf{e} := \mathbf{u} - \mathbf{v}$ and $\eta := \pi - q$ is a weak solution of

$$\begin{aligned} \partial_t \mathbf{e} - \operatorname{div}(\mathbf{TDe}) + \boldsymbol{\nabla}\eta &= 0 \quad \text{on } I \times \Omega, \\ \operatorname{div} \mathbf{e} &= 0 \quad \text{on } I \times \Omega, \\ \mathbf{e}(0) &= 0 \quad \text{on } \Omega. \end{aligned} \tag{4.24}$$

Since $\mathbf{e} \in L^\infty(I, L^2_0(\Omega))$, $\boldsymbol{\nabla}\mathbf{e} \in L^2(I, L^2_0(\Omega))$, and $\pi \in L^2(I, L^2_0(\Omega))$, we can use $\mathbf{e}$ as a testfunction. This implies (compare with the derivation of the a priori estimate above)

$$\tfrac{1}{2} d_t \|\mathbf{e}\|_2^2 + \langle \mathbf{TDe}, \mathbf{De} \rangle = 0.$$

Integration over $[0, t]$ implies for almost all $t \in I$

$$\tfrac{1}{2}\|\mathbf{e}(t)\|_2^2 + \int_0^t \langle \mathbf{TDe}(\tau), \mathbf{De}(\tau) \rangle \, d\tau = \tfrac{1}{2}\|\mathbf{e}(0)\|_2^2 = 0.$$

Since $\langle \mathbf{TDe}(\tau), \mathbf{De}(\tau) \rangle \geq \gamma_1 \|\mathbf{De}(\tau)\|_2^2 \geq 0$ this implies $\mathbf{e} \equiv 0$. This concludes the proof for $\gamma_2 = 1$, $\varepsilon = 0$.

Case $\gamma_2 = 1, \varepsilon > 0$: Let us now assume that $\varepsilon > 0$. We will proceed very similar to the case $\varepsilon = 0$. The major difference is that we have to use different eigenfunctions: Let ω_j be a set of eigenfunctions of the scalar operator $-\Delta$ with corresponding eigenvalues λ_j, which is orthonormal with respect to the $L^2(\Omega)$ scalar product. Let $X_N := \operatorname{span}\{\omega_1, \ldots, \omega_N\}$. Note that by definition $\langle \omega_j, 1 \rangle = 0$ for all $j \in \mathbb{N}$. Define $P^N : L^2(\Omega) \to X_N^d, \mathbf{u} \mapsto P^N \mathbf{u}$ by

$$P^N \mathbf{u} := \Big(\sum_{j=1}^N \langle u_k, \omega_j \rangle \omega_j \Big)_{k=1,\ldots,d},$$

then

$$\lambda_j \langle \omega_j, (P^N \mathbf{u})_k \rangle = \langle \boldsymbol{\nabla}\omega_j, \boldsymbol{\nabla}((P^N \mathbf{u})_k) \rangle. \tag{4.25}$$

and the projections $P^N : W^{s,2}(\Omega) \to W_0^{s,2}(\Omega)$ are uniformly continuous for $0 \leq s \leq 2$. Moreover P^N is a selfadjoint, orthogonal projection with respect to the $L^2(\Omega)$ scalar product. Due to the periodicity $\cup_{N=1}^\infty X_N$ is dense in any $W_0^{s,2}(\Omega)$, $s \in \mathbb{R}$. We use the ansatz

$$\mathbf{u}^N(t, x) := \sum_{j,k=1}^N \alpha_{jk}^N(t) \boldsymbol{\omega}_{\mathbf{j}}(x), \qquad \pi^N(t) := \tfrac{1}{\varepsilon} \Delta^{-1} \operatorname{div} \mathbf{u}^N(t)$$

with $\boldsymbol{\omega}_{\mathbf{j}} = (\omega_{j_k})_{k=1,\ldots,d}$ and look for coefficients $\alpha_{jk}^N : I \to \mathbb{R}$, such that $\mathbf{u}^N$ solves the system

$$\partial_t \langle \mathbf{u}^N, \boldsymbol{\omega}_{\mathbf{j}} \rangle + \langle \mathbf{TDu}^N, \mathbf{D}\boldsymbol{\omega}_{\mathbf{j}} \rangle + \tfrac{1}{\varepsilon} \langle \boldsymbol{\nabla}\Delta^{-1} \operatorname{div} \mathbf{u}^N, \boldsymbol{\omega}_{\mathbf{j}} \rangle = -\langle \mathbf{G}, \boldsymbol{\nabla}\boldsymbol{\omega}_{\mathbf{j}} \rangle \tag{4.26}$$

for all $\mathbf{j} \in \{1, \ldots, N\}^d$ and initial condition $\mathbf{u}^N(0) = P^N \mathbf{u}_0$. Define $\mathbf{M}$, $\mathbf{R}$, and $\mathbf{Q}$ by $M_{\mathbf{j},\mathbf{s}} := \langle \boldsymbol{\omega}_{\mathbf{j}}, \boldsymbol{\omega}_{\mathbf{s}} \rangle$, $R_{\mathbf{j},\mathbf{s}} := \langle \mathbf{TD}\boldsymbol{\omega}_{\mathbf{j}}, \mathbf{D}\boldsymbol{\omega}_{\mathbf{s}} \rangle$, and $Q_{\mathbf{j},\mathbf{s}} := -\langle \Delta^{-1} \operatorname{div} \boldsymbol{\omega}_{\mathbf{j}}, \operatorname{div} \boldsymbol{\omega}_{\mathbf{s}} \rangle$ for $\mathbf{j}, \mathbf{s} \in \{1, \ldots, N\}^d$. Let $\boldsymbol{\alpha}^N := (\alpha_{\mathbf{j}}^N)_{\mathbf{j} \in \{1,\ldots,N\}^d}$, then (4.26) is equivalent to

$$\mathbf{M}\partial_t \boldsymbol{\alpha}^N + \mathbf{R}\boldsymbol{\alpha}^N + \tfrac{1}{\varepsilon}\mathbf{Q}\boldsymbol{\alpha}^N = -(\langle \mathbf{G}, \boldsymbol{\nabla}\boldsymbol{\omega}_{\mathbf{j}} \rangle)_{\mathbf{j} \in \{1,\ldots,N\}^d} \tag{4.27}$$

with initial data $\alpha_{\mathbf{j}}^N(0) = \langle \mathbf{u}_0, \boldsymbol{\omega}_{\mathbf{j}}\rangle$. Note that $\mathbf{M}$, $\mathbf{R}$, and $\mathbf{Q}$ are positive definite. Since especially $\mathbf{M}$ is positive definite, the system (4.27) is solvable for small times $I^* := [0, T^*]$ with $\boldsymbol{\alpha}^N \in C^1([0,T^*])$. To get existence for large times T we need an a priori estimate: We multiply (4.27) by $(\boldsymbol{\alpha}^N)^T$ from the left side, then

$$\begin{aligned}\tfrac{1}{2}\partial_t\big((\boldsymbol{\alpha}^N)^T\mathbf{M}\boldsymbol{\alpha}^T\big) + (\alpha^N)^T\mathbf{R}\boldsymbol{\alpha}^N + \tfrac{1}{\varepsilon}(\boldsymbol{\alpha}^N)^T\mathbf{Q}\boldsymbol{\alpha}^N \\ = -(\boldsymbol{\alpha}^N)^T(\langle \mathbf{G}, \boldsymbol{\nabla}\boldsymbol{\omega}_{\mathbf{s}}\rangle)_{\mathbf{s}\in\{1,\dots,N\}^d}.\end{aligned}$$

Let $\pi^N := \frac{1}{\varepsilon}\Delta^{-1}\operatorname{div}\mathbf{u}^N$. Since

$$\tfrac{1}{\varepsilon}(\boldsymbol{\alpha}^N)^T\mathbf{Q}\boldsymbol{\alpha}^N = -\tfrac{1}{\varepsilon}\langle\Delta^{-1}\operatorname{div}\mathbf{u}^N, \operatorname{div}\mathbf{u}^N\rangle = -\tfrac{1}{\varepsilon}\langle\varepsilon\pi^N, \varepsilon\Delta\pi^N\rangle = \varepsilon\|\boldsymbol{\nabla}\pi^N\|_2^2,$$

we can rewrite the estimate for $\boldsymbol{\alpha}^N$ in terms of $\mathbf{u}^N$ and π^N as

$$\tfrac{1}{2}\partial_t\|\mathbf{u}^N\|_2^2 + \langle\mathbf{TDu}^N, \mathbf{Du}^N\rangle + \varepsilon\|\boldsymbol{\nabla}\pi^N\|_2^2 = -\langle\mathbf{G}, \boldsymbol{\nabla}\mathbf{u}^N\rangle.$$

Analogously to the case $\varepsilon = 0$ this implies

$$\begin{aligned}&\sup_{I^*}\|\mathbf{u}^N\|_2^2 + \gamma_1(1-\mu)\|\mathbf{Du}^N\|^2_{L^2(I^*,L^2(\Omega))} + 2\varepsilon\|\boldsymbol{\nabla}\pi^N\|_2^2 \\ &\qquad\le \tfrac{1}{\gamma_1}\|\mathbf{G}^{\mathrm{sym}}\|^2_{L^2(I,L^2(\Omega))} + \tfrac{C}{\gamma_1\mu}\|\mathbf{G}^{\mathrm{anti}}\|^2_{L^2(I,L^2(\Omega))} + C\,\|\mathbf{u}_0\|_2^2.\end{aligned} \tag{4.28}$$

The only difference is the extra information about $2\varepsilon\|\boldsymbol{\nabla}\pi^N\|_2^2$. As in the case $\varepsilon = 0$ this enables us to extend the interval of existence from I^* to I and also to transfer the a priori estimates from I^* to I, i.e. there holds

$$\begin{aligned}&\sup_{I}\|\mathbf{u}^N\|_2^2 + \gamma_1(1-\mu)\|\mathbf{Du}^N\|^2_{L^2(I,L^2(\Omega))} + 2\varepsilon\|\boldsymbol{\nabla}\pi^N\|_2^2 \\ &\qquad\le \tfrac{1}{\gamma_1}\|\mathbf{G}^{\mathrm{sym}}\|^2_{L^2(I,L^2(\Omega))} + \tfrac{C}{\gamma_1\mu}\|\mathbf{G}^{\mathrm{anti}}\|^2_{L^2(I,L^2(\Omega))} + C\,\|\mathbf{u}_0\|_2^2.\end{aligned} \tag{4.29}$$

with constants independent of ε and T. As in the case $\varepsilon = 0$ the estimates (4.29) immediately imply

$$\gamma_1\|\boldsymbol{\nabla}\mathbf{u}^N\|^2_{L^2(I,L^2(\Omega))} \le \tfrac{C}{\gamma_1}\|\mathbf{G}\|^2_{L^2(I,L^2(\Omega))} + C\|\mathbf{u}_0\|_2^2. \tag{4.30}$$

Since P^N is an orthogonal, selfadjoint projection with respect to the $L^2(\Omega)$ scalar product, there holds for all $\|\boldsymbol{\varphi}\|_{L^2(I,W_0^{1,2}(\Omega))} \le 1$

$$\begin{aligned}\int_I\langle\partial_t\mathbf{u}^N, \boldsymbol{\varphi}\rangle\,dt &= \int_I\langle P^N\partial_t\mathbf{u}^N, \boldsymbol{\varphi}\rangle\,dt = \int_I\langle\partial_t\mathbf{u}^N, P^N\boldsymbol{\varphi}\rangle\,dt \\ &\overset{(4.26)}{=} \int_I -\langle\mathbf{TDu}^N, \mathbf{D}P^N\boldsymbol{\varphi}\rangle + \langle\mathbf{G}, \boldsymbol{\nabla}P^N\boldsymbol{\varphi}\rangle + \tfrac{1}{\varepsilon}\langle\Delta^{-1}\operatorname{div}\mathbf{u}^N, \operatorname{div}P^N\boldsymbol{\varphi}\rangle\,dt.\end{aligned}$$

Thus (recall $\gamma_2 = 1$)

$$\begin{aligned}&\Big|\int_I\langle\partial_t\mathbf{u}^N, \boldsymbol{\varphi}\rangle\,dt\Big| \\ &\quad\le \big((1+\tfrac{1}{\varepsilon})\|\boldsymbol{\nabla}\mathbf{u}^N\|_{L^2(I,L^2(\Omega))} + \|\mathbf{G}\|_{L^2(I,L^2(\Omega))}\big)\|P^N\boldsymbol{\varphi}\|_{L^2(I,W_0^{1,2}(\Omega))} \\ &\quad\le \big((1+\tfrac{1}{\varepsilon})\|\boldsymbol{\nabla}\mathbf{u}^N\|_{L^2(I,L^2(\Omega))} + \|\mathbf{G}\|_{L^2(I,L^2(\Omega))}\big)C\,\|\boldsymbol{\varphi}\|_{L^2(I,W_0^{1,2}(\Omega))} \\ &\quad\le C\,\big((1+\tfrac{1}{\varepsilon})\|\boldsymbol{\nabla}\mathbf{u}^N\|_{L^2(I,L^2(\Omega))} + \|\mathbf{G}\|_{L^2(I,L^2(\Omega))}\big) \\ &\quad\overset{(4.30)}{\le} C\,(1+\tfrac{1}{\varepsilon})\Big(\tfrac{1}{\gamma_1}\|\mathbf{G}\|_{L^2(I,L^2(\Omega))} + \tfrac{1}{\sqrt{\gamma_1}}\|\mathbf{u}_0\|_2\Big).\end{aligned}$$

Contrary to the case $\varepsilon = 0$ this estimate depends on $\frac{1}{\varepsilon}$. But this fact does not matter when passing (for a fixed $\varepsilon > 0$) to the limit $N \to \infty$. Analogously to the case $\varepsilon = 0$ it follows that

$$\partial_t \langle \mathbf{u}, \boldsymbol{\omega}_\mathbf{j} \rangle + \langle \mathbf{T}\mathbf{D}\mathbf{u}, \mathbf{D}\boldsymbol{\omega}_\mathbf{j} \rangle + \tfrac{1}{\varepsilon} \langle \boldsymbol{\nabla} \Delta^{-1} \operatorname{div} \mathbf{u}, \boldsymbol{\omega}_\mathbf{j} \rangle = -\langle \mathbf{G}, \boldsymbol{\nabla} \boldsymbol{\omega}_\mathbf{j} \rangle \quad \text{in } \mathcal{D}'(I) \tag{4.31}$$

for all $\boldsymbol{\omega}_\mathbf{j}$. Certainly the estimate for $\mathbf{u}^N$ and $\pi^N = \frac{1}{\varepsilon} \Delta^{-1} \operatorname{div} \mathbf{u}^N$ transfer to $\mathbf{u}$ and $\pi = \frac{1}{\varepsilon} \Delta^{-1} \operatorname{div} \mathbf{u}$, so by (4.29)

$$\begin{aligned} \|\mathbf{u}\|^2_{L^\infty(I,L^2(\Omega))} + (1-\mu)\gamma_1 \|\mathbf{D}\mathbf{u}\|^2_{L^2(I,L^2(\Omega))} + 2\varepsilon \, \|\boldsymbol{\nabla}\pi\|^2_{L^2(I,L^2_0(\Omega))} \\ \leq \tfrac{1}{\gamma_1} \|\mathbf{G}^{\mathrm{sym}}\|^2_{L^2(I^*,L^2(\Omega))} + \tfrac{C}{\gamma_1 \mu} \|\mathbf{G}^{\mathrm{anti}}\|^2_{L^2(I^*,L^2(\Omega))} + C\|\mathbf{u}_0\|_2^2, \end{aligned} \tag{4.32}$$

Since $\cup_{N=1}^\infty X_N$ is dense in any $W_0^{s,2}(\Omega)$ and by the a priori estimates for $\mathbf{u}^N$, $\partial_t \mathbf{u}^N$, $\boldsymbol{\nabla}\mathbf{u}^N$, which transfer from $\mathbf{u}^N$ to $\mathbf{u}$, there follows that (4.31) also holds in the space $\mathcal{D}(I, (W_0^{1,2}(\Omega))')$. (The dependence of the norm on $\frac{1}{\varepsilon}$ does not matter.) As in the case $\varepsilon = 0$ we further deduce from $\mathbf{u}^N(0) = P^N \mathbf{u}_0 \to \mathbf{u}_0$ in $L^2(\Omega)$ that $\mathbf{u}(0) = \mathbf{u}_0$ and that the initial data is continuously assumed in $L^2_0(\Omega)$. Since $\operatorname{div} \mathbf{u} = \varepsilon \Delta \pi$ and $\boldsymbol{\nabla}\mathbf{u} \in L^2(I, L^2_0(\Omega))$ there also holds $\pi \in L^2(I, L^2_0(\Omega))$ (norm may depend on ε).

Overall we have shown that $\mathbf{u}, \pi$ is a weak solution of (4.4), which satisfies (4.5).

Let us now prove uniqueness. Let us assume that $\mathbf{v}, q$ is another solution to the same problem, then $\mathbf{e} := \mathbf{u} - \mathbf{v}$ and $\eta := \pi - q$ is a weak solution of

$$\begin{aligned} \partial_t \mathbf{e} - \operatorname{div}(\mathbf{T}\mathbf{D}\mathbf{e}) + \boldsymbol{\nabla}\eta &= 0 && \text{on } I \times \Omega, \\ \operatorname{div} \mathbf{e} &= \varepsilon \Delta \eta && \text{on } I \times \Omega, \\ \mathbf{e}(0) &= 0 && \text{on } \Omega. \end{aligned} \tag{4.33}$$

Since $\mathbf{e} \in L^\infty(I, L^2_0(\Omega))$, $\boldsymbol{\nabla}\mathbf{e} \in L^2(I, L^2_0(\Omega))$, and $\operatorname{div} \mathbf{e} = \varepsilon \Delta \eta$ there holds $\pi \in L^2(I, W_0^{2,2}(\Omega))$. Thus we can test the first equation of (4.33) with $\mathbf{e}$ and use

$$\langle \boldsymbol{\nabla}\pi, \mathbf{e} \rangle = -\langle \pi, \operatorname{div} \mathbf{e} \rangle = -\varepsilon \langle \pi, \Delta\pi \rangle = \varepsilon \|\boldsymbol{\nabla}\pi\|_2^2$$

to deduce

$$\tfrac{1}{2} d_t \|\mathbf{e}\|_2^2 + \langle \mathbf{T}\mathbf{D}\mathbf{e}, \mathbf{D}\mathbf{e} \rangle + \varepsilon \|\boldsymbol{\nabla}\pi\|_2^2 = 0.$$

Integration over $[0, t]$ implies for almost all $t \in I$

$$\tfrac{1}{2} \|\mathbf{e}(t)\|_2^2 + \int_0^t \langle \mathbf{T}\mathbf{D}\mathbf{e}(\tau), \mathbf{D}\mathbf{e}(\tau) \rangle + \varepsilon \|\boldsymbol{\nabla}\pi(\tau)\|_2^2 \, d\tau = \tfrac{1}{2} \|\mathbf{e}(0)\|_2^2 = 0.$$

Since $\langle \mathbf{T}\mathbf{D}\mathbf{e}(\tau), \mathbf{D}\mathbf{e}(\tau) \rangle \geq \gamma_1 \|\mathbf{D}\mathbf{e}(\tau)\|_2^2 \geq 0$ this implies $\mathbf{e} \equiv 0$, which proves uniqueness.

It remains to prove (4.6) for the special case $\mathbf{T} = \mathrm{Id}^{\mathrm{sym}}$: From (4.4) we deduce in $\mathcal{D}'(I \times \Omega)$

$$\operatorname{div} \operatorname{div} \mathbf{G} = \operatorname{div}(\partial_t \mathbf{u} - \operatorname{div}(\mathrm{Id}^{\mathrm{sym}} \mathbf{D}\mathbf{u}) + \boldsymbol{\nabla}\pi).$$

Since

$$\begin{aligned} \operatorname{div} \operatorname{div}(\mathrm{Id}^{\mathrm{sym}} \mathbf{D}\mathbf{u}) &= \operatorname{div} \operatorname{div}(\mathbf{D}\mathbf{u}) = \operatorname{div} \big(\tfrac{1}{2} \Delta \mathbf{u} + \tfrac{1}{2} \boldsymbol{\nabla}(\operatorname{div} \mathbf{u}) \big) \\ &= \tfrac{1}{2} \big(\operatorname{div}(\Delta \mathbf{u}) + \Delta(\operatorname{div} \mathbf{u}) \big) = \Delta(\operatorname{div} \mathbf{u}), \end{aligned}$$

this implies

$$\begin{aligned}\operatorname{div}\operatorname{div}\mathbf{G} &= \partial_t(\operatorname{div}\mathbf{u}) - \Delta(\operatorname{div}\mathbf{u}) + \Delta\pi \\ &= \varepsilon\partial_t\Delta\pi - \varepsilon\Delta^2\pi + \Delta\pi.\end{aligned}$$

So due to the periodicity and $\langle \pi, 1\rangle = 0$

$$\Delta^{-1}\operatorname{div}\operatorname{div}\mathbf{G} = \varepsilon\partial_t\pi - \varepsilon\Delta\pi + \pi. \tag{4.34}$$

Since $\operatorname{div}\mathbf{u} = \varepsilon\Delta\pi$ (in the distributional sense), $\mathbf{u} \in C(I, L^2_0(\Omega))$ and $\operatorname{div}\mathbf{u}(0) = \operatorname{div}\mathbf{u}_0 = 0$, we have $\pi(0) = 0$.

Since $\pi \in L^2(I, W^{1,2}(\Omega))$ and $\Delta^{-1}\operatorname{div}\operatorname{div}\mathbf{G} \in L^2(I, L^2(\Omega))$, the function π (norms depending on ε) is an admissible test function. Thus

$$\begin{aligned}\tfrac{\varepsilon}{2}d_t\|\pi\|_2^2 + \varepsilon\|\boldsymbol{\nabla}\pi\|_2^2 + \|\pi\|_2^2 &\le \|\Delta^{-1}\operatorname{div}\operatorname{div}\mathbf{G}\|_2\|\pi\|_2 \le C\,\|\mathbf{G}\|_2\|\pi\|_2 \\ &\le C\,\|\mathbf{G}\|_2^2 + \tfrac{1}{2}\|\pi\|_2^2.\end{aligned}$$

Integration over I yields

$$\|\pi\|_{L^2(I,L^2(\Omega))} \le C\,\|\mathbf{G}\|_{L^2(I,L^2(\Omega))}$$

with constants independent of ε and T. This proves (4.6).

So far we have proven the lemma for the special case $\gamma_2 = 1$. It remains to check the case $\gamma_2 \neq 1$. This will be done by scaling in time: Define

$$\begin{aligned} \widetilde{T} &:= \gamma_2 T, & & \\ \widetilde{I} &:= [0, \gamma_2 T], & \widetilde{\varepsilon} &:= \gamma_2\varepsilon, \\ \widetilde{\gamma}_1 &:= \gamma_1/\gamma_2 & \widetilde{\gamma}_2 &:= 1, \\ \widetilde{\mathbf{u}}(t,x) &:= \mathbf{u}(t/\gamma_2, x), & \widetilde{\pi}(t,x) &:= \tfrac{1}{\gamma_2}\pi(t/\gamma_2, x), \\ \widetilde{\mathbf{G}}(t,x) &:= \tfrac{1}{\gamma_2}\mathbf{G}(t/\gamma_2, x), & \widetilde{\mathbf{T}}(t,x) &:= \tfrac{1}{\gamma_2}\mathbf{T}(t/\gamma_2, x), \\ \widetilde{\mathbf{u}}_0 &:= \mathbf{u}_0, & & \end{aligned} \tag{4.35}$$

then $\widetilde{\mathbf{u}}, \widetilde{\pi}$ solves (4.4) with $\mathbf{G}, \mathbf{T}, \mathbf{u}_0$ replaced by $\widetilde{\mathbf{G}}, \widetilde{\mathbf{T}}, \widetilde{\mathbf{u}}_0$. Since $\widetilde{\gamma}_2 = 1$ we can apply the calculations above for $\gamma_2 = 1$ to our scaled problem. Recall that the constants in the a priori estimates did not depend on T nor ε. Thus there will be no implicit dependence of the a priori estimates on γ_2 via $\widetilde{T}$ or $\widetilde{\varepsilon}$. So we have

$$\begin{aligned} \|\widetilde{\mathbf{u}}\|^2_{L^\infty(\widetilde{I},L^2(\Omega))} &+ (1-\mu)\widetilde{\gamma}_1\|\mathbf{D}\widetilde{\mathbf{u}}\|^2_{L^2(\widetilde{I},L^2(\Omega))} \\ &\le \tfrac{1}{\widetilde{\gamma}_1}\|\widetilde{\mathbf{G}}^{\mathrm{sym}}\|^2_{L^2(\widetilde{I},L^2(\Omega))} + \tfrac{C}{\widetilde{\gamma}_1\mu}\|\widetilde{\mathbf{G}}^{\mathrm{anti}}\|^2_{L^2(\widetilde{I},L^2(\Omega))} + C\|\widetilde{\mathbf{u}}_0\|_2^2, \\ \widetilde{\gamma}_1\|\boldsymbol{\nabla}\widetilde{\mathbf{u}}\|^2_{L^2(\widetilde{I},L^2(\Omega))} &\le \tfrac{C}{\widetilde{\gamma}_1}\|\widetilde{\mathbf{G}}\|^2_{L^2(\widetilde{I},L^2(\Omega))} + C\|\widetilde{\mathbf{u}}_0\|_2^2 \end{aligned} \tag{4.36}$$

and additionally for the special case $\varepsilon = 0$ or $\mathbf{T} = \gamma_2\,\mathrm{Id}^{\mathrm{sym}}$, i.e. $\widetilde{\varepsilon} = 0$ or $\widetilde{\mathbf{T}} = \mathrm{Id}^{\mathrm{sym}}$,

$$\|\widetilde{\pi}\|_{L^2(\widetilde{I},L^2(\Omega))} \le C\left(\tfrac{1}{\widetilde{\gamma}_1}\|\widetilde{\mathbf{G}}\|_{L^2(\widetilde{I},L^2(\Omega))} + \tfrac{1}{\sqrt{\widetilde{\gamma}_1}}\|\widetilde{\mathbf{u}}_0\|_2\right). \tag{4.37}$$

This implies

$$\begin{aligned} \|\mathbf{u}\|^2_{L^\infty(I,L^2(\Omega))} &+ (1-\mu)\gamma_1\|\mathbf{D}\mathbf{u}\|^2_{L^2(I,L^2(\Omega))} \\ &\le \tfrac{1}{\gamma_1}\|\mathbf{G}^{\mathrm{sym}}\|^2_{L^2(I,L^2(\Omega))} + \tfrac{C}{\gamma_1\mu}\|\mathbf{G}^{\mathrm{anti}}\|^2_{L^2(I,L^2(\Omega))} + C\,\|\mathbf{u}_0\|_2^2, \\ \gamma_1\|\boldsymbol{\nabla}\mathbf{u}\|^2_{L^2(I,L^2(\Omega))} &\le \tfrac{C}{\gamma_1}\|\mathbf{G}\|^2_{L^2(I,L^2(\Omega))} + C\,\|\mathbf{u}_0\|_2^2, \end{aligned} \tag{4.38}$$

for all $\mu > 0$ and additionally for the special case $\varepsilon = 0$ or $\mathbf{T} = \gamma_2 \operatorname{Id}^{\mathrm{sym}}$

$$\tag{4.39} \tfrac{1}{\sqrt{\gamma_2}} \|\pi\|_{L^2(I,L^2(\Omega))} \le C \left(\tfrac{\sqrt{\gamma_2}}{\gamma_1} \|\mathbf{G}\|_{L^2(I,L^2(\Omega))} + \tfrac{\sqrt{\gamma_2}}{\sqrt{\gamma_1}} \|\mathbf{u}_0\|_2 \right)$$

This concludes the proof of the lemma (4.1). □

The next lemma provides an $L^r(I \times \Omega)$ result.

Lemma 4.2. *Let $\mathbf{u}_0 \in B^{1-\frac{2}{r}}_{r,\mathrm{div}}(\Omega)$, $\langle \mathbf{u}_0, 1 \rangle = 0$, and $\mathbf{G} \in L^r(I, L^r(\Omega; \mathbb{R}^{d\times d}))$ for a fixed $r > 2$. Further let $\mathbf{T} = \operatorname{Id}^{\mathrm{sym}}$. Then for all $\varepsilon \ge 0$ there exists a unique weak solution $\mathbf{u}, \pi$ of (4.4) such that*

$$\begin{aligned} \|\mathbf{u}\|_{C(I,B^{1-\frac{2}{r}}_r(\Omega))} &\le C \left(\|\mathbf{G}\|_{L^r(I,L^r(\Omega))} + \|\mathbf{u}_0\|_{B^{1-\frac{2}{r}}_r(\Omega)} \right), \\ \|\nabla \mathbf{u}\|_{L^r(I,L^r(\Omega))} &\le C \left(\|\mathbf{G}\|_{L^r(I,L^r(\Omega))} + \|\mathbf{u}_0\|_{B^{1-\frac{2}{r}}_r(\Omega)} \right), \\ \|\pi\|_{L^r(I,L^r(\Omega))} &\le C \, \|\mathbf{G}\|_{L^r(I,L^r(\Omega))}, \end{aligned}$$

where C does not depend on ε.

Proof. **Case $\varepsilon = 0$:** Let $\mathbf{u}, \pi$ denote the unique solution of lemma 4.1. Then π satisfies

$$\begin{aligned} \Delta \pi &= \operatorname{div}(-\partial_t \mathbf{u} + \operatorname{div}(\mathbf{D}\mathbf{u}) + \operatorname{div} \mathbf{G}) \\ &= -\partial_t(\operatorname{div} \mathbf{u}) + \tfrac{1}{2}\big(\Delta(\operatorname{div} \mathbf{u}) + \operatorname{div}(\Delta \mathbf{u})\big) + \operatorname{div}\operatorname{div} \mathbf{G} \\ &= -\partial_t(\operatorname{div} \mathbf{u}) + \Delta(\operatorname{div} \mathbf{u}) + \operatorname{div}\operatorname{div} \mathbf{G} \\ &= \operatorname{div}\operatorname{div} \mathbf{G}. \end{aligned}$$

So by the L^r theory of the Laplacian

$$\|\pi\|_{L^r(I,L^r(\Omega))} \le C \, \|\mathbf{G}\|_{L^r(I,L^r(\Omega))},$$

where C does not depend on T. On the other hand by (4.1) $\mathbf{u}$ solves

$$\begin{aligned} \mathbf{H} := \operatorname{div} \mathbf{G} - \nabla \pi &= \partial_t \mathbf{u} - \operatorname{div}(\mathbf{D}\mathbf{u}) \\ &= \partial_t \mathbf{u} + \tfrac{1}{2}\Delta \mathbf{u} + \tfrac{1}{2}\nabla(\operatorname{div} \mathbf{u}) \\ &= \partial_t \mathbf{u} + \tfrac{1}{2}\Delta \mathbf{u} \end{aligned}$$

with initial data $\mathbf{u}(0) = \mathbf{u}_0$. The L^r theory of the heat equation (see Grubb [**Gru01**] corollary 2.3 and Grubb [**Gru95**] corollary 4.5) now implies

$$\begin{aligned} &\|\mathbf{u}\|_{C(I,B^{1-\frac{2}{r}}_r(\Omega))} + \|\nabla \mathbf{u}\|_{L^r(I,L^r(\Omega))} \\ &\quad \le C \, \|\mathbf{H}\|_{L^r(I,W^{-1,r}(\Omega))} + C \, \|\mathbf{u}_0\|_{B^{1-\frac{2}{r}}_r(\Omega)} \\ &\quad \le C \, \|\mathbf{G}\|_{L^r(I,L^r(\Omega))} + C \, \|\pi\|_{L^r(I,L^r(\Omega))} + C \, \|\mathbf{u}_0\|_{B^{1-\frac{2}{r}}_r(\Omega)} \\ &\quad \le C \, \|\mathbf{G}\|_{L^r(I,L^r(\Omega))} + C \, \|\mathbf{u}_0\|_{B^{1-\frac{2}{r}}_r(\Omega)}, \end{aligned}$$

which proves the lemma for $\varepsilon = 0$.

Case $\varepsilon > 0$: In the proof of lemma 4.1 we have seen that π satisfies

$$\tag{4.40} \Delta^{-1} \operatorname{div}\operatorname{div} \mathbf{G} = \varepsilon \partial_t \pi - \varepsilon \Delta \pi + \pi.$$

Define $g := \pi |\pi|^{r-2}$. Since $\varepsilon \Delta \pi = \operatorname{div} \mathbf{u}$ and $\mathbf{u} \in L^2(I, W^{1,2}_0(\Omega))$, there holds $\pi \in L^2(I, W^{2,2}_0(\Omega))$. Moreover, $\mathbf{G} \in L^2(I, L^2(\Omega))$, so (4.40) implies $\partial_t \pi \in L^2(I, L^2_0(\Omega))$.

As a consequence $\pi \in C(I, W_0^{1,2}(\Omega))$. If additionally there holds $\pi \in C(I, W_0^{1,r}(\Omega))$, then g is an admissible test function for (4.40). This gives

$$\varepsilon\langle \partial_t \pi, \pi|\pi|^{r-2}\rangle + \varepsilon\langle \nabla\pi, \nabla(\pi|\pi|^{r-2})\rangle + \|\pi\|_r^r = \langle \Delta^{-1} \operatorname{div}\operatorname{div} \mathbf{G}, \pi|\pi|^{r-2}\rangle.$$

Note that

$$(\nabla\pi)\big(\nabla(\pi|\pi|^{r-2})\big) = (r-1)|\nabla\pi|^2|\pi|^{r-2},$$

so

$$\tfrac{\varepsilon}{r}\partial_t\|\pi\|_r^r + \varepsilon(r-1)\int_\Omega |\nabla\pi|^2|\pi|^{r-2}\,dx + \|\pi\|_r^r \leq C\,\|\mathbf{G}\|_r\|\pi\|_r^{r-1}.$$

We integrate over the time, neglect the positive terms involving ε, and use $\pi(0) = \frac{1}{\varepsilon}\Delta^{-1}\operatorname{div}\mathbf{u}_0 = 0$ as in lemma 4.1, then

$$\|\pi\|_{L^r(I,L^r(\Omega))} \leq C\,\|\mathbf{G}\|_{L^r(I,L^r(\Omega))}, \tag{4.41}$$

where C does neither depend on T nor ε. If $\pi \notin C(I, W_0^{1,r}(\Omega))$, then we mollify (4.40) in space to pass from $\pi \in C(I, W_0^{1,2}(\Omega))$ to the desired regularity, do all the calculations above, and pass to the limit in the last inequality. Thus we have proven the desired regularity of π so far. Moreover, we deduce from (4.40)

$$\varepsilon\partial_t\pi - \varepsilon\Delta\pi = \Delta^{-1}\operatorname{div}\operatorname{div}\mathbf{G} - \pi =: \mathbf{K}. \tag{4.42}$$

We have already shown that $\mathbf{K} \in L^r(I, L^r(\Omega))$ with

$$\|\mathbf{K}\|_{L^r(I,L^r(\Omega))} \leq C\,\|\mathbf{G}\|_{L^r(I,L^r(\Omega))} + \|\pi\|_{L^r(I,L^r(\Omega))} \leq C\,\|\mathbf{G}\|_{L^r(I,L^r(\Omega))}.$$

The L^r theory of the heat equation (see Grubb [**Gru01**] corollary 2.3 and Grubb [**Gru95**] corollary 4.5) now implies

$$\varepsilon\|\pi\|_{L^r(I,W^{2,r}(\Omega))} \leq C\,\|\mathbf{G}\|_{L^r(I,L^r(\Omega))}. \tag{4.43}$$

On the other hand by (4.1) $\mathbf{u}$ solves

$$\begin{aligned}\partial_t\mathbf{u} - \tfrac12\Delta\mathbf{u} &= \partial_t\mathbf{u} - \operatorname{div}(\mathbf{D}\mathbf{u}) + \tfrac12\nabla(\operatorname{div}\mathbf{u})\\ &= \partial_t\mathbf{u} - \operatorname{div}(\mathbf{D}\mathbf{u}) + \tfrac{\varepsilon}{2}\nabla(\Delta\pi)\\ &= \operatorname{div}\mathbf{G} - \nabla\pi + \tfrac{\varepsilon}{2}\nabla(\Delta\pi).\end{aligned} \tag{4.44}$$

Due to the regularity of $\mathbf{G}$, (4.41), and (4.43) we have

$$\|\operatorname{div}\mathbf{G} - \nabla\pi + \tfrac{\varepsilon}{2}\nabla(\Delta\pi)\|_{L^r(I,W^{-1,r}(\Omega))} \leq C\,\|\mathbf{G}\|_{L^r(I,L^r(\Omega))}.$$

The L^r theory of the heat equation (see Grubb [**Gru01**] corollary 2.3 and Grubb [**Gru95**] corollary 4.5) applied to (4.44) now implies

$$\begin{aligned}&\|\mathbf{u}\|_{C(I,B_r^{1-\frac{2}{r}}(\Omega))} + \|\nabla\mathbf{u}\|_{L^r(I,L^r(\Omega))}\\ &\qquad\leq C\,\|\operatorname{div}\mathbf{G} - \nabla\pi + \tfrac{\varepsilon}{2}\nabla(\Delta\pi)\|_{L^r(I,W^{-1,r}(\Omega))} + C\,\|\mathbf{u}_0\|_{B_r^{1-\frac{2}{r}}(\Omega)}\\ &\qquad\leq C\,\|\mathbf{G}\|_{L^r(I,L^r(\Omega))} + C\,\|\mathbf{u}_0\|_{B_r^{1-\frac{2}{r}}(\Omega)}.\end{aligned}$$

This proves the lemma for $\varepsilon > 0$. □

The next corollary will combines the strength of lemma 4.1 and lemma 4.2 to optimize the continuity constants in 4.2:

Corollary 4.3. *Let* $r>2$*,* $\mathbf{u}_0\in B^{1-\frac{2}{r}}_{r,\mathrm{div}}(\Omega)$*,* $\langle \mathbf{u}_0,1\rangle=0$*, and* $\mathbf{G}\in L^r(I,L^r(\Omega;\mathbb{R}^{d\times d}))$*. Further let* $\mathbf{T}=\mathrm{Id}^{\mathrm{sym}}$*. Then there exists* $K\geq 1$*, such that for all* $\varepsilon\geq 0$ *there exists a unique weak solution* $\mathbf{u},\pi$ *of* (4.4) *with*

$$\begin{aligned}
&\|\mathbf{u}\|_{C(I,B_s^{1-\frac{2}{s}}(\Omega))}\\
&\quad\leq K^{\frac{s-2}{2}}\Big(\|\mathbf{G}^{\mathrm{sym}}\|_{L^s(I,L^s(\Omega))}+\tfrac{C}{\sqrt{\mu}}\|\mathbf{G}^{\mathrm{anti}}\|_{L^s(I,L^s(\Omega))}+C\,\|\mathbf{u}_0\|_{B_s^{1-\frac{2}{s}}(\Omega))}\Big),\\
&\|\mathbf{D}\mathbf{u}\|_{L^s(I,L^s(\Omega))}\\
&\quad\leq K^{\frac{s-2}{2}}\tfrac{1}{\sqrt{1-\mu}}\Big(\|\mathbf{G}^{\mathrm{sym}}\|_{L^s(I,L^s(\Omega))}+\tfrac{C}{\sqrt{\mu}}\|\mathbf{G}^{\mathrm{anti}}\|_{L^s(I,L^s(\Omega))}+C\,\|\mathbf{u}_0\|_{B_s^{1-\frac{2}{s}}(\Omega))}\Big),
\end{aligned}$$

$$\|\pi\|_{L^r(I,L^r(\Omega))}\leq C\,\|\mathbf{G}\|_{L^r(I,L^r(\Omega))},$$

for all $2\leq s\leq r$ *and all* $0<\mu<1$*.*

Proof. Note that lemma 4.1 and lemma 4.2 show that for all μ with $0<\mu<1$ there holds

$$\begin{aligned}
&\|\mathbf{u}\|_{C(I,L^2(\Omega))}\\
&\qquad\leq \|\mathbf{G}^{\mathrm{sym}}\|_{L^2(I,L^2(\Omega))}+\tfrac{C}{\sqrt{\mu}}\|\mathbf{G}^{\mathrm{anti}}\|_{L^2(I,L^2(\Omega))}+C\,\|\mathbf{u}_0\|_2,\\
&\|\mathbf{D}\mathbf{u}\|_{L^2(I,L^2(\Omega))}\\
&\qquad\leq \tfrac{1}{\sqrt{1-\mu}}\Big(\|\mathbf{G}^{\mathrm{sym}}\|_{L^2(I,L^2(\Omega))}+\tfrac{C}{\sqrt{\mu}}\|\mathbf{G}^{\mathrm{anti}}\|_{L^2(I,L^2(\Omega))}+C\,\|\mathbf{u}_0\|_2\Big)
\end{aligned}$$

and

$$\begin{aligned}
&\|\mathbf{u}\|_{C(I,B_r^{1-\frac{2}{r}}(\Omega))}\\
&\qquad\leq \|\mathbf{G}^{\mathrm{sym}}\|_{L^r(I,L^r(\Omega))}+\tfrac{C}{\sqrt{\mu}}\|\mathbf{G}^{\mathrm{anti}}\|_{L^r(I,L^r(\Omega))}+C\,\|\mathbf{u}_0\|_{B_r^{1-\frac{2}{r}}(\Omega))},\\
&\|\mathbf{D}\mathbf{u}\|_{L^r(I,L^r(\Omega))}\\
&\qquad\leq \tfrac{1}{\sqrt{1-\mu}}\Big(\|\mathbf{G}^{\mathrm{sym}}\|_{L^r(I,L^r(\Omega))}+\tfrac{C}{\sqrt{\mu}}\|\mathbf{G}^{\mathrm{anti}}\|_{L^r(I,L^r(\Omega))}+C\,\|\mathbf{u}_0\|_{B_r^{1-\frac{2}{r}}(\Omega))}\Big).
\end{aligned}$$

Thus the interpolation theorem of Riesz–Thorin immediately proves the estimates for $\mathbf{u}$ and $\mathbf{D}\mathbf{u}$. The estimate for the pressure π follows directly from lemma 4.2. This concludes the proof of the corollary. □

Let us now consider the general case with $\mathbf{T}\not\equiv\mathrm{Id}^{\mathrm{sym}}$.

Lemma 4.4. *Let* $r>2$*,* $\mathbf{u}_0\in B^{1-\frac{2}{r}}_{r,\mathrm{div}}(\Omega)$*,* $\mathbf{G}\in L^r(I,L^r(\Omega;\mathbb{R}^{d\times d}))$*,* $\langle\mathbf{u}_0,1\rangle=0$*, and let* $\mathbf{T}$ *be as described above with ellipticity constants* γ_1,γ_2*. Then there exists a constant* $\kappa>0$ *(independent of* γ_1 *and* γ_2*), such that for all* s *with*

$$2\leq s\leq \min\Big\{r,2+\kappa\frac{\gamma_1}{\gamma_2}\Big\}$$

and all $\varepsilon \geq 0$ *there exists a unique weak solution* $\mathbf{u}, \pi$ *of* (4.4)*, such that*

$$
\begin{aligned}
\sqrt{\gamma_2}\,\|\mathbf{u}\|_{C(I,B_s^{1-\frac{2}{s}}(\Omega))} &\leq \tfrac{C}{\gamma_1}\Big(\|\mathbf{G}^{\mathrm{sym}}\|_{L^s(I,L^s(\Omega))} \\
&\qquad + \tfrac{C}{\sqrt{\gamma_1}}\|\mathbf{G}^{\mathrm{anti}}\|_{L^s(I,L^s(\Omega))} + C\,\sqrt{\gamma_2}\,\|\mathbf{u}_0\|_{B_s^{1-\frac{2}{s}}(\Omega))}\Big),\\
\|\mathbf{D}\mathbf{u}\|_{L^s(I,L^s(\Omega))} &\leq \tfrac{C}{\gamma_1}\Big(\|\mathbf{G}^{\mathrm{sym}}\|_{L^s(I,L^s(\Omega))} \\
&\qquad + \tfrac{C}{\sqrt{\gamma_1}}\|\mathbf{G}^{\mathrm{anti}}\|_{L^s(I,L^s(\Omega))} + C\,\sqrt{\gamma_2}\,\|\mathbf{u}_0\|_{B_s^{1-\frac{2}{s}}(\Omega))}\Big),\\
\|\pi\|_{L^s(I,L^s(\Omega))} &\leq \tfrac{C\,\gamma_2}{\gamma_1}\big(\|\mathbf{G}\|_{L^s(I,L^s(\Omega))} + \sqrt{\gamma_2}\,\|\mathbf{u}_0\|_{B_s^{1-\frac{2}{s}}(\Omega)}\big),
\end{aligned}
\tag{4.45}
$$

where the constants do not depend on ε *nor* s*.*

PROOF. The uniqueness and the existence of a weak solution $\mathbf{u}, \pi$ follow from lemma 4.1.

To prove the L^r estimates let us first assume that $\gamma_2 = 1$, $\mathbf{G}, \mathbf{T} \in C^\infty(I \times \Omega)$, and $\mathbf{u}_0 \in C^\infty(\Omega)$ with $\langle \mathbf{u}_0, 1\rangle = 0$. Later we will show how to get rid of these pre-assumptions.

For $h > 0$ let ∂_k^h be the discrete partial derivative in direction x_k, $k \in \{1, \dots, d\}$. Then $\mathbf{v}^h := \partial_k^h \mathbf{u}$, $\rho^h := \partial_k^h \pi$ solves

$$
\begin{aligned}
\partial_t \mathbf{v}^h - \operatorname{div}(\mathbf{T}\mathbf{D}\mathbf{v}^h) + \boldsymbol{\nabla}\rho^h &= \operatorname{div}(\partial_k^h \mathbf{G} + (\partial_k^h \mathbf{T})\boldsymbol{\nabla}\mathbf{u}) && \text{on } I \times \Omega,\\
\operatorname{div} \mathbf{v}^h &= \varepsilon\Delta\rho^h && \text{on } I \times \Omega,\\
\mathbf{v}^h(0) &= (\partial_k^h \mathbf{u}_0) && \text{on } \Omega.
\end{aligned}
$$

Thus by lemma 4.1

$$
\begin{aligned}
&\|\partial_k^h \mathbf{u}\|_{C(I,L^2(\Omega))} + \|\partial_k^h \mathbf{u}\|_{L^2(I,W^{1,2}(\Omega))} + \|\partial_k^h \pi\|_{L^2(I,L^2(\Omega))}\\
&\quad\leq C\,\big(\|\partial_k^h \mathbf{G}\|_{L^2(I,L^2(\Omega))} + \|\partial_k^h \mathbf{T}\|_{L^\infty(I\times\Omega)}\|\boldsymbol{\nabla}\mathbf{u}\|_{L^2(I,L_0^2(\Omega))} + \|\partial_k^h \mathbf{u}_0\|_2\big)\\
&\quad\leq C\,\big(\|\boldsymbol{\nabla}\mathbf{G}\|_{L^2(I,L^2(\Omega))} + \|\boldsymbol{\nabla}\mathbf{T}\|_{L^\infty(I\times\Omega)}\|\boldsymbol{\nabla}\mathbf{u}\|_{L^2(I,L_0^2(\Omega))} + \|\boldsymbol{\nabla}\mathbf{u}_0\|_2\big).
\end{aligned}
$$

It follows by $h \to 0$ that

$$
\begin{aligned}
&\|\boldsymbol{\nabla}\mathbf{u}\|_{C(I,L^2(\Omega))} + \|\boldsymbol{\nabla}\mathbf{u}\|_{L^2(I,W^{1,2}(\Omega))} + \|\boldsymbol{\nabla}\pi\|_{L^2(I,L^2(\Omega))}\\
&\quad\leq C\,\big(\|\boldsymbol{\nabla}\mathbf{G}\|_{L^2(I,L^2(\Omega))} + \|\boldsymbol{\nabla}\mathbf{T}\|_{L^\infty(I\times\Omega)}\|\boldsymbol{\nabla}\mathbf{u}\|_{L^2(I,L_0^2(\Omega))} + \|\boldsymbol{\nabla}\mathbf{u}_0\|_2\big)\\
&\quad< \infty,
\end{aligned}
$$

which implies $\mathbf{u} \in C(I, W_0^{1,2}(\Omega)) \cap L^2(I, W_0^{2,2}(\Omega))$, justifying the following calculations. Let us rewrite (4.4) as

$$
\begin{aligned}
\partial_t \mathbf{u} - \operatorname{div}(\mathrm{Id}^{\mathrm{sym}}\,\mathbf{D}\mathbf{u}) + \boldsymbol{\nabla}\pi &= \operatorname{div}\mathbf{G} + \operatorname{div}((\mathbf{T} - \mathrm{Id}^{\mathrm{sym}})\mathbf{D}\mathbf{u}) && \text{on } I \times \Omega,\\
\operatorname{div}\mathbf{u} &= \varepsilon\Delta\pi && \text{on } I \times \Omega,\\
\mathbf{u}(0) &= \mathbf{u}_0 && \text{on } \Omega,
\end{aligned}
\tag{4.46}
$$

Since all eigenvalues of $\mathbf{T}$ are within $[\gamma_1, \gamma_2] = [\gamma_1, 1]$, the eigenvalues of $\mathbf{T} - \mathrm{Id}^{\mathrm{sym}}$ are within $[-1 + \gamma_1, 0]$. Thus

$$
|(\mathbf{T} - \mathrm{Id})\mathbf{D}\mathbf{u}| \leq (1 - \gamma_1)|\mathbf{D}\mathbf{u}|.
$$

Thus

$$\|(\mathbf{T}-\mathrm{Id})\mathbf{Du}\|_{L^s(I,L^s(\Omega))} \leq (1-\gamma_1)\|\mathbf{Du}\|_{L^s(I,L^s(\Omega))}. \tag{4.47}$$

Let $\mathbf{H} := \mathbf{G} + (\mathbf{T}-\mathrm{Id}^{\mathrm{sym}})\mathbf{Du}$, then

$$\begin{aligned}\mathbf{H}^{\mathrm{sym}} &:= \mathbf{G}^{\mathrm{sym}} + (\mathbf{T}-\mathrm{Id}^{\mathrm{sym}})\mathbf{Du},\\ \mathbf{H}^{\mathrm{anti}} &:= \mathbf{G}^{\mathrm{anti}}.\end{aligned}$$

Applying corollary 4.3 (with its constant $K \geq 1$) to (4.46) we get for all s with $2 \leq s \leq \min\{r,3\}$ and all $0<\mu<1$

$$\begin{aligned}\|\mathbf{u}\|_{C(I,B_s^{1-\frac{2}{s}}(\Omega))} \leq K^{\frac{s-2}{2}}\Big(&\|\mathbf{H}^{\mathrm{sym}}\|_{L^s(I,L^s(\Omega))}\\ &+\tfrac{C}{\sqrt{\mu}}\|\mathbf{H}^{\mathrm{anti}}\|_{L^s(I,L^s(\Omega))} + C\,\|\mathbf{u}_0\|_{B_s^{1-\frac{2}{s}}(\Omega))}\Big),\end{aligned} \tag{4.48}$$

$$\begin{aligned}\|\mathbf{Du}\|_{L^s(I,L^s(\Omega))} \leq K^{\frac{s-2}{2}}\tfrac{1}{\sqrt{1-\mu}}\Big(&\|\mathbf{H}^{\mathrm{sym}}\|_{L^s(I,L^s(\Omega))}\\ &+\tfrac{C}{\sqrt{\mu}}\|\mathbf{H}^{\mathrm{anti}}\|_{L^s(I,L^s(\Omega))} + C\,\|\mathbf{u}_0\|_{B_s^{1-\frac{2}{s}}(\Omega))}\Big),\end{aligned} \tag{4.49}$$

$$\|\pi\|_{L^r(I,L^r(\Omega))} \leq C\,\|\mathbf{H}\|_{L^r(I,L^r(\Omega))}, \tag{4.50}$$

From (4.49), the definition of $\mathbf{H}$, and (4.47) there follows

$$\begin{aligned}\|\mathbf{Du}\|_{L^s(I,L^s(\Omega))} \leq K^{\frac{s-2}{2}}\tfrac{1}{\sqrt{1-\mu}}\Big(&\|\mathbf{G}^{\mathrm{sym}}\|_{L^s(I,L^s(\Omega))} + (1-\gamma_1)\|\mathbf{Du}\|_{L^s(I,L^s(\Omega))}\\ &+\tfrac{C}{\sqrt{\mu}}\|\mathbf{G}^{\mathrm{anti}}\|_{L^s(I,L^s(\Omega))} + C\,\|\mathbf{u}_0\|_{B_s^{1-\frac{2}{s}}(\Omega))}\Big).\end{aligned} \tag{4.51}$$

Fix $\mu := \gamma_1/2$. If $s \to 2^+$, then

$$K^{\frac{s-2}{s}}\tfrac{1-\gamma_1}{\sqrt{1-\mu}} = K^{\frac{s-2}{s}}\tfrac{1-\gamma_1}{\sqrt{1-\gamma_1/2}} \searrow \tfrac{1-\gamma_1}{\sqrt{1-\gamma_1/2}} < 1-\tfrac{\gamma_1}{2}.$$

So there exists $s_0 > 2$, such that for all $2 < s < s_0$ there holds $K^{\frac{s-2}{s}}\frac{1-\gamma_1}{\sqrt{1-\mu}} < 1-\frac{\gamma_1}{2}$ and $K^{\frac{s-2}{s}} < 2$. For such s we deduce from (4.51)

$$\begin{aligned}\tfrac{\gamma_1}{2}\|\mathbf{Du}\|_{L^s(I,L^s(\Omega))} &\leq K^{\frac{s-2}{2}}\tfrac{1}{\sqrt{1-\gamma_1/2}}\Big(\|\mathbf{G}^{\mathrm{sym}}\|_{L^s(I,L^s(\Omega))}\\ &\qquad+\tfrac{C}{\sqrt{\gamma_1}}\|\mathbf{G}^{\mathrm{anti}}\|_{L^s(I,L^s(\Omega))} + C\,\|\mathbf{u}_0\|_{B_s^{1-\frac{2}{s}}(\Omega))}\Big)\\ &\leq C\,\Big(\|\mathbf{G}^{\mathrm{sym}}\|_{L^s(I,L^s(\Omega))}\\ &\qquad+\tfrac{C}{\sqrt{\gamma_1}}\|\mathbf{G}^{\mathrm{anti}}\|_{L^s(I,L^s(\Omega))} + C\,\|\mathbf{u}_0\|_{B_s^{1-\frac{2}{s}}(\Omega))}\Big),\end{aligned}$$

where we have used that $K^{\frac{s-2}{s}} \leq K \leq C$ and $\frac{1}{\sqrt{1-\gamma_1/2}} < 2$. Thus

$$\begin{aligned}\|\pi\|_{L^s(I,L^s(\Omega))} &\leq C\,\big((1+\tfrac{2C}{\gamma_1})\|\mathbf{G}\|_{L^s(I,L^s(\Omega))} + \tfrac{2C}{\gamma_1}\|\mathbf{u}_0\|_{B_s^{1-\frac{2}{s}}(\Omega)}\big)\\ &\leq \tfrac{C}{\gamma_1}\big(\|\mathbf{G}\|_{L^s(I,L^s(\Omega))} + \|\mathbf{u}_0\|_{B_s^{1-\frac{2}{s}}(\Omega)}\big).\end{aligned}$$

This proves the estimates for $\mathbf{Du}$ and π but we still have to determine a nice bound for s_0 in terms of γ_1. Recall that we require s_0 to satisfy $K^{\frac{s_0-2}{s_0}}\frac{1-\gamma_1}{\sqrt{1-\gamma_1/2}} < 1-\frac{\gamma_1}{2}$, which is equivalent to

$$K^{\frac{s_0-2}{s_0}} < \frac{(1-\frac{\gamma_1}{2})^{\frac{3}{2}}}{1-\gamma_1}$$

Since

$$1+\tfrac{\gamma_1}{4} < \frac{(1-\frac{\gamma_1}{2})^{\frac{3}{2}}}{1-\gamma_1}$$

for all $0<\gamma_1<1$ it suffices to find s_0, such that

$$K^{\frac{s_0-2}{s_0}} < 1+\frac{\gamma_1}{4}.$$

Let $f(s) := K^{\frac{s-2}{s}}$, then $f(2)=1$. Furthermore f is Lipschitz on $[2,\infty)$, i.e. $|f'(s)| = |\ln(K)K^{\frac{s-2}{s}}\frac{2}{s^2}| \le \ln(K)\frac{K}{2}$. Since $K\ge 1$ the function f is monotonously increasing. So for all $s>2$

$$f(s) < 1+(s-2)\frac{\ln(K)K}{2}.$$

Let $\kappa := \frac{1}{2\ln(K)K}$ and $s_0 := 2+\kappa\gamma_1$, then for all s with $2<s<s_0$ there holds

$$\begin{aligned} K^{\frac{s-2}{s}} &= f(s) < f(s_0) < 1+(s_0-2)\tfrac{\ln(K)K}{2} \\ &= 1+\kappa\gamma_1\tfrac{\ln(K)K}{2} = 1+\tfrac{\gamma_1}{4}. \end{aligned}$$

This proves the correct choice of s_0 and κ. So we have proven (note $\gamma_2=1$) that (4.51) holds for all s with

$$2 < s < 2+\kappa\frac{\gamma_1}{\gamma_2} \qquad \text{with } \kappa = \frac{1}{2\ln(K)K}, \tag{4.52}$$

i.e.

$$\begin{aligned} \|\mathbf{Du}\|_{L^s(I,L^s(\Omega))} \le \tfrac{C}{\gamma_1}\Big(&\|\mathbf{G}^{\mathrm{sym}}\|_{L^s(I,L^s(\Omega))} \\ &+ \tfrac{C}{\sqrt{\gamma_1}}\|\mathbf{G}^{\mathrm{anti}}\|_{L^s(I,L^s(\Omega))} + C\,\|\mathbf{u}_0\|_{B_s^{1-\frac{2}{s}}(\Omega))}\Big), \end{aligned}$$

For such s there follows from (4.48), $\mu=\gamma_1/2$, $K^{\frac{s-2}{s}} \le K \le C$, and (4.51)

$$\begin{aligned} \|\mathbf{u}\|_{C(I,B_s^{1-\frac{2}{s}}(\Omega))} &\le C\Big(\|\mathbf{G}^{\mathrm{sym}}\|_{L^s(I,L^s(\Omega))} + (1-\gamma_1)\|\mathbf{Du}\|_{L^s(I,L^s(\Omega))} \\ &\qquad + \tfrac{C}{\sqrt{\gamma_1}}\|\mathbf{G}^{\mathrm{anti}}\|_{L^s(I,L^s(\Omega))} + C\,\|\mathbf{u}_0\|_{B_s^{1-\frac{2}{s}}(\Omega))}\Big). \\ &\le \tfrac{C}{\gamma_1}\Big(\|\mathbf{G}^{\mathrm{sym}}\|_{L^s(I,L^s(\Omega))} \\ &\qquad + \tfrac{C}{\sqrt{\gamma_1}}\|\mathbf{G}^{\mathrm{anti}}\|_{L^s(I,L^s(\Omega))} + C\,\|\mathbf{u}_0\|_{B_s^{1-\frac{2}{s}}(\Omega))}\Big). \end{aligned}$$

This proves the missing estimate for $\mathbf{u}$. Overall we have proven the lemma for $\gamma_2=1$, $\mathbf{G},\mathbf{T}\in C^\infty(I\times\Omega)$, and $\mathbf{u}_0\in C^\infty(\Omega)$. If $\gamma_2=1$ and $\mathbf{G},\mathbf{T},\mathbf{u}_0$ are not smooth, choose

$\mathbf{G}^n$, $\mathbf{T}^n$, $\mathbf{u}_0^n$ smooth, such that

$$\begin{aligned} \mathbf{G}^n &\to \mathbf{G} && \text{in } L^r(I, L^r(\Omega)), \\ \mathbf{T}^n &\to \mathbf{T} && \text{a.e. in } I \times \Omega, \\ \mathbf{T}^n &\overset{*}{\rightharpoonup} \mathbf{T} && \text{in } L^\infty(I \times \Omega), \\ \mathbf{u}_0^n &\to \mathbf{u}_0 && \text{in } B_r^{1-\frac{2}{r}}(\Omega), \end{aligned}$$

and for all $\mathbf{B} \in \mathbb{R}^{d\times d}$

$$\begin{gathered} \|\mathbf{G}^n\|_{L^r(I,L^r(\Omega))} \le 2\,\|\mathbf{G}\|_{L^r(I,L^r(\Omega))}, \\ \|\mathbf{u}_0^n\|_{B_r^{1-\frac{2}{r}}(\Omega)} \le 2\,\|\mathbf{u}_0\|_{B_r^{1-\frac{2}{r}}(\Omega)}, \\ \gamma_1|\mathbf{B}^{\mathrm{sym}}|^2 \le \sum_{jk,lm} T^n_{jk,lm} B_{jk} B_{lm} \le \gamma_2 |\mathbf{B}^{\mathrm{sym}}|^2, \end{gathered}$$

where $\mathbf{T}^n$ has the same symmetry properties as $\mathbf{T}$. Let $\mathbf{u}^n, \pi^n$ be the solutions of

$$\begin{aligned} \partial_t \mathbf{u}^n - \operatorname{div}(\mathbf{T}^n \mathbf{D}\mathbf{u}^n) + \boldsymbol{\nabla}\pi^n &= \operatorname{div}\mathbf{G}^n && \text{on } I \times \Omega, \\ \operatorname{div}\mathbf{u}^n &= \varepsilon\Delta\pi^n && \text{on } I \times \Omega, \\ \mathbf{u}^n(0) &= \mathbf{u}_0^n && \text{on } \Omega. \end{aligned}$$

Then $\mathbf{u}^n, \pi^n$ satisfy (4.45) with $\mathbf{T}, \mathbf{G}, \mathbf{u}_0$ replaced by $\mathbf{T}^n, \mathbf{G}^n, \mathbf{u}_0^n$. Since (4.45) is robust under the limit $n \to \infty$, this proves that (4.45) holds for $\mathbf{u}, \pi$. This concludes the proof of lemma 4.4 for $\gamma_2 = 1$. The arbitrary case $\gamma_2 \neq 1$ can be deduced from the case γ_2 by the transformation (4.35) (scaling in time). This proves the lemma. □

REMARK 4.5. *Note that it is not necessary for the right-hand side of (4.4) to have the form* $\operatorname{div}\mathbf{G}$ *with* $\mathbf{G} \in L^r(I, L^r(\Omega, \mathbb{R}^{d\times d}))$. *It is also possible to admit* $\mathbf{f}$ *with* $\mathbf{f} \in L^r(I, (W^{1,r}(\Omega))')$ *as a right-hand side. This can be done by the following procedure. Let* $\mathbf{v}$ *be the solution of* $-\Delta\mathbf{v} = \mathbf{f}$ *and define* $\mathbf{G} := -\boldsymbol{\nabla}\mathbf{v}$. *So* $\operatorname{div}\mathbf{G} = -\Delta\mathbf{v} = \mathbf{f}$ *and there holds*

$$\|\mathbf{G}\|_{L^r(I,L^r(\Omega))} \le C\,\|\mathbf{f}\|_{L^r(I,(W_0^{1,r}(\Omega))')}.$$

But by using $\mathbf{f}$ *rather than* $\mathbf{G}$ *we cannot distinguish between* $\mathbf{G}^{\mathrm{sym}}$ *and* $\mathbf{G}^{\mathrm{anti}}$ *anymore. Thus the estimates for* $\mathbf{u}, \mathbf{D}\mathbf{u}, \pi$ *read as follows*

$$\begin{aligned} \sqrt{\gamma_2}\,\|\mathbf{u}\|_{C(I,B_s^{1-\frac{2}{s}}(\Omega))} &\le \tfrac{C}{\sqrt{\gamma_1}^3}\|\mathbf{f}\|_{L^s(I,(W_0^{1,s}(\Omega))')} + C\,\tfrac{\sqrt{\gamma_2}}{\gamma_1}\,\|\mathbf{u}_0\|_{B_s^{1-\frac{2}{s}}(\Omega))}\Big), \\ \|\mathbf{D}\mathbf{u}\|_{L^s(I,L^s(\Omega))} &\le \tfrac{C}{\sqrt{\gamma_1}^3}\|\mathbf{f}\|_{L^s(I,(W_0^{1,s}(\Omega))')} + C\,\tfrac{\sqrt{\gamma_2}}{\gamma_1}\,\|\mathbf{u}_0\|_{B_s^{1-\frac{2}{s}}(\Omega))}\Big), \\ \|\pi\|_{L^s(I,L^s(\Omega))} &\le \tfrac{C\,\gamma_2}{\gamma_1}\,\Big(\|\mathbf{f}\|_{L^s(I,(W_0^{1,s}(\Omega))')} + \sqrt{\gamma_2}\,\|\mathbf{u}_0\|_{B_s^{1-\frac{2}{s}}(\Omega)}\Big), \end{aligned} \tag{4.53}$$

Let us also make a little remark on the stationary case, which is a lot easier than the parabolic one. Consider the system

$$\begin{aligned} -\operatorname{div}(\mathbf{T}\mathbf{D}\mathbf{u}) + \boldsymbol{\nabla}\pi &= \operatorname{div}\mathbf{G} && \text{on } \Omega, \\ \operatorname{div}\mathbf{u} &= \varepsilon\Delta\pi && \text{on } \Omega, \end{aligned} \tag{4.54}$$

with the same $\mathbf{T}$ is as above. Then the following lemma holds.

Lemma 4.6. *Let $r > 2$, $\mathbf{G} \in L^r(\Omega;\mathbb{R}^{d\times d})$, and let $\mathbf{T}$ be as described above. Then there exists a constant $\kappa > 0$ (independent of γ_1 and γ_2), such that for all s with*

$$2 \le s \le \min\left\{r, 2 + \kappa\frac{\gamma_1}{\gamma_2}\right\}$$

and all $\varepsilon \ge 0$ there exists a unique weak solution $\mathbf{u}, \pi$ of (4.54), such that

$$\|\mathbf{D}\mathbf{u}\|_{L^s(\Omega)} \le \tfrac{C\sqrt{\gamma_2}}{\sqrt{\gamma_1}^3}\,\|\mathbf{G}\|_{L^s(\Omega)},$$
$$\|\pi\|_{L^s(\Omega)} \le \tfrac{C\,\gamma_2}{\gamma_1}\,\|\mathbf{G}\|_{L^s(\Omega)},$$

where the constants do not depend on ε, γ_1, γ_2, and s.

Proof. The proof of this result is just like the one of lemma 4.2 and corollary 4.3. It is based on the stationary versions of the intermediate results of lemma 4.1, lemma 4.2, corollary 4.3, and lemma 4.4. The proof is far simpler, since instead of referring to the heat equation, we can immediately apply the theory of the Laplacian equation. The modifications due to $\varepsilon > 0$ do not differ from the parabolic case treated above. To pass from $\gamma_2 = 1$ to $\gamma_2 \ne 1$ we use the scaling

$$\begin{aligned} \widetilde{\varepsilon} &:= \gamma_2\varepsilon, & & \\ \widetilde{\gamma}_1 &:= \gamma_1/\gamma_2 & \widetilde{\gamma}_2 &:= 1, \\ \widetilde{\mathbf{u}}(x) &:= \mathbf{u}(x), & \widetilde{\pi}(x) &:= \tfrac{1}{\gamma_2}\pi(x), \\ \widetilde{\mathbf{G}}(x) &:= \tfrac{1}{\gamma_2}\mathbf{G}(\gamma_2, x), & \widetilde{\mathbf{T}}(x) &:= \tfrac{1}{\gamma_2}\mathbf{T}(\gamma_2, x), \end{aligned} \tag{4.55}$$

which differs slightly from the scaling in the parabolic case. □

Remark 4.7. *Note that, as in the parabolic case, it is not necessary in lemma 4.6 for the right-hand side of (4.54) to have the form $\operatorname{div}\mathbf{G}$ with $\mathbf{G} \in L^r(\Omega, \mathbb{R}^{d\times d})$. It is also possible to admit $\mathbf{f}$ with $\mathbf{f} \in (W_0^{1,r}(\Omega))'$ as a right-hand side. (Compare with remark 4.5).*

3. Shear Dependent Flow — Strong Solutions

As a further intermediate step to the $C^{1,\alpha}(I \times \Omega)$ regularity of the systems (4.1), (4.2), and (4.3) we will now examine suitable approximations of these systems. Especially we will replace the extra stress $\mathbf{S}$ in these systems by its A–approximation $\mathbf{S}^A$. We will see that these approximated systems have strong solutions. Hereby we refer to the term *strong solution* as a weak solution $\mathbf{u}, \pi$ which additionally satisfies $\mathbf{u} \in L^q(I, W^{2,2}(\Omega))$ for some $1 < q < \infty$ and $\partial_t\mathbf{u} \in L^\infty(I, L^2(\Omega))$. We say $\mathbf{u}, \pi$ is a weak solution if and only if $\mathbf{u} \in C(I, L^2_0(\Omega))$, $\mathbf{D}\mathbf{u} \in L^{p(\cdot)}(I \times \Omega)$, $\pi \in L^{p_0'}(I, L_0^{p_0'}(\Omega))$, and $\mathbf{u}, \pi$ is a solution in the sense of distributions.

Let us remind that from here to the end of the chapter all considerations are restricted to the two–space–dimensional case, i.e. $d = 2$.

Instead of examining both versions of pressure stabilization at the same time, we will begin with the system

$$\begin{aligned} \partial_t\mathbf{u} - \operatorname{div}(\mathbf{S}(\mathbf{D}\mathbf{u})) + ((\mathbf{u} - \varepsilon\boldsymbol{\nabla}\pi)\cdot\boldsymbol{\nabla})\mathbf{u} + \boldsymbol{\nabla}\pi &= \mathbf{f} && \text{on } I \times \Omega, \\ \operatorname{div}\mathbf{u} &= \varepsilon\Delta\pi && \text{on } I \times \Omega, \\ \mathbf{u}(0) &= \mathbf{u}_0 && \text{on } \Omega. \end{aligned} \tag{4.56}$$

Note that this already includes the non–pressure stabilized case. Later on we will sketch how to proceed if the convective term is replaced by $(\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u}+\frac{1}{2}(\operatorname{div}\mathbf{u})\mathbf{u}$. But first, we will prove existence of strong solutions of the following approximated system

$$\begin{aligned}\partial_t\mathbf{u}-\operatorname{div}(\mathbf{S}^A(\mathbf{D}\mathbf{u}))+((\mathbf{u}-\varepsilon\boldsymbol{\nabla}\pi)\cdot\boldsymbol{\nabla})\mathbf{u}+\boldsymbol{\nabla}\pi=\mathbf{f}\qquad&\text{on } I\times\Omega,\\ \operatorname{div}\mathbf{u}=\varepsilon\Delta\pi\quad&\text{on } I\times\Omega,\\ \mathbf{u}(0)=\mathbf{u}_0\qquad&\text{on }\Omega.\end{aligned}\tag{4.57}$$

LEMMA 4.8. *Let $p:\Omega\to(1,2]$ be uniformly Lipschitz continuous, i.e. $p\in W^{1,\infty}(I\times\Omega)$, with $1<p_\infty\le p_0\le 2$. Let $\mathbf{S}$ be induced by a p–potential F,Φ, which additionally satisfies (3.27). Let $\mathbf{u}_0\in W^{2,2}(\Omega)\cap W_0^{1,2}(\Omega)$ and $\mathbf{f}\in L^2(I,W_0^{1,2}(\Omega))$ and $\partial_t\mathbf{f}\in L^2(I,L_0^2(\Omega))$. Then there exists a constant $\varepsilon_0>0$, such that for all $A\ge 1$ and all ε with $0\le\varepsilon\le\varepsilon_0$ the approximated system (4.57) has a weak solution $\mathbf{u}^A,\pi^A$ with*

$$\begin{aligned}\|\boldsymbol{\nabla}\mathbf{u}^A\|_{L^\infty(I,L^2(\Omega))}+\|\partial_t\mathbf{u}^A\|_{L^\infty(I,L^2(\Omega))}&\le C,\\ \|\mathcal{I}_\Phi{}^A(\mathbf{u}^A)\|_{L^1(I)}+\|\mathcal{J}_\Phi{}^A(\mathbf{u}^A)\|_{L^1(I)}&\le C,\\ \|\mathbf{u}^A\|_{C(I,W^{1,2p_\infty}(\Omega))}&\le C,\end{aligned}\tag{4.58}$$

where C does not depend on A. (Especially $\mathbf{u}^A,\pi^A$ is a strong solution of (4.57).)

PROOF. As in lemma 4.1 we will prove existence by means of the Galerkin method distinguishing the cases $\varepsilon=0$ and $\varepsilon>0$. Again we start with the case $\varepsilon=0$. For the sake of readability we will omit the index A of the functions $\mathbf{u}^A$ and π^A.

Case $\varepsilon=0$: Let $\{\boldsymbol{\omega}^r\}$ denote the set consisting of eigenfunctions of the Stokes operator $\mathbb{S}$. Let λ_r be the corresponding eigenvalues and define X_N by $X_N=\operatorname{span}\{\boldsymbol{\omega}^1,\dots,\boldsymbol{\omega}^N\}$. Let us recall that we are only looking for (space periodic) solutions with mean value zero. So the $\boldsymbol{\omega}^r$ all fulfill $\langle\boldsymbol{\omega}^r,1\rangle=0$. Define $P^N\mathbf{u}=\sum_{r=1}^N\langle\mathbf{u},\boldsymbol{\omega}^r\rangle\boldsymbol{\omega}^r$. Then

$$\lambda_r\langle\mathbf{u}^N,\boldsymbol{\omega}^r\rangle=\langle\mathbf{u}^N,\mathbb{S}\boldsymbol{\omega}^r\rangle=\langle\boldsymbol{\nabla}\mathbf{u}^N,\boldsymbol{\nabla}\boldsymbol{\omega}^r\rangle$$

and the $P^N:W_0^{s,2}\to(X_N,\|\cdot\|_{s,2})$ are uniformly continuous for all $0\le s\le 2$. (See [**MNRR96**] for a proof.)

Let us define $\mathbf{u}^N(t,x)=\sum_{r=1}^N c_r^N(t)\boldsymbol{\omega}^r(x)$ and $\mathbf{f}^N=P^N\mathbf{f}$, where the coefficients $c_r^N(t)$ solve the Galerkin system (for all $1\le r\le N$)

$$\begin{aligned}\big\langle\partial_t\mathbf{u}^N,\boldsymbol{\omega}^r\big\rangle+\langle\mathbf{S}^A(\mathbf{D}(\mathbf{u}^N)),\mathbf{D}\boldsymbol{\omega}^r\rangle+\langle(\mathbf{u}^N\cdot\boldsymbol{\nabla})\mathbf{u}^N,\boldsymbol{\omega}^r\rangle&=\langle\mathbf{f}^N,\boldsymbol{\omega}^r\rangle,\\ \mathbf{u}^N(0)&=P^N\mathbf{u}_0.\end{aligned}\tag{4.59}$$

Since the matrix $\langle\boldsymbol{\omega}_j,\boldsymbol{\omega}_k\rangle$ with $j,k=1,\dots,N$ is positive definite, this can be rewritten as a system of ordinary differential equations. This in turn fulfills the Carathéodory conditions and is therefore solvable locally in time, i.e. on a small time interval $I^*=[0,T^*)$. Since $\mathbf{f},\partial_t\mathbf{f}\in L^2(I^*,L_0^2(\Omega))$, $\mathbf{f}^N=P^N\mathbf{f}$, and $\partial_t\mathbf{f}^N=P^N(\partial_t\mathbf{f})$, we have $\mathbf{f}^N,\partial_t\mathbf{f}^N\in L^2(I^*,X_N)$, which implies $\partial_t c_r^N,\partial_t^2 c_r^N\in L^2(I^*)$ (norms may depend on N). This implies $\mathbf{u}^N,\partial_t\mathbf{u}^N,\partial_t^2\mathbf{u}^N\in L^2(I^*,X_N)$. To ensure solvability for large times at least for this finite dimensional problem we have to establish a first a priori estimate.

Since $\mathbf{u}^N,\partial_t\mathbf{u}^N,\partial_t^2\mathbf{u}^N\in L^2(I^*,X_N)$, we can test the Galerkin system (4.59) with $\mathbf{u}^N$ and get

$$\tfrac{1}{2}d_t\|\mathbf{u}^N\|_2^2+\langle\mathbf{S}^A(\mathbf{D}\mathbf{u}^N),\mathbf{D}\mathbf{u}^N\rangle=\langle\mathbf{f}^N,\mathbf{u}^N\rangle.$$

Note that $\langle (\mathbf{u}^N \cdot \boldsymbol{\nabla})\mathbf{u}^N, \mathbf{u}^N \rangle = 0$ due to $\operatorname{div} \mathbf{u}^N = 0$. The coercivity of $\mathbf{S}^A$ implies

$$\tfrac{1}{2} d_t \|\mathbf{u}^N\|_2^2 + A^{p-2} \|\mathbf{D}\mathbf{u}^N\|_2^2 + |\mathbf{D}\mathbf{u}^N|_{p(\cdot)} \le \|\mathbf{f}^N\|_2^2 + \|\mathbf{u}^N\|_2^2 + C\,|\Omega|.$$

By Gronwall's lemma and $\|\mathbf{f}\|_{L^2(I,L^2(\Omega))} \le C$

$$\tfrac{1}{2} \max_{I^*} \|\mathbf{u}^N\|_2^2 + \int_{I^*}\int_{\Omega} |\mathbf{D}\mathbf{u}^N|^p \,dx\,dt \le C. \tag{4.60}$$

This implies (independent of T^* with $T^* < T$)

$$\|c_r^N\|_{L^\infty(I^*)} \le C. \tag{4.61}$$

As a consequence we can iterate Carathéodory's theorem to push the solvability of the Galerkin system (4.59) up to any fixed time interval $I = [0, T)$. (Compare with the proof lemma 4.1.)

Since inequality (4.60) remains valid for I^* replaced by I, there holds (independently of N)

$$\|\mathbf{u}^N\|_{L^\infty(I,L^2(\Omega))} + A^{p-2}\|\mathbf{D}\mathbf{u}^N\|_2^2 + \|\mathbf{D}\mathbf{u}^N\|_{L^{p(\cdot)}(I\times\Omega)} \le C.$$

We got the first a priori estimate by using $\mathbf{u}^N$ as a test function. To derive our second a priori estimate we want to use $\mathbb{S}\mathbf{u}^N$ as a test function. The special choice of base functions $\boldsymbol{\omega}^r$ ensures that we do not leave X_N, the space of admissible test functions: More explicitly we multiply the r-th equation of the Galerkin system (4.59) by $\lambda_r c_r^N$, use the definition of the $\boldsymbol{\omega}^r$, λ^r, and sum up over $r = 1, \dots, N$ to obtain

$$\langle \partial_t \mathbf{u}^N, \mathbb{S}\mathbf{u}^N\rangle - \langle \mathbf{S}^A(\mathbf{D}\mathbf{u}^N), \mathbf{D}\mathbb{S}\mathbf{u}^N\rangle - \langle (\mathbf{u}^N \cdot \boldsymbol{\nabla})\mathbf{u}^N, \mathbb{S}\mathbf{u}^N\rangle = \langle \boldsymbol{\nabla}\mathbf{f}^N, \boldsymbol{\nabla}\mathbf{u}^N\rangle.$$

Due to the space periodicity $\mathbb{S}\boldsymbol{\omega}^r = -\Delta\boldsymbol{\omega}^r$ for all r, so

$$\tfrac{1}{2} d_t \|\boldsymbol{\nabla}\mathbf{u}^N\|_2^2 - \langle \mathbf{S}^A(\mathbf{D}\mathbf{u}^N), \mathbf{D}\Delta\mathbf{u}^N\rangle - \langle (\mathbf{u}^N \cdot \boldsymbol{\nabla})\mathbf{u}^N, \Delta\mathbf{u}^N\rangle = \langle \boldsymbol{\nabla}\mathbf{f}^N, \boldsymbol{\nabla}\mathbf{u}^N\rangle.$$

Let us simplify the third term on the left-hand side:

$$\begin{aligned}
\langle (\mathbf{u}^N \cdot \boldsymbol{\nabla})\mathbf{u}^N, -\Delta\mathbf{u}^N\rangle &= \sum_{ijk} \langle -\mathbf{u}_i^N \partial_i \mathbf{u}_j^N, \partial_k^2 \mathbf{u}_j^N\rangle \\
&= \sum_{ijk} \langle \partial_k \mathbf{u}_i^N \partial_i \mathbf{u}_j^N, \partial_k \mathbf{u}_j^N\rangle + \sum_{ijk} \langle \mathbf{u}_i^N, \partial_i(\tfrac{1}{2}(\partial_k \mathbf{u}_j^N)^2)\rangle \\
&= \sum_{ijk} \langle \partial_k \mathbf{u}_i^N \partial_i \mathbf{u}_j^N, \partial_k \mathbf{u}_j^N\rangle - \sum_{ijk} \langle \partial_i \mathbf{u}_i^N, (\tfrac{1}{2}(\partial_k \mathbf{u}_j^N)^2)\rangle \\
&= \sum_{ijk} \langle \partial_k \mathbf{u}_i^N \partial_i \mathbf{u}_j^N, \partial_k \mathbf{u}_j^N\rangle, \qquad \text{since } \operatorname{div} \mathbf{u}^N = 0.
\end{aligned}$$

Since $\operatorname{div} \mathbf{u}^N = 0$ there holds $\partial_1 u_1^N = -\partial_2 u_2^N$. This implies that in our special two dimensional case we have the following pointwise identity:

$$\sum_{ijk} (\partial_k \mathbf{u}_i^N)(\partial_i \mathbf{u}_j^N)(\partial_k \mathbf{u}_j^N) = 0.$$

In other words, if $\mathbf{v}$ is a divergence free function, then $\Delta\mathbf{v}$ is in the two dimensional space periodic case orthogonal to $(\mathbf{v} \cdot \boldsymbol{\nabla})\mathbf{v}$. This observation is quite crucial for the calculations, since it enables us to test with both $\mathbf{u}^N$ and $-\Delta\mathbf{u}^N$ without having to control the convective term. As a result we will get existence of a solution for large times without restrictions to p_∞. Note that this is not the case for the non–space periodic setting, where one has either to restrict oneself to small times or to some lower bound for p_∞ in order to control the convection. So far we have shown

$$\tfrac{1}{2}d_t\|\boldsymbol{\nabla}\mathbf{u}^N\|_2^2 - \langle \mathbf{S}^A(\mathbf{D}\mathbf{u}^N), \mathbf{D}\Delta\mathbf{u}^N\rangle = \langle \boldsymbol{\nabla}\mathbf{f}^N, \boldsymbol{\nabla}\mathbf{u}^N\rangle.$$

Let us simplify the second term on the left-hand side:

$$\begin{aligned}
\langle \mathbf{S}^A(\mathbf{D}\mathbf{u}^N), -\mathbf{D}\Delta\mathbf{u}^N\rangle &= \sum_r \langle \partial_r(\mathbf{S}^A(\mathbf{D}\mathbf{u}^N)), \partial_r\mathbf{D}\mathbf{u}^N\rangle \\
&= \sum_{rkl} \big\langle \partial_r\big((\partial_{kl}\Phi^A)(\mathbf{D}\mathbf{u}^N)\big), \partial_r D_{kl}\mathbf{u}^N\big\rangle \\
&= \sum_{rijkl} \big\langle (\partial_{ij}\partial_{kl}\Phi^A)(\mathbf{D}\mathbf{u}^N)\partial_r D_{ij}\mathbf{u}^N, \partial_r D_{kl}\mathbf{u}^N\big\rangle \\
&\quad + \sum_{rkl} \big\langle (\partial_r\partial_{kl}\Phi^A)(\mathbf{D}\mathbf{u}^N), \partial_r D_{kl}\mathbf{u}^N\big\rangle \\
&\overset{(3.28),(3.15)}{\geq} \mathcal{I}_\Phi{}^A(\mathbf{u}^N) - C\int_\Omega (\widetilde{D}\mathbf{u}^N)^{p-1}\ln(\widetilde{D}\mathbf{u}^N)|\boldsymbol{\nabla}\mathbf{D}\mathbf{u}^N|\,dx,
\end{aligned}$$

where we have used $\|\boldsymbol{\nabla}p\|_\infty \leq C$. Using (3.49) we get

$$\langle \mathbf{S}^A(\mathbf{D}\mathbf{u}^N), -\mathbf{D}\Delta\mathbf{u}^N\rangle \geq \tfrac{1}{2}\mathcal{I}_\Phi{}^A(\mathbf{u}^N) - C\int_\Omega (\widetilde{D}\mathbf{u}^N)^p\ln^2(\widetilde{D}\mathbf{u}^N)\,dx.$$

This gives

$$\tfrac{1}{2}d_t\|\boldsymbol{\nabla}\mathbf{u}^N\|_2^2 + \tfrac{1}{2}\mathcal{I}_\Phi{}^A(\mathbf{u}^N) \leq |\langle \boldsymbol{\nabla}\mathbf{f}^N, \boldsymbol{\nabla}\mathbf{u}^N\rangle| + C\,\big\|(\widetilde{D}\mathbf{u}^N)^{\frac{p}{2}}\ln(\widetilde{D}\mathbf{u}^N)\big\|_2^2. \tag{4.62}$$

Let $2 < s < 3$, then with (3.51) we deduce

$$\begin{aligned}
\big\|(\widetilde{D}\mathbf{u}^N)^{\frac{p}{2}}\ln(\widetilde{D}\mathbf{u}^N)\big\|_2^2 &\leq \big\|(\widetilde{D}\mathbf{u}^N)^{\frac{p}{2}}\big\|_s^s \\
&\leq \big\|(\widetilde{D}\mathbf{u}^N)^{\frac{p}{2}}\big\|_2^2\big\|(\widetilde{D}\mathbf{u}^N)^{\frac{p}{2}}\big\|_{1,2}^{s-2} \\
&\leq C_\varepsilon\big\|(\widetilde{D}\mathbf{u}^N)^{\frac{p}{2}}\big\|_2^{\frac{4}{4-s}} + \varepsilon\big\|(\widetilde{D}\mathbf{u}^N)^{\frac{p}{2}}\big\|_{1,2}^2 \\
&\leq C_\varepsilon\big\|(\widetilde{D}\mathbf{u}^N)^{\frac{p}{2}}\big\|_2^{\frac{4}{4-s}} + 2\,\varepsilon\big\|(\widetilde{D}\mathbf{u}^N)^{\frac{p}{2}}\big\|_2^2 + 2\,\varepsilon\big\|\boldsymbol{\nabla}\big((\widetilde{D}\mathbf{u}^N)^{\frac{p}{2}}\big)\big\|_2^2 \\
&\leq C_\varepsilon\big\|(\widetilde{D}\mathbf{u}^N)^{\frac{p}{2}}\big\|_2^{\frac{4}{4-s}} + 2\,\varepsilon\big\|(\widetilde{D}\mathbf{u}^N)^{\frac{p}{2}}\big\|_2^2 \\
&\quad + \varepsilon\, C\,\mathcal{I}_\Phi(\mathbf{u}^N) + \varepsilon\, C\int_\Omega (\widetilde{D}\mathbf{u}^N)^p\ln^2\widetilde{D}\mathbf{u}^N\,dx.
\end{aligned}$$

This proves

$$\begin{aligned}
\big\|(\widetilde{D}\mathbf{u}^N)^{\frac{p}{2}}\ln(\widetilde{D}\mathbf{u}^N)\big\|_2^2 &\leq C\big\|(\widetilde{D}\mathbf{u}^N)^{\frac{p}{2}}\big\|_2^{\frac{4}{4-s}} + \big\|(\widetilde{D}\mathbf{u}^N)^{\frac{p}{2}}\big\|_2^2 + \mathcal{I}_\Phi(\mathbf{u}^N) \\
&\leq C + C\big\|(\widetilde{D}\mathbf{u}^N)^{\frac{p}{2}}\big\|_2^{\frac{4}{4-s}} + \mathcal{I}_\Phi(\mathbf{u}^N).
\end{aligned}$$

Since $2 < s < 3$, we have $\frac{4}{4-s} < 4$. Thus

$$\big\|(\widetilde{D}\mathbf{u}^N)^{\frac{p}{2}}\ln(\widetilde{D}\mathbf{u}^N)\big\|_2^2 \leq C + C\,\big\|\boldsymbol{\nabla}\mathbf{u}^N\big\|_2^2\big\|(\widetilde{D}\mathbf{u}^N)^{\frac{p}{2}}\big\|_2^2 + \mathcal{I}_\Phi(\mathbf{u}^N). \tag{4.63}$$

Thus (3.51) and (4.62) imply

$$\begin{aligned}\tfrac{1}{2}d_t\|\nabla\mathbf{u}^N\|_2^2 + c\left\|(\widetilde{D}\mathbf{u}^N)^{\frac{p}{2}}\ln(\widetilde{D}\mathbf{u}^N)\right\|_2^2 + c\left\|\nabla\big((\widetilde{D}\mathbf{u}^N)^{\frac{p}{2}}\big)\right\|_2^2 + c\,\mathcal{I}_\Phi{}^A(\mathbf{u}^N)\\ \le |\langle\nabla\mathbf{f}^N,\nabla\mathbf{u}^N\rangle| + C + C\left\|\nabla\mathbf{u}^N\right\|_2^2\left\|(\widetilde{D}\mathbf{u}^N)^{\frac{p}{2}}\right\|_2^2\\ \le \|\nabla\mathbf{f}^N\|_2^2 + \|\nabla\mathbf{u}^N\|_2^2 + C + C\left\|\nabla\mathbf{u}^N\right\|_2^2|\widetilde{D}\mathbf{u}^N|_{p(\cdot)}.\end{aligned}\tag{4.64}$$

Since $|\widetilde{D}\mathbf{u}^N|_{p(\cdot)} \in L^1(I)$, an application of Gronwall's inequality gives

$$\|\nabla\mathbf{u}^N\|_{L^\infty(I,L^2(\Omega))}^2 + \left\|(\widetilde{D}\mathbf{u}^N)^{\frac{p}{2}}\ln(\widetilde{D}\mathbf{u}^N)\right\|_{L^2(I,L^2(\Omega))}^2\tag{4.65}$$

$$+\left\|\nabla\big((\widetilde{D}\mathbf{u}^N)^{\frac{p}{2}}\big)\right\|_{L^2(I,L^2(\Omega))}^2 + \left\|\mathcal{I}_\Phi{}^A(\mathbf{u}^N)\right\|_{L^1(I)} \le C_N,\tag{4.66}$$

where the constant C_N depends on $\|\mathbf{f}^N\|_{L^2(I,W^{1,2}(\Omega))}$, $\|\nabla\mathbf{u}_0^N\|_2$, and T. But since $\mathbf{f}^N = P^N\mathbf{f}$, $\mathbf{u}_0^N = P^N\mathbf{u}_0$, the projections P^N are uniformly continuous on $W_0^{1,2}(\Omega)$, and the norms $\|\mathbf{f}^N\|_{L^2(I,W^{1,2}(\Omega))}$ and $\|\nabla\mathbf{u}_0^N\|_2$ are finite, we see that C_N is bounded uniformly with respect to N. Overall uniformly in N:

$$\begin{aligned}\|\nabla\mathbf{u}^N\|_{L^\infty(I,L^2(\Omega))}^2 + \left\|(\widetilde{D}\mathbf{u}^N)^{\frac{p}{2}}\ln(\widetilde{D}\mathbf{u}^N)\right\|_{L^2(I,L^2(\Omega))}^2\\ +\left\|\nabla\big((\widetilde{D}\mathbf{u}^N)^{\frac{p}{2}}\big)\right\|_{L^2(I,L^2(\Omega))}^2 + \left\|\mathcal{I}_\Phi{}^A(\mathbf{u}^N)\right\|_{L^1(I)} \le C.\end{aligned}\tag{4.67}$$

Thus by lemma 3.14

$$\begin{aligned}\|\nabla^2\mathbf{u}^N\|_{L^2(I,L^{\frac{4}{4-p_\infty}}(\Omega))}^2 &\le C\int_I \mathcal{I}_\Phi{}^A(\mathbf{u}^N)\left\|(\widetilde{D}\mathbf{u}^N)^{\frac{2-p}{2}}\right\|_{\frac{4}{2-p_\infty}}^2 dt\\ &\le C\int_I \mathcal{I}_\Phi{}^A(\mathbf{u}^N)\left\|\widetilde{D}\mathbf{u}^N\right\|_2^{2-p_\infty} dt\\ &\le C\,\|\mathcal{I}_\Phi{}^A(\mathbf{u}^N)\|_{L^1(I)}\big(1 + \|\nabla\mathbf{u}^N\|_{L^\infty(I,L^2(\Omega))}\big)^{2-p_\infty}\\ &\le C.\end{aligned}\tag{4.68}$$

Let us take the time derivative of the Galerkin system (4.59):

$$\left\langle\partial_t^2\mathbf{u}^N,\boldsymbol{\omega}^r\right\rangle + \langle\partial_t\mathbf{S}^A(\mathbf{D}(\mathbf{u}^N)),\mathbf{D}\boldsymbol{\omega}^r\rangle + \langle\partial_t((\mathbf{u}^N\cdot\nabla)\mathbf{u}^N),\boldsymbol{\omega}^r\rangle = \langle\partial_t\mathbf{f}^N,\boldsymbol{\omega}^r\rangle,$$

for $1 \le r \le N$. Since $\mathbf{u}^N, \partial_t\mathbf{u}^N, \partial_t^2\mathbf{u}^N \in L^2(I,X_n)$, this makes sense and we can even test with $\partial_t\mathbf{u}^N$:

$$\begin{aligned}\tfrac{1}{2}d_t\|\partial_t\mathbf{u}^N\|_2^2 + \left\langle\partial_t(\mathbf{S}^A(\mathbf{D}(\mathbf{u}^N))),\partial_t\mathbf{D}\mathbf{u}^N\right\rangle\\ +\langle\partial_t((\mathbf{u}^N\cdot\nabla)\mathbf{u}^N),\partial_t\mathbf{u}^N\rangle = \langle\partial_t\mathbf{f}^N,\partial_t\mathbf{u}^N\rangle.\end{aligned}$$

Once again the second term on the left-hand side gives us a positive term, namely

$$\begin{aligned}
\big\langle \partial_t(\mathbf{S}^A(\mathbf{D}\mathbf{u}^N)), \partial_t \mathbf{D}\mathbf{u}^N \big\rangle &= \sum_{ikl} \big\langle \partial_t\big((\partial_{kl}\Phi^A)(\mathbf{D}\mathbf{u}^N)\big), \partial_t D_{kl}\mathbf{u}^N \big\rangle \\
&= \sum_{ikl} \big\langle (\partial_{ij}\partial_{kl}\Phi^A)(\mathbf{D}\mathbf{u}^N)\partial_t D_{ij}\mathbf{u}^N, \partial_t D_{kl}\mathbf{u}^N \big\rangle \\
&\quad + \sum_{ikl} \big\langle (\partial_t\partial_{kl}\Phi^A)(\mathbf{D}\mathbf{u}^N), \partial_t D_{kl}\mathbf{u}^N \big\rangle \\
&\overset{(3.15),(3.15)}{\geq} \mathcal{J}_\Phi{}^A(\mathbf{u}^N) - C\int_\Omega (\widetilde{D}\mathbf{u}^N)^{p-1}\ln(\widetilde{D}\mathbf{u}^N)|\partial_t \mathbf{D}\mathbf{u}^N|\,dx \\
&\overset{(3.50)}{\geq} \tfrac{1}{2}\mathcal{J}_\Phi{}^A(\mathbf{u}^N) - C\int_\Omega (\widetilde{D}\mathbf{u}^N)^{p}\ln^2(\widetilde{D}\mathbf{u}^N)\,dx.
\end{aligned}$$

This yields

$$\begin{aligned}
d_t\|\partial_t\mathbf{u}^N\|_2^2 + \mathcal{J}_\Phi{}^A(\mathbf{u}) &\leq C\left(|\langle \partial_t((\mathbf{u}^N\cdot\boldsymbol{\nabla})\mathbf{u}^N), \partial_t\mathbf{u}^N\rangle| + \|\partial_t \mathbf{f}\|_2^2\right) \\
&\quad + C\,\|\partial_t\mathbf{u}^N\|_2^2 + C\,\big\|(\widetilde{D}\mathbf{u}^N)^{\frac{p}{2}}\ln(\widetilde{D}\mathbf{u}^N)\big\|_2^2,
\end{aligned} \tag{4.69}$$

where we have used $\|\partial_t \mathbf{f}^N\|_2 = \|P^N(\partial_t\mathbf{f})\|_2 \leq C\|\partial_t\mathbf{f}\|_2$. Due to (4.67) we can control all but the first term on the right-hand side. Since $\operatorname{div}\mathbf{u}^N = 0$, we have

$$|\langle \partial_t((\mathbf{u}^N\cdot\boldsymbol{\nabla})\mathbf{u}^N), \partial_t\mathbf{u}^N\rangle| = |\langle((\partial_t\mathbf{u}^N)\cdot\boldsymbol{\nabla})\mathbf{u}^N, \partial_t\mathbf{u}^N\rangle|,$$

where we have used $|\langle(\mathbf{u}^N\cdot\boldsymbol{\nabla})\partial_t\mathbf{u}^N, \partial_t\mathbf{u}^N\rangle| = 0$. Now (4.67) implies

$$|\langle \partial_t((\mathbf{u}^N\cdot\boldsymbol{\nabla})\mathbf{u}^N), \partial_t\mathbf{u}^N\rangle| \leq \|\boldsymbol{\nabla}\mathbf{u}^N\|_2\|\partial_t\mathbf{u}^N\|_4^2 \leq C\,\|\partial_t\mathbf{u}^N\|_4^2. \tag{4.70}$$

Further we conclude from lemma 3.14 that

$$\|\partial_t\boldsymbol{\nabla}\mathbf{u}^N\|^2_{\frac{4}{4-p_\infty}} \leq C\,\mathcal{J}_\Phi{}^A(\mathbf{u}^N)\big(1 + \|\boldsymbol{\nabla}\mathbf{u}^N\|_{L^\infty(I,L^2(\Omega))}\big)^{2-p_\infty} \leq C\,\mathcal{J}_\Phi{}^A(\mathbf{u}^N). \tag{4.71}$$

Since $2 < 4 < \frac{4}{2-p_\infty}$, we can estimate the convective term from (4.70) as

$$\begin{aligned}
|\langle \partial_t((\mathbf{u}^N\cdot\boldsymbol{\nabla})\mathbf{u}^N), \partial_t\mathbf{u}^N\rangle| &\leq C\,\|\partial_t\mathbf{u}^N\|_4^2 \\
&\leq \delta\|\partial_t\mathbf{u}^N\|^2_{\frac{4}{2-p_\infty}} + C_\delta\|\partial_t\mathbf{u}^N\|_2^2 \\
&\leq \delta C\,\|\partial_t\boldsymbol{\nabla}\mathbf{u}^N\|^2_{\frac{4}{4-p_\infty}} + C_\delta\|\partial_t\mathbf{u}^N\|_2^2 \\
&\overset{(4.71)}{\leq} \delta C\,\mathcal{J}_\Phi{}^A(\mathbf{u}^N) + C_\delta\|\partial_t\mathbf{u}^N\|_2^2.
\end{aligned}$$

So overall for small (but fixed) $\delta > 0$ (4.69) simplifies to

$$d_t\|\partial_t\mathbf{u}^N\|_2^2 + \tfrac{1}{2}\,\mathcal{J}_\Phi{}^A(\mathbf{u}) \leq C\left(\|\partial_t\mathbf{f}\|_2^2 + \|\partial_t\mathbf{u}^N\|_2^2 + \|\widetilde{D}\mathbf{u}^N\|^2_{\frac{4}{4-p_\infty}} + 1\right). \tag{4.72}$$

To bound $(\partial_t \mathbf{u}^N)(0)$ let $\varphi \in L^2(\Omega)$ with $\|\varphi\|_2 \le 1$, then

$$
\begin{aligned}
|\langle \partial_t \mathbf{u}^N(0), \varphi\rangle| &= |\langle \partial_t \mathbf{u}_0^N, P^N \varphi\rangle| \\
&= |\langle \operatorname{div} \mathbf{S}^A(\mathbf{D}\mathbf{u}_0^N)) + (\mathbf{u}_0^N \cdot \boldsymbol{\nabla})\mathbf{u}_0^N - \mathbf{f}(0)^N, P^N \varphi\rangle| \\
&\le \|\operatorname{div} \mathbf{S}^A(\mathbf{D}\mathbf{u}_0^N)\|_2 + \|(\mathbf{u}_0^N \cdot \boldsymbol{\nabla})\mathbf{u}_0^N\|_2 + \|\mathbf{f}(0)^N\|_2 \\
&\le \|\boldsymbol{\nabla}\mathbf{S}^A(\mathbf{D}\mathbf{u}_0^N)\|_2 + C\,\|\mathbf{u}_0\|_{2,2}^2 + \|\mathbf{f}(0)^N\|_2 \\
&\overset{(3.34),(3.28)}{\le} C\,\|\boldsymbol{\nabla}\mathbf{D}\mathbf{u}_0^N\|_2 + C\,\|(\widetilde{D}\mathbf{u}_0)^{p-1}\ln(\widetilde{D}\mathbf{u}^N)\|_2 \\
&\quad + C\,\|\mathbf{u}_0\|_{2,2}^2 + \|\mathbf{f}(0)^N\|_2 \\
&\le C\,\|\boldsymbol{\nabla}\mathbf{D}\mathbf{u}_0^N\|_2 + C\,\|\widetilde{D}\mathbf{u}^N\|_3^3 + C\,\|\mathbf{u}_0\|_{2,2}^2 + \|\mathbf{f}(0)^N\|_2 \\
&\le C\,(\|\mathbf{u}_0\|_{2,2} + \|\mathbf{u}_0\|_{2,2}^3 + + \|\mathbf{u}_0\|_{2,2}^2) + \|\mathbf{f}(0)\|_2 \le C.
\end{aligned}
$$

Here we have used that $\mathbf{f} \in L^2(I, W^{1,2}(\Omega))$ and that $\partial_t \mathbf{f} \in L^2(I, L^2(\Omega))$ implies $\mathbf{f} \in C(I, L^2(\Omega))$. Thus $\|(\partial_t \mathbf{u}^N)(0)\|_2 \le C$.

Due to (4.68) and the regularity of $\mathbf{f}$ we can apply Gronwall's inequality to (4.72), so

$$\|\partial_t \mathbf{u}^N\|_{L^\infty(I,L^2(\Omega))}^2 + \left\|\mathcal{J}_\Phi{}^A(\mathbf{u}^N)\right\|_{L^1(I)} \le C. \tag{4.73}$$

Summarized we have shown the following a priori estimates

$$\|\boldsymbol{\nabla}\mathbf{u}^N\|_{L^\infty(I,L^2(\Omega))}^2 + \left\|\mathcal{I}_\Phi{}^A(\mathbf{u}^N)\right\|_{L^1(I)} \le C, \tag{4.74}$$

$$\|\partial_t \mathbf{u}^N\|_{L^\infty(I,L^2(\Omega))}^2 + \left\|\mathcal{J}_\Phi{}^A(\mathbf{u}^N)\right\|_{L^1(I)} \le C, \tag{4.75}$$

$$\|\boldsymbol{\nabla}^2\mathbf{u}^N\|_{L^2(I,L^{\frac{4}{4-p_\infty}}(\Omega))} + \|\partial_t \boldsymbol{\nabla}\mathbf{u}^N\|_{L^2(I,L^{\frac{4}{4-p_\infty}}(\Omega))} \le C. \tag{4.76}$$

With the embedding $L^2(I, L^{\frac{4}{4-p_\infty}}(\Omega)) \hookrightarrow L^2(I, L^{\frac{4}{2-p_\infty}}(\Omega))$ and the interpolation

$$[L^\infty(I, L^2(\Omega)), L^2(I, L^{\frac{4}{2-p_\infty}}(\Omega))]_{\frac{2}{2+p_\infty}}$$

further deduce

$$\|\boldsymbol{\nabla}\mathbf{u}^N\|_{L^{2+p_\infty}(I\times\Omega)} \le C. \tag{4.77}$$

Thus there exists a subsequence (still denoted by $\mathbf{u}^N$) and a function $\mathbf{u}$, such that

$$\boldsymbol{\nabla}\mathbf{u}^N \overset{*}{\rightharpoonup} \boldsymbol{\nabla}\mathbf{u} \quad \text{in } L^\infty(I, L^2(\Omega)), \tag{4.78}$$

$$\partial_t \mathbf{u}^N \overset{*}{\rightharpoonup} \partial_t \mathbf{u} \quad \text{in } L^\infty(I, L^2(\Omega)), \tag{4.79}$$

$$\boldsymbol{\nabla}^2\mathbf{u}^N \rightharpoonup \boldsymbol{\nabla}^2\mathbf{u} \quad \text{in } L^2(I, L^{\frac{4}{4-p_\infty}}(\Omega)), \tag{4.80}$$

$$\partial_t \boldsymbol{\nabla}\mathbf{u}^N \rightharpoonup \partial_t \boldsymbol{\nabla}\mathbf{u} \quad \text{in } L^2(I, L^{\frac{4}{4-p_\infty}}(\Omega)). \tag{4.81}$$

Since $W^{1,\frac{4}{4-p_\infty}}(\Omega) \hookrightarrow\hookrightarrow L^4(\Omega)$ we can use the lemma of Aubin Lions to deduce (for a subsequence)

$$\boldsymbol{\nabla}\mathbf{u}^N \to \boldsymbol{\nabla}\mathbf{u} \quad \text{in } L^2(I, L^4(\Omega))$$

and

$$\boldsymbol{\nabla}\mathbf{u}^N \to \boldsymbol{\nabla}\mathbf{u} \quad \text{a.e. in } I \times \Omega. \tag{4.82}$$

From $W^{1,4}(\Omega) \to C(\Omega)$ we further deduce

$$\mathbf{u}^N \to \mathbf{u} \qquad \text{in } L^2(I, C(\Omega)).$$

As a consequence of the derived convergences we have

$$(\mathbf{u}^N \cdot \boldsymbol{\nabla})\mathbf{u}^N \to (\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{u} \qquad \text{in } L^1(I, L^4(\Omega)).$$

It only remains to pass to the limit in the main part, i.e. $-\operatorname{div}(\mathbf{S}^A(\mathbf{D}\mathbf{u}^N))$. First observe that (3.11), (3.36), and (4.77) imply

$$\begin{aligned} &\|\mathbf{S}^A(\mathbf{D}\mathbf{u}^N)\|_{L^{2+p_\infty}(I\times\Omega)} \\ &\qquad \le \|\mathbf{S}(\mathbf{D}\mathbf{u}^N)\|_{L^{2+p_\infty}(I\times\Omega)} + \|(\mathbf{S}-\mathbf{S}^A)(\mathbf{D}\mathbf{u}^N)\|_{L^{2+p_\infty}(I\times\Omega)} \\ &\qquad \le C\,\|(\widetilde{D}\mathbf{u}^N)^{p-1}\|_{L^{2+p_\infty}(I\times\Omega)} + C\,A^{p-2}\|\mathbf{D}\mathbf{u}^N\|_{L^{2+p_\infty}(I\times\Omega)} \\ &\qquad \le C. \end{aligned} \tag{4.83}$$

On the other hand $\mathbf{D}\mathbf{u}^N \to \mathbf{D}\mathbf{u}$ a.e. in $I \times \Omega$, so

$$\mathbf{S}^A(\mathbf{D}\mathbf{u}^N) \to \mathbf{S}^A(\mathbf{D}\mathbf{u}) \qquad \text{a.e. in } I\times\Omega \tag{4.84}$$

due to the continuity properties of $\mathbf{S}^A$ and p. Now Vitali's theorem, (4.83), and (4.84) imply

$$\mathbf{S}^A(\mathbf{D}\mathbf{u}^N) \to \mathbf{S}^A(\mathbf{D}\mathbf{u}) \qquad \text{a.e. in } L^1(I\times\Omega). \tag{4.85}$$

We still have to show that the limit function $\mathbf{u}$ is a weak solution of the system (4.56). For this choose $\boldsymbol{\omega}^r \in X_N$ and $\varphi \in C_0^\infty(I)$. Then we can conclude from (4.59), the convergence properties of $\mathbf{u}^N$ derived above, and (4.85) that

$$\int_I \varphi\Big(\big\langle\partial_t\mathbf{u}, \boldsymbol{\omega}^r\big\rangle + \langle\mathbf{S}^A(\mathbf{D}(\mathbf{u})), \mathbf{D}\boldsymbol{\omega}^r\rangle + \langle(\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u}, \boldsymbol{\omega}^r\rangle\Big)\,dt = \int_I \varphi\langle\mathbf{f}, \boldsymbol{\omega}^r\rangle dt.$$

Furthermore $\mathbf{u}$ fulfills (see above)

$$\|\partial_t\mathbf{u}\|_{L^\infty(I,L^2(\Omega))} + \|\mathbf{S}^A(\mathbf{D}\mathbf{u})\|_{L^{2+p_\infty}(I\times\Omega)} + \|(\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u}\|_{L^1(I,L^4(\Omega))} \le C.$$

Since $\{\boldsymbol{\omega}^1, \boldsymbol{\omega}^2, \dots\}$ is dense in $W^{2,2}_{\operatorname{div},0}(\Omega)$, we deduce that

$$\int_I \varphi\Big(\big\langle\partial_t\mathbf{u}, \boldsymbol{\omega}\big\rangle + \langle\mathbf{S}^A(\mathbf{D}(\mathbf{u})), \mathbf{D}\boldsymbol{\omega}\rangle + \langle(\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u}, \boldsymbol{\omega}\rangle\Big)\,dt = \int_I \varphi\langle\mathbf{f}, \boldsymbol{\omega}\rangle dt$$

is fulfilled for all $\boldsymbol{\omega} \in W^{s,2}_{\operatorname{div},0}(\Omega)$, especially for all $\boldsymbol{\omega} \in \mathcal{V}_0$. Note that

$$\big\langle\partial_t\mathbf{u}, \boldsymbol{\omega}\big\rangle + \langle\mathbf{S}^A(\mathbf{D}(\mathbf{u})), \mathbf{D}\boldsymbol{\omega}\rangle + \langle(\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u}, \boldsymbol{\omega}\rangle,\ \langle\mathbf{f}, \boldsymbol{\omega}\rangle \in L^1(I),$$

so

$$\big\langle\partial_t\mathbf{u}, \boldsymbol{\omega}\big\rangle + \langle\mathbf{S}^A(\mathbf{D}(\mathbf{u})), \mathbf{D}\boldsymbol{\omega}\rangle + \langle(\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u}, \boldsymbol{\omega}\rangle = \langle\mathbf{f}, \boldsymbol{\omega}\rangle \tag{4.86}$$

for all $\boldsymbol{\omega} \in \mathcal{V}_0$ and a.a. $t \in I$. It remains to show that $\mathbf{u}(0) = \mathbf{u}_0$. But this follows from the parabolic embedding

$$\begin{aligned} \|P^N\mathbf{u}_0 - \mathbf{u}(0)\|_2 &= \|\mathbf{u}^N(0) - \mathbf{u}(0)\|_2 \\ &\le C\,\underbrace{\|\mathbf{u}^N - \mathbf{u}\|^{\frac12}_{L^2(I,L^2(\Omega))}}_{\to 0}\,\underbrace{\|\partial_t\mathbf{u}^N - \partial_t\mathbf{u}\|^{\frac12}_{L^2(I,L^2(\Omega))}}_{\le C} \to 0. \end{aligned} \tag{4.87}$$

Since $P^N \mathbf{u}_0 \to \mathbf{u}_0$ in $L^2(\Omega)$ we get $\mathbf{u}(0) = \mathbf{u}_0$. Overall we have shown by (4.86) and (4.87) that $\mathbf{u}$ satisfies (5.1) in the weak sense. The convergence (4.78) and (4.79) further ensure that $\mathbf{u}$ satisfies

$$\|\nabla \mathbf{u}\|_{L^\infty(I,L^2(\Omega))} + \|\partial_t \mathbf{u}\|_{L^\infty(I,L^2(\Omega))} \le C.$$

It remains to prove the boundedness of $\|\mathcal{I}_\Phi{}^A(\mathbf{u})\|_{L^1(I)}$ and $\|\mathcal{J}_\Phi{}^A(\mathbf{u})\|_{L^1(I)}$. Define $H : I \times \Omega \times \mathbb{R}^{d\times d} \times \mathbb{R}^{d\times d}$ by

$$H(t,x,\mathbf{y},\mathbf{z}) := \sum_{jk\alpha\beta} (\partial_{\alpha\beta}\partial_{jk}\Phi^A)(t,x,\mathbf{y})\, z_{\alpha\beta}\, z_{jk},$$

then

(a) $H \ge 0$,
(b) H is measurable in (t,x) for all $\mathbf{y}, \mathbf{z}$,
(c) H is continuous in $\mathbf{z}$ and $\mathbf{y}$ for almost every $(t,x) \in I' \times \Omega$,
(d) H is convex in $\mathbf{z}$ for all $\mathbf{y}$ and almost every $(t,x) \in I' \times \Omega$.

Furthermore

$$\|\mathcal{J}_\Phi{}^A(\mathbf{u}^N)\|_{L^1(I',L^1(\Omega))} = \Big\| H(\mathbf{D}\mathbf{u}^N, \partial_t \mathbf{D}\mathbf{u}^N)\Big\|_{L^1(I'\times\Omega)}, \tag{4.88}$$

$$\|\mathcal{I}_\Phi{}^A(\mathbf{u}^N)\|_{L^1(I',L^1(\Omega))} = \Big\| \sum_{k=1}^d H(\mathbf{D}\mathbf{u}^N, \partial_k \mathbf{D}\mathbf{u}^N)\Big\|_{L^1(I'\times\Omega)}. \tag{4.89}$$

From (4.80), (4.81), and (4.82) we deduce that

$$\begin{aligned} \nabla^2 \mathbf{u}^N &\rightharpoonup \nabla^2 \mathbf{u} && \text{in } L^1(I'\times\Omega),\\ \partial_t \nabla \mathbf{u}^N &\rightharpoonup \partial_t \nabla \mathbf{u} && \text{in } L^1(I'\times\Omega),\\ \nabla \mathbf{u}^N &\to \nabla \mathbf{u} && \text{in } L^1(I'\times\Omega). \end{aligned}$$

Thus from the semicontinuity theorem of De Giorgi ([GMS98], pg. 132), (4.67), (4.73), (4.88), and (4.89) we deduce

$$\|\mathcal{I}_\Phi{}^A(\mathbf{u})\|_{L^1(I')} + \|\mathcal{J}_\Phi{}^A(\mathbf{u})\|_{L^1(I')} \le C. \tag{4.90}$$

From (4.77) we further deduce

$$\|\nabla \mathbf{u}\|_{L^{2+p_\infty}(I\times\Omega)} \le C.$$

Hence $\|(\widetilde{D}\mathbf{u})^{\frac{p}{2}} \ln(\widetilde{D}\mathbf{u})\|^2_{L^2(I',L^2(\Omega))} \le C$. From this and lemma 3.12 we deduce

$$\big\|\nabla\big((\widetilde{D}\mathbf{u})^{\frac{p}{2}}\big)\big\|^2_{L^2(I',L^2(\Omega))} + \big\|\partial_t\big((\widetilde{D}\mathbf{u})^{\frac{p}{2}}\big)\big\|^2_{L^2(I',L^2(\Omega))} \le C.$$

Parabolic interpolation therefore (see theorem 8.21) implies

$$\big\|(\widetilde{D}\mathbf{u})^{\frac{p}{2}}\big\|_{C(I,W^{\frac12,2}(\Omega))} = \big\|(\widetilde{D}\mathbf{u})^{\frac{p}{2}}\big\|_{C(I,[W^{1,2}(\Omega),L^2(\Omega)]_{\frac12,2})} \le C.$$

Thus

$$\big\|(\widetilde{D}\mathbf{u})^{\frac{p_\infty}{2}}\big\|_{C(I,L^4(\Omega))} \le C.$$

Hence Korn's inequality implies

$$\big\|\mathbf{u}\big\|_{C(I,W^{1,2p_\infty}(\Omega))} \le C.$$

This proves the lemma for $\varepsilon = 0$.

Case $\varepsilon > 0$: Suppose now that $\varepsilon > 0$. Instead of repeating all the steps above for this new situation we will only point out the differences to the case $\varepsilon = 0$.

As in the case $\varepsilon = 0$ we will prove existence by means of the Galerkin method. But instead of using the eigenfunctions of the Stokes operator we will proceed as in lemma 4.1. So let ω_j be the system of eigenfunctions of the scalar operator $-\Delta$ with corresponding eigenvalues λ_j, $\langle \omega_j, 1\rangle = 0$, $X_N := \operatorname{span}\{\omega_1, \dots, \omega_N\}$, and $X_N^2 := \operatorname{span}\{\boldsymbol{\omega}_{jk} : j,k = 1,\dots,N\}$ with $\boldsymbol{\omega}_{jk} := (\omega_j, \omega_k)^T$. Then the projections defined by $P^N : L^2(\Omega) \to X_N^2$

$$P^N \mathbf{u} := \sum_{j,k=1}^{N} \langle \mathbf{u}, \boldsymbol{\omega}_{jk}\rangle \boldsymbol{\omega}_{jk}$$

have the same continuity properties as in the case $\varepsilon = 0$ (compare with lemma 4.1). We use the ansatz

$$\mathbf{u}^N(t,x) := \sum_{j,k=1}^{N} \alpha_{jk}^N(t) \boldsymbol{\omega}_{jk}(x), \qquad \pi^N(t) := \tfrac{1}{\varepsilon}\Delta^{-1} \operatorname{div} \mathbf{u}^N(t)$$

with coefficients $\alpha_{jk}^N : I \to \mathbb{R}$. Then as in lemma 4.1 we get approximative solutions $\mathbf{u}^N, \pi^N$ of the Galerkin system

$$\begin{aligned} d_t\langle \mathbf{u}^N, \boldsymbol{\omega}_{rs}\rangle + \langle \mathbf{S}^A(\mathbf{D}\mathbf{u}^N), \boldsymbol{\nabla}\boldsymbol{\omega}_{rs}\rangle + \tfrac{1}{\varepsilon}\langle \boldsymbol{\nabla}\Delta^{-1} \operatorname{div} \mathbf{u}^N, \boldsymbol{\omega}_{rs}\rangle & \\ + \langle ((\mathbf{u}^N - \varepsilon\boldsymbol{\nabla}\pi^N)\cdot\boldsymbol{\nabla})\mathbf{u}^N, \boldsymbol{\omega}_{rs}\rangle &= \langle \mathbf{f}^N, \boldsymbol{\omega}_{rs}\rangle, \\ \mathbf{u}^N(0) &= P^N\mathbf{u}_0 \end{aligned} \tag{4.91}$$

for all $r, s = 1, \dots, N$ for small times $I^* = [0, T^*]$. But as long as we can derive a priori estimates for $\mathbf{u}^N$ (depending on T but not on T^*) bounding also the coefficients α_{jk}^N, this existence can be extended to the large time interval $[0, T]$ (compare the proof of lemma 4.1). Thus we will only point out how to get this a priori estimate.

Using $\mathbf{u}^N \in X_N^2$ as a test function for the Galerkin system (4.91) we get (compare lemma 4.1)

$$\begin{aligned} \tfrac{1}{2} d_t\|\mathbf{u}^N\|_2^2 + \langle \mathbf{S}^A(\mathbf{D}\mathbf{u}^N), \mathbf{D}\mathbf{u}^N\rangle + \varepsilon\|\boldsymbol{\nabla}\pi^N\|_2^2 & \\ + \langle ((\mathbf{u}^N - \varepsilon\boldsymbol{\nabla}\pi^N)\cdot\boldsymbol{\nabla})\mathbf{u}^N, \mathbf{u}^N\rangle &= \langle \mathbf{f}^N, \mathbf{u}^N\rangle. \end{aligned}$$

Note that

$$\langle ((\mathbf{u}^N - \varepsilon\boldsymbol{\nabla}\pi^N)\cdot\boldsymbol{\nabla})\mathbf{u}^N, \mathbf{u}^N\rangle = 0,$$

since $\operatorname{div}(\mathbf{u}^N - \varepsilon\boldsymbol{\nabla}\pi^N) = \operatorname{div}\mathbf{u}^N - \varepsilon\Delta\pi^N = 0$. Thus

$$\tfrac{1}{2} d_t\|\mathbf{u}^N\|_2^2 + \langle \mathbf{S}^A(\mathbf{D}\mathbf{u}^N), \mathbf{D}\mathbf{u}^N\rangle + \varepsilon\|\boldsymbol{\nabla}\pi^N\|_2^2 = \langle \mathbf{f}^N, \mathbf{u}^N\rangle.$$

Analogously to the case $\varepsilon = 0$ we derive from this

$$\|\mathbf{u}^N\|_{L^\infty(I^*,L^2(\Omega))} + \|\mathbf{D}\mathbf{u}^N\|_{L^{p(\cdot)}(I^*\times\Omega)} + \varepsilon\|\boldsymbol{\nabla}\pi^N\|_{L^2(I^*,L^2(\Omega))} \le C,$$

where C does not depend on T^*. This boundedness of $\|\mathbf{u}^N\|_{L^\infty(I^*,L^2(\Omega))}$ implies boundedness of $\|\alpha_{rs}\|_{L^\infty(I^*)}$ independently of T^*, which in turn enables us to extend the time interval I^* to I by retaining the a priori estimates for $\mathbf{u}^N$ and π^N.

The next step differing from the case $\varepsilon = 0$ is the testing with $-\Delta\mathbf{u}^N$ in order to derive higher order a priori estimates:

Using $-\Delta\mathbf{u}^N \in X_N^2$ as a test function for the Galerkin system (4.91) we get

$$\tfrac{1}{2}d_t\|\nabla\mathbf{u}^N\|_2^2 - \langle\mathbf{S}^A(\mathbf{D}\mathbf{u}^N),\mathbf{D}\Delta\mathbf{u}^N\rangle - \tfrac{1}{\varepsilon}\langle\nabla\Delta^{-1}\operatorname{div}\mathbf{u}^N,\Delta\mathbf{u}^N\rangle$$
$$-\langle((\mathbf{u}^N-\varepsilon\nabla\pi^N)\cdot\nabla)\mathbf{u}^N,\Delta\mathbf{u}^N\rangle = \langle\nabla\mathbf{f}^N,\nabla\mathbf{u}^N\rangle.$$

Note that due to the space periodicity

$$-\tfrac{1}{\varepsilon}\langle\nabla\Delta^{-1}\operatorname{div}\mathbf{u}^N,\Delta\mathbf{u}^N\rangle = \tfrac{1}{\varepsilon}\|\operatorname{div}\mathbf{u}^N\|_2^2 = \varepsilon\|\Delta\pi^N\|_2^2.$$

Hence

$$\tfrac{1}{2}d_t\|\nabla\mathbf{u}^N\|_2^2 - \langle\mathbf{S}^A(\mathbf{D}\mathbf{u}^N),\mathbf{D}\Delta\mathbf{u}^N\rangle + \varepsilon\|\Delta\pi^N\|_2^2$$
$$\le \langle\nabla\mathbf{f}^N,\nabla\mathbf{u}^N\rangle + \big|\langle((\mathbf{u}^N-\varepsilon\nabla\pi^N)\cdot\nabla)\mathbf{u}^N,\Delta\mathbf{u}^N\rangle\big|.$$

Analogously to the case $\varepsilon = 0$ (compare (4.64)) we deduce from this

$$\begin{aligned}(4.92)\quad &\tfrac{1}{2}d_t\|\nabla\mathbf{u}^N\|_2^2 + c\,\big\|(\widetilde{D}\mathbf{u}^N)^{\frac{p}{2}}\ln(\widetilde{D}\mathbf{u}^N)\big\|_2^2\\ &+c\,\big\|\nabla\big((\widetilde{D}\mathbf{u}^N)^{\frac{p}{2}}\big)\big\|_2^2 + c\,\mathcal{I}_\Phi{}^A(\mathbf{u}^N) + \varepsilon\|\Delta\pi^N\|_2^2\\ &\quad\le \|\nabla\mathbf{f}^N\|_2^2 + C\left(|\widetilde{D}\mathbf{u}^N|_{p(\cdot)}+1\right)\|\nabla\mathbf{u}^N\|_2^2\\ &\qquad+\big|\langle((\mathbf{u}^N-\varepsilon\nabla\pi^N)\cdot\nabla)\mathbf{u}^N,\Delta\mathbf{u}^N\rangle\big|.\end{aligned}$$

Different from the case $\varepsilon = 0$ the convective term does not vanish and has to be controlled. Since $\|\varepsilon\nabla^2\pi^N\|_3 \le R\,\|\varepsilon\Delta\pi^N\|_3 = C\,\|\operatorname{div}\mathbf{u}^N\|_3$, it is not difficult to bound the convective term like

$$\begin{aligned}\big|\langle((\mathbf{u}^N-\varepsilon\nabla\pi^N)\cdot\nabla)\mathbf{u}^N,\Delta\mathbf{u}^N\rangle\big| &\le\cdots\le \|\mathbf{u}^N-\varepsilon\nabla\pi^N\|_{1,3}\|\nabla\mathbf{u}^N\|_3^2\\ &\le C\,\|\nabla\mathbf{u}^N\|_3^3\\ &\le\cdots\le \delta\mathcal{I}_\Phi{}^A(\mathbf{u}^N) + C_\delta(1+\|\nabla\mathbf{u}^N\|_2^R)\end{aligned}$$

for some constant $R > 2$. But this would only imply an a priori estimate for small times, since $R > 2$ restricts us to apply a local version of Gronwall's inequality (see lemma 8.7). In order to derive an a priori estimate for the whole time interval I, we will have to use the extra information $\varepsilon\|\Delta\pi^N\|_2^2$, the smallness of ε and the special structure of the convection in two dimensions (space periodic): Note that

$$\begin{aligned}&\langle((\mathbf{u}^N-\varepsilon\nabla\pi^N)\cdot\nabla)\mathbf{u}^N,\Delta\mathbf{u}^N\rangle\\ &\quad=\sum_r\langle((\mathbf{u}^N-\varepsilon\nabla\pi^N)\cdot\nabla)\mathbf{u}^N,\partial_r^2\mathbf{u}^N\rangle\\ &\quad=\sum_r\langle((\partial_r\mathbf{u}^N-\varepsilon\partial_r\nabla\pi^N)\cdot\nabla)\mathbf{u}^N,\partial_r\mathbf{u}^N\rangle\\ &\qquad+\sum_r\langle((\mathbf{u}^N-\varepsilon\nabla\pi^N)\cdot\nabla)\partial_r\mathbf{u}^N,\partial_r\mathbf{u}^N\rangle\\ &\quad=\sum_r\langle((\partial_r\mathbf{u}^N-\varepsilon\partial_r\nabla\pi^N)\cdot\nabla)\mathbf{u}^N,\partial_r\mathbf{u}^N\rangle\\ &\qquad-\sum_r\langle(\operatorname{div}(\mathbf{u}^N-\varepsilon\nabla\pi^N))\partial_r\mathbf{u}^N,\partial_r\mathbf{u}^N\rangle\\ &\quad=\sum_r\langle((\partial_r\mathbf{u}^N-\varepsilon\partial_r\nabla\pi^N)\cdot\nabla)\mathbf{u}^N,\partial_r\mathbf{u}^N\rangle.\end{aligned}$$

Let $\mathbf{v}^N := \mathbf{u}^N - \varepsilon\boldsymbol{\nabla}\pi^N$, then

$$\langle((\mathbf{u}^N - \varepsilon\boldsymbol{\nabla}\pi^N)\cdot\boldsymbol{\nabla})\mathbf{u}^N, \Delta\mathbf{u}^N\rangle = \sum_r \langle(\partial_r\mathbf{v}^N\cdot\boldsymbol{\nabla})\mathbf{u}^N, \partial_r\mathbf{u}^N\rangle$$
$$= \sum_{jkr}\int_\Omega (\partial_r v_j^N)(\partial_j u_k^N)(\partial_r u_k^N)\,dx.$$

In the proof for $\varepsilon = 0$ we have seen that $\operatorname{div}\mathbf{v}^N = 0$ implies in our special two dimensional setting

$$\sum_{jkr}(\partial_r v_j^N)(\partial_j v_k^N)(\partial_r v_k^N) = 0.$$

Thus

$$\begin{aligned}
&\langle((\mathbf{u}^N - \varepsilon\boldsymbol{\nabla}\pi^N)\cdot\boldsymbol{\nabla})\mathbf{u}^N, \Delta\mathbf{u}^N\rangle \\
&\quad= \sum_{jkr}\int_\Omega (\partial_r v_j^N)(\partial_j u_k^N)(\partial_r u_k^N) - (\partial_r v_j^N)(\partial_j v_k^N)(\partial_r v_k^N)\,dx \\
&\quad= \sum_{jkr}\int_\Omega (\partial_r v_j^N)(\partial_j(u_k - v_k^N))(\partial_r u_k^N) + (\partial_r v_j^N)(\partial_j v_k^N)(\partial_r(u_k^N - v_k^N))\,dx.
\end{aligned}$$

Hence with $\mathbf{u}^N - \mathbf{v}^N = \varepsilon\boldsymbol{\nabla}\pi^N$ we get

$$\|\boldsymbol{\nabla}\mathbf{v}^N\|_r \le \|\boldsymbol{\nabla}\mathbf{u}^N\|_r + \varepsilon\|\boldsymbol{\nabla}\pi\|_r \le \|\boldsymbol{\nabla}\mathbf{u}^N\|_r + C\,\|\operatorname{div}\mathbf{u}^N\|_r \le C\,\|\boldsymbol{\nabla}\mathbf{u}^N\|_r$$

for all $1 < r < \infty$. There follows for $r > \frac{2p_\infty}{p_\infty - 1}$

$$\begin{aligned}
&\big|\langle((\mathbf{u}^N - \varepsilon\boldsymbol{\nabla}\pi^N)\cdot\boldsymbol{\nabla})\mathbf{u}^N, \Delta\mathbf{u}^N\rangle\big| \\
&\quad\le \left|\sum_{jkr}\int_\Omega (\partial_r v_j^N)(\varepsilon\partial_j\partial_k\pi^N)(\partial_r u_k^N) + (\partial_r v_j^N)(\partial_j v_k^N)(\varepsilon\partial_r\partial_k\pi^N)\,dx\right| \\
&\quad\le \varepsilon\,C\,\|\boldsymbol{\nabla}^2\pi^N\|_2\|\boldsymbol{\nabla}\mathbf{v}^N\|_4\|\boldsymbol{\nabla}\mathbf{u}^N\|_4 + \varepsilon\,C\,\|\boldsymbol{\nabla}^2\pi^N\|_2\|\boldsymbol{\nabla}\mathbf{v}^N\|_4^2 \\
&\quad\le \varepsilon\,C\,\|\boldsymbol{\nabla}^2\pi^N\|_2\|\boldsymbol{\nabla}\mathbf{u}^N\|_4^2 \\
&\quad\le \varepsilon\,C\,\|\boldsymbol{\nabla}^2\pi^N\|_2\|\boldsymbol{\nabla}\mathbf{u}^N\|_2^{\frac{r-4}{r-2}}\|\boldsymbol{\nabla}\mathbf{u}^N\|_r^{\frac{r}{r-2}} \\
&\quad\le \varepsilon\,C\,\|\Delta\pi^N\|_2\|\boldsymbol{\nabla}\mathbf{u}^N\|_2^{\frac{r-4}{r-2}}\|\boldsymbol{\nabla}\mathbf{u}^N\|_r^{\frac{r}{r-2}} \\
&\quad\le \delta\|\boldsymbol{\nabla}\mathbf{u}^N\|_r^{p_\infty} + C_\delta\big(\varepsilon\,\|\Delta\pi^N\|_2\big)^{\frac{p_\infty r - 2p_\infty}{p_\infty r - 2p_\infty - r}}\|\boldsymbol{\nabla}\mathbf{u}^N\|_2^{\frac{p_\infty(r-4)}{p_\infty r - 2p_\infty - r}} \\
&\quad\le \delta\|\boldsymbol{\nabla}\mathbf{u}^N\|_r^{p} + \tfrac{\varepsilon}{2}\|\Delta\pi^N\|_2^2 + \varepsilon C_\delta\big(\varepsilon\,\|\Delta\pi^N\|_2\big)^{\frac{2r}{p_\infty r - 2p_\infty - r}}\|\boldsymbol{\nabla}\mathbf{u}^N\|_2^{\frac{2p_\infty(r-4)}{p_\infty r - 2p_\infty - r}}.
\end{aligned}$$

Since $\varepsilon\|\Delta\pi^N\|_2 \le C\,\|\operatorname{div}\mathbf{u}^N\|_2 \le C\,\|\boldsymbol{\nabla}\mathbf{u}^N\|_2$, this implies

$$\begin{aligned}
&\big|\langle((\mathbf{u}^N - \varepsilon\boldsymbol{\nabla}\pi^N)\cdot\boldsymbol{\nabla})\mathbf{u}^N, \Delta\mathbf{u}^N\rangle\big| \\
&\qquad\le \delta\|\boldsymbol{\nabla}\mathbf{u}^N\|_r^{p_\infty} + \varepsilon\|\Delta\pi^N\|_2^2 + \varepsilon C_\delta\|\boldsymbol{\nabla}\mathbf{u}^N\|_2^{\frac{2(p_\infty r - 4p_\infty + r)}{p_\infty r - 2p_\infty - r}}.
\end{aligned}$$

Fix $r := \frac{4p_\infty}{p_\infty - 1} > \frac{2p_\infty}{p_\infty - 1}$, then

$$
\begin{aligned}
&\big|\langle((\mathbf{u}^N - \varepsilon\boldsymbol{\nabla}\pi^N)\cdot\boldsymbol{\nabla})\mathbf{u}^N, \Delta\mathbf{u}^N\rangle\big| \\
&\quad\le \delta\|\boldsymbol{\nabla}\mathbf{u}^N\|^{p_\infty}_{\frac{4p_\infty}{p_\infty-1}} + \tfrac{\varepsilon}{2}\|\Delta\pi^N\|_2^2 + \varepsilon C_\delta\|\boldsymbol{\nabla}\mathbf{u}^N\|_2^{\frac{8}{p_\infty-1}} \\
&\quad\le \delta C\,\|\mathbf{D}\mathbf{u}^N\|^{p_\infty}_{\frac{4p_\infty}{p_\infty-1}} + \tfrac{\varepsilon}{2}\|\Delta\pi^N\|_2^2 + \varepsilon C_\delta\|\boldsymbol{\nabla}\mathbf{u}^N\|_2^{\frac{8}{p_\infty-1}} \\
&\quad\le \delta C\,\big\|(\widetilde{D}\mathbf{u}^N)^{\frac{p}{2}}\big\|^2_{\frac{8}{p_\infty-1}} + \tfrac{\varepsilon}{2}\|\Delta\pi^N\|_2^2 + \varepsilon C_\delta\|\boldsymbol{\nabla}\mathbf{u}^N\|_2^{\frac{8}{p_\infty-1}} \\
&\quad\le \delta C\,\big\|\boldsymbol{\nabla}\big((\widetilde{D}\mathbf{u}^N)^{\frac{p}{2}}\big)\big\|_2^2 + \delta C\,\big\|(\widetilde{D}\mathbf{u}^N)^{\frac{p}{2}}\big\|_2^2 + \tfrac{\varepsilon}{2}\|\Delta\pi^N\|_2^2 + \varepsilon C_\delta\|\boldsymbol{\nabla}\mathbf{u}^N\|_2^{\frac{8}{p_\infty-1}}.
\end{aligned}
$$

So with (4.92) and the estimate $\|\boldsymbol{\nabla}\mathbf{f}^N\|_2 \le C\|\boldsymbol{\nabla}\mathbf{f}\|_2$, we get

$$
\begin{aligned}
&\tfrac12 d_t\|\boldsymbol{\nabla}\mathbf{u}^N\|_2^2 + c\,\big\|(\widetilde{D}\mathbf{u}^N)^{\frac{p}{2}}\ln(\widetilde{D}\mathbf{u}^N)\big\|_2^2 \\
&+c\,\big\|\boldsymbol{\nabla}\big((\widetilde{D}\mathbf{u}^N)^{\frac{p}{2}}\big)\big\|_2^2 + c\,\mathcal{I}_\Phi{}^A(\mathbf{u}^N) + \tfrac{\varepsilon}{2}\|\Delta\pi^N\|_2^2 \\
&\qquad\le C\,\|\boldsymbol{\nabla}\mathbf{f}^N\|_2^2 + C\,\big(|\widetilde{D}\mathbf{u}^N|_{p(\cdot)} + 1\big)\,\|\boldsymbol{\nabla}\mathbf{u}^N\|_2^2 + \varepsilon C\|\boldsymbol{\nabla}\mathbf{u}^N\|_2^{\frac{8}{p_\infty-1}}.
\end{aligned}
\tag{4.93}
$$

Since $\|\boldsymbol{\nabla}\mathbf{f}\|_{L^2(I,L^2(\Omega))} + |\mathbf{D}\mathbf{u}^N|_{L^{p(\cdot)}(I\times\Omega)} \le C$, there exists by lemma 8.8 a constant $\varepsilon_0 > 0$, such that if ε suitable small, i.e. $0 \le \varepsilon < \varepsilon_0$, then

$$
\begin{aligned}
&\|\boldsymbol{\nabla}\mathbf{u}^N\|^2_{L^\infty(I,L^2(\Omega))} + \big\|(\widetilde{D}\mathbf{u}^N)^{\frac{p}{2}}\ln(\widetilde{D}\mathbf{u}^N)\big\|^2_{L^2(I,L^2(\Omega))} \\
&+\big\|\boldsymbol{\nabla}\big((\widetilde{D}\mathbf{u}^N)^{\frac{p}{2}}\big)\big\|^2_{L^2(I,L^2(\Omega))} + c\,\big\|\mathcal{I}_\Phi{}^A(\mathbf{u}^N)\big\|_{L^1(I)} + \tfrac{\varepsilon}{2}\|\Delta\pi^N\|^2_{L^2(I,L^2(\Omega))} \le C.
\end{aligned}
$$

This proves that, when testing with $-\Delta\mathbf{u}^N$ in the case $0 < \varepsilon < \varepsilon_0$, we get the same a priori estimate as for $\varepsilon = 0$.

It remains to show how to obtain (4.73) for $0 < \varepsilon \le \varepsilon_0$. The only difference to the case $\varepsilon = 0$ is that we get the extra information $\varepsilon\|\partial_t\boldsymbol{\nabla}\pi\|_2^2$ and have to control

$$\langle\partial_t(((\mathbf{u}^N - \varepsilon\boldsymbol{\nabla}\pi^N)\cdot\boldsymbol{\nabla})\mathbf{u}^N), \partial_t\mathbf{u}^N\rangle.$$

Since $\operatorname{div}(\mathbf{u}^N - \varepsilon\boldsymbol{\nabla}\pi^N) = 0$, we get

$$
\begin{aligned}
\langle\partial_t(((\mathbf{u}^N - \varepsilon\boldsymbol{\nabla}\pi^N)\cdot\boldsymbol{\nabla})\mathbf{u}^N), \partial_t\mathbf{u}^N\rangle &= \langle((\partial_t\mathbf{u}^N - \varepsilon\partial_t\boldsymbol{\nabla}\pi^N)\cdot\boldsymbol{\nabla})\mathbf{u}^N, \partial_t\mathbf{u}^N\rangle \\
&\le \|\boldsymbol{\nabla}\mathbf{u}^N\|_2\|\partial_t\mathbf{u}^N - \varepsilon\partial_t\boldsymbol{\nabla}\pi^N\|_4\|\partial_t\mathbf{u}^N\|_4.
\end{aligned}
$$

Further $\operatorname{div}(\partial_t\mathbf{u}^N) = \varepsilon\Delta(\partial_t\pi^N)$, so $\|\varepsilon\partial_t\boldsymbol{\nabla}\pi^N\|_4 \le C\,\|\partial_t\mathbf{u}^N\|_4$ and

$$\langle\partial_t(((\mathbf{u}^N - \varepsilon\boldsymbol{\nabla}\pi^N)\cdot\boldsymbol{\nabla})\mathbf{u}^N), \partial_t\mathbf{u}^N\rangle \le C\,\|\boldsymbol{\nabla}\mathbf{u}^N\|_2\|\partial_t\mathbf{u}^N\|_4^2.$$

From this point forth all calculations for $\varepsilon = 0$ also hold for $0 < \varepsilon < \varepsilon_0$. This proves the statement. ∎

LEMMA 4.9. *Lemma 4.8 remains valid if* $((\mathbf{u}^A - \varepsilon\boldsymbol{\nabla}\pi^A)\cdot\boldsymbol{\nabla})\mathbf{u}^A$ *is replaced by* $(\mathbf{u}^A\cdot\boldsymbol{\nabla})\mathbf{u}^A + \frac12(\operatorname{div}\mathbf{u}^A)\mathbf{u}^A$.

PROOF. If $\varepsilon = 0$, then $\operatorname{div}\mathbf{u}^A = 0$, so the two versions of the convective term do agree and there is nothing to show. If $\varepsilon > 0$, then we will prove that we can derive the same a priori estimates for the system with $(\mathbf{u}^A\cdot\boldsymbol{\nabla})\mathbf{u}^A + \frac12(\operatorname{div}\mathbf{u}^A)\mathbf{u}^A$ as we have done for the one with $((\mathbf{u}^A - \varepsilon\boldsymbol{\nabla}\pi^A)\cdot\boldsymbol{\nabla})\mathbf{u}^A$ in lemma 4.8. Based on this everything else will be just as in the proof of lemma 4.8, so we focus on the three a priori estimates.

Note that the Galerkin system corresponding to the modified system is

$$
\begin{aligned}
&d_t\langle \mathbf{u}^N, \boldsymbol{\omega}_{rs}\rangle + \langle \mathbf{S}^A(\mathbf{D}\mathbf{u}^N), \boldsymbol{\nabla}\boldsymbol{\omega}_{rs}\rangle + \tfrac{1}{\varepsilon}\langle \boldsymbol{\nabla}\Delta^{-1}\operatorname{div}\mathbf{u}^N, \boldsymbol{\omega}_{rs}\rangle \\
&\qquad + \langle (\mathbf{u}^N\cdot\boldsymbol{\nabla})\mathbf{u}^N + \tfrac{1}{2}(\operatorname{div}\mathbf{u}^N)\mathbf{u}^N, \boldsymbol{\omega}_{rs}\rangle = \langle \mathbf{f}^N, \boldsymbol{\omega}_{rs}\rangle, \\
&\qquad \mathbf{u}^N(0) = P^N\mathbf{u}_0
\end{aligned} \tag{4.94}
$$

for all $r, s = 1, \ldots, N$. When we use $\mathbf{u}^N \in X_N^2$ as a test function, then the convective term gives

$$
\begin{aligned}
\langle (\mathbf{u}^N\cdot\boldsymbol{\nabla})\mathbf{u}^N + \tfrac{1}{2}(\operatorname{div}\mathbf{u}^N)\mathbf{u}^N, \mathbf{u}^N\rangle &= \sum_{jk}\int_\Omega u_j^N(\partial_j u_k^N)u_k^N + \tfrac{1}{2}(\partial_j u_j^N)u_k^N u_k^N\,dx \\
&= \sum_{jk}\int_\Omega u_j^N(\partial_j u_k^N)u_k^N - \tfrac{1}{2}u_j^N\partial_j(u_k^N u_k^N)\,dx \\
&= 0.
\end{aligned}
$$

Thus exactly as in lemma 4.8 the convective part tested with $\mathbf{u}^N$ vanishes. Hence in analogy we get

$$
\|\mathbf{u}^N\|_{L^\infty(I,L^2(\Omega))} + \|\mathbf{D}\mathbf{u}^N\|_{L^{p(\cdot)}(I\times\Omega)} \le C.
$$

When we use $-\Delta\mathbf{u}^N \in X_N^2$ as a test function for the Galerkin system the calculations are more complicated. Note that

$$
\begin{aligned}
&\langle (\mathbf{u}^N\cdot\boldsymbol{\nabla})\mathbf{u}^N, -\Delta\mathbf{u}^N\rangle \\
&\quad = -\sum_{jkr}\int_\Omega u_j^N(\partial_j u_k^N)(\partial_r^2 u_k^N)\,dx \\
&\quad = \sum_{jkr}\int_\Omega (\partial_r u_j^N)(\partial_j u_k^N)(\partial_r u_k^N) + u_j^N(\partial_j\partial_r u_k^N)(\partial_r u_k^N)\,dx \\
&\quad = \sum_{jkr}\int_\Omega (\partial_r u_j^N)(\partial_j u_k^N)(\partial_r u_k^N) + \tfrac{1}{2}u_j^N\partial_j\big((\partial_r u_k^N)^2\big)\,dx \\
&\quad = \sum_{jkr}\int_\Omega (\partial_r u_j^N)(\partial_j u_k^N)(\partial_r u_k^N) - \tfrac{1}{2}(\partial_j u_j^N)(\partial_r u_k^N)^2\,dx \\
&\quad = \sum_{jkr}\int_\Omega (\partial_r u_j^N)(\partial_j u_k^N)(\partial_r u_k^N) - \tfrac{1}{2}(\operatorname{div}\mathbf{u}^N)|\boldsymbol{\nabla}\mathbf{u}^N|^2\,dx.
\end{aligned}
$$

So the convective term tested with $\Delta\mathbf{u}^N$ reads

$$
\begin{aligned}
&\langle (\mathbf{u}^N\cdot\boldsymbol{\nabla})\mathbf{u}^N + \tfrac{1}{2}(\operatorname{div}\mathbf{u}^N)\mathbf{u}^N, -\Delta\mathbf{u}^N\rangle \\
&\qquad = \sum_{jkr}\int_\Omega (\partial_r u_j^N)(\partial_j u_k^N)(\partial_r u_k^N)\,dx \\
&\qquad\quad - \tfrac{1}{2}\int_\Omega (\operatorname{div}\mathbf{u}^N)|\boldsymbol{\nabla}\mathbf{u}^N|^2\,dx - \tfrac{1}{2}\langle (\operatorname{div}\mathbf{u}^N)\mathbf{u}^N, \Delta\mathbf{u}^N\rangle.
\end{aligned} \tag{4.95}
$$

Let $\mathbf{v}^N := \mathbf{u}^N - \varepsilon\boldsymbol{\nabla}\pi^N$, then $\operatorname{div}\mathbf{v}^N = 0$. So as in lemma 4.8 (two-dimensional, space periodic!)

$$\sum_{jkr}(\partial_r v_j^N)(\partial_j v_k^N)(\partial_r v_k^N) = 0.$$

Thus

$$\begin{aligned}\Big|\sum_{jkr}&(\partial_r u_j^N)(\partial_j u_k^N)(\partial_r u_k^N)\Big|\\ &= \Big|\sum_{jkr}(\partial_r u_j^N)(\partial_j u_k^N)(\partial_r u_k^N) - (\partial_r v_j^N)(\partial_j v_k^N)(\partial_r v_k^N)\Big|\\ &= \Big|\sum_{jkr}(\varepsilon\partial_r\partial_j\pi^N)(\partial_j v^N)(\partial_r v_k^N) + (\partial_r v_j^N)(\varepsilon\partial_j\partial_k\pi^N)(\partial_r u_k^N)\\ &\qquad + (\partial_r v_j^N)(\partial_j v_k^N)(\varepsilon\partial_r\partial_k\pi^N)\Big|.\end{aligned}$$

Choose $r := \frac{4p_\infty}{p_\infty - 1}$ as in lemma 4.8, then by $\varepsilon\Delta\pi^N = \operatorname{div}\mathbf{u}^N$ we have $\|\mathbf{v}^N\|_r \leq \|\mathbf{u}^N\|_r + \|\varepsilon\boldsymbol{\nabla}\pi^N\|_r \leq C\,\|\boldsymbol{\nabla}\mathbf{u}^N\|_r$ and

$$\begin{aligned}\Big|\sum_{jkr}&\int_\Omega(\partial_r u_j^N)(\partial_j u_k^N)(\partial_r u_k^N)\,dx - \tfrac{1}{2}\int_\Omega(\operatorname{div}\mathbf{u}^N)|\boldsymbol{\nabla}\mathbf{u}^N|^2\,dx\Big|\\ &\leq \Big|\sum_{jkr}\int_\Omega(\partial_r u_j^N)(\partial_j u_k^N)(\partial_r u_k^N)\,dx\Big| + \Big|\tfrac{1}{2}\int_\Omega(\varepsilon\Delta\pi^N)|\boldsymbol{\nabla}\mathbf{u}^N|^2\,dx\Big|\\ &\leq C\,\|\varepsilon\boldsymbol{\nabla}^2\pi^N\|_2\|\boldsymbol{\nabla}\mathbf{u}^N\|_2^{\frac{r-4}{r-2}}\|\boldsymbol{\nabla}\mathbf{u}^N\|_r^{\frac{r}{r-2}}.\end{aligned}$$

But this expression can be controlled for small ε, i.e. $0 < \varepsilon \leq \varepsilon_0$ as we have seen in lemma 4.8. So from (4.95) and the estimate above we conclude that from the original expression

$$\langle(\mathbf{u}^N\cdot\boldsymbol{\nabla})\mathbf{u}^N + \tfrac{1}{2}(\operatorname{div}\mathbf{u}^N)\mathbf{u}^N, -\Delta\mathbf{u}^N\rangle$$

we have only left to control

$$\tfrac{1}{2}\langle(\operatorname{div}\mathbf{u}^N)\mathbf{u}^N, -\Delta\mathbf{u}^N\rangle.$$

This will be done as follows:

$$\begin{aligned}\big|\langle(\operatorname{div}&\,\mathbf{u}^N)\mathbf{u}^N, -\Delta\mathbf{u}^N\rangle\big|\\ &\leq \int_\Omega|\varepsilon\boldsymbol{\nabla}^2\pi^N||\mathbf{u}^N||\Delta\mathbf{u}^N|\,dx\\ &\leq \delta\int_\Omega(\widetilde{D}\mathbf{u}^N)^{p-2}|\boldsymbol{\nabla}^2\mathbf{u}^N|^2\,dx + C_\delta\int_\Omega(\widetilde{D}\mathbf{u}^N)^{2-p}|\varepsilon\boldsymbol{\nabla}^2\pi^N|^2|\mathbf{u}^N|^2\,dx\\ &\leq \delta C\,\mathcal{I}_\Phi{}^A(\mathbf{u}^N) + C_\delta\int_\Omega(\widetilde{D}\mathbf{u}^N)^{2-p}|\varepsilon\boldsymbol{\nabla}^2\pi^N|^2|\mathbf{u}^N|^2\,dx.\end{aligned}$$

Then

$$
\begin{aligned}
&\big|\langle(\operatorname{div}\mathbf{u}^N)\mathbf{u}^N, -\Delta\mathbf{u}^N\rangle\big| - \tfrac{1}{10}\mathcal{I}_\Phi{}^A(\mathbf{u}^N)\\
&\quad\le C\big\|(\widetilde{D}\mathbf{u}^N)^{2-p}\big\|_{\frac{2}{2-p_\infty}}\|\varepsilon\boldsymbol{\nabla}^2\pi^N\|^2_{\frac{4}{p_\infty}}\|\mathbf{u}^N\|^2_\infty\\
&\quad\le C\big(1+\big\|\boldsymbol{\nabla}\mathbf{u}^N\big\|_2\big)^{2-p_\infty}\|\varepsilon\boldsymbol{\nabla}^2\pi^N\|^2_{\frac{4}{p_\infty}}\|\mathbf{u}^N\|^2_\infty\\
&\quad\le C\big(1+\big\|\boldsymbol{\nabla}\mathbf{u}^N\big\|_2\big)^{2-p_\infty}\|\varepsilon\boldsymbol{\nabla}^2\pi^N\|_2\|\varepsilon\boldsymbol{\nabla}^2\pi^N\|_{\frac{2}{p_\infty-1}}\|\mathbf{u}^N\|^2_\infty\\
&\quad\le C\big(1+\big\|\boldsymbol{\nabla}\mathbf{u}^N\big\|_2\big)^{2-p_\infty}\|\varepsilon\boldsymbol{\nabla}^2\pi^N\|_2\|\operatorname{div}\mathbf{u}^N\|_{\frac{2}{p_\infty-1}}\|\mathbf{u}^N\|^2_\infty\\
&\quad\le C\big(1+\big\|\boldsymbol{\nabla}\mathbf{u}^N\big\|_2\big)^{2-p_\infty}\|\varepsilon\Delta\pi^N\|_2\|\boldsymbol{\nabla}\mathbf{u}^N\|_{\frac{2}{p_\infty-1}}\|\mathbf{u}^N\|^2_\infty\\
&\quad\le \tfrac{\varepsilon}{4}\|\Delta\pi^N\|_2^2+\varepsilon\, C\big(1+\big\|\boldsymbol{\nabla}\mathbf{u}^N\big\|_2\big)^{2(2-p_\infty)}\|\boldsymbol{\nabla}\mathbf{u}^N\|^2_{\frac{2}{p_\infty-1}}\|\mathbf{u}^N\|^4_\infty\\
&\quad\le \tfrac{\varepsilon}{4}\|\Delta\pi^N\|_2^2+\varepsilon\, C\big(1+\big\|\boldsymbol{\nabla}\mathbf{u}^N\big\|_2\big)^{2(2-p_\infty)}\|\boldsymbol{\nabla}\mathbf{u}^N\|_2^{\frac{3-p_\infty}{2}}\|\boldsymbol{\nabla}\mathbf{u}^N\|^{\frac{1+p_\infty}{2}}_{\frac{2(p_\infty-1)}{5p_\infty-7}}\|\mathbf{u}^N\|^4_\infty\\
&\quad\le \tfrac{\varepsilon}{4}\|\Delta\pi^N\|_2^2+\varepsilon\, C\big(1+\big\|\boldsymbol{\nabla}\mathbf{u}^N\big\|_2\big)^{2(2-p_\infty)}\|\boldsymbol{\nabla}\mathbf{u}^N\|_2^{\frac{3-p_\infty}{2}}\|\boldsymbol{\nabla}\mathbf{u}^N\|^{\frac{1+3p_\infty}{4}}_{\frac{2(p_\infty-1)}{5p_\infty-7}}\|\mathbf{u}^N\|^{\frac{17-p_\infty}{4}}_\infty\\
&\quad\le \tfrac{\varepsilon}{4}\|\Delta\pi^N\|_2^2+\varepsilon\, C\big(1+\big\|\boldsymbol{\nabla}\mathbf{u}^N\big\|_2\big)^{\frac{39-11p_\infty}{4}}\|\boldsymbol{\nabla}\mathbf{u}^N\|^{\frac{1+3p_\infty}{4}}_{\frac{2(p_\infty-1)}{5p_\infty-7}}\\
&\quad\le \tfrac{\varepsilon}{4}\|\Delta\pi^N\|_2^2+\varepsilon\, C_\delta\big(1+\big\|\boldsymbol{\nabla}\mathbf{u}^N\big\|_2\big)^{\frac{p_\infty(39-11p_\infty)}{p_\infty-1}}+\delta\varepsilon\|\boldsymbol{\nabla}\mathbf{u}^N\|^{p_\infty}_{\frac{2(p_\infty-1)}{5p_\infty-7}}\\
&\quad\le \tfrac{\varepsilon}{4}\|\Delta\pi^N\|_2^2+\varepsilon\, C_\delta\big(1+\big\|\boldsymbol{\nabla}\mathbf{u}^N\big\|_2\big)^{\frac{p_\infty(39-11p_\infty)}{p_\infty-1}}+\delta\varepsilon\|\mathbf{D}\mathbf{u}^N\|^{p_\infty}_{\frac{2(p_\infty-1)}{5p_\infty-7}}\\
&\quad\le \tfrac{\varepsilon}{4}\|\Delta\pi^N\|_2^2+\varepsilon\, C_\delta\big(1+\big\|\boldsymbol{\nabla}\mathbf{u}^N\big\|_2\big)^{\frac{p_\infty(39-11p_\infty)}{p_\infty-1}}+\delta\varepsilon\|(\widetilde{D}\mathbf{u}^N)^{\frac{p}{2}}\|^2_{\frac{4(p_\infty-1)}{p_\infty(5p_\infty-7)}}\\
&\quad\le \tfrac{\varepsilon}{4}\|\Delta\pi^N\|_2^2+\varepsilon\, C_\delta\big(1+\big\|\boldsymbol{\nabla}\mathbf{u}^N\big\|_2\big)^{\frac{p_\infty(39-11p_\infty)}{p_\infty-1}}+\delta\varepsilon\, C\|\boldsymbol{\nabla}(\widetilde{D}\mathbf{u}^N)^{\frac{p}{2}}\|_2^2+|\widetilde{D}\mathbf{u}^N|_{p(\cdot)}.
\end{aligned}
$$

As in lemma 4.8 we absorb the second term into $\mathcal{I}_\Phi{}^A(\mathbf{u}^N)$. The first term can also be absorbed by the left-hand side. The last term is in $L^1(I)$, while the third term can be controlled for small $\varepsilon > 0$ with the help of lemma 8.8. Overall we have shown that the convective term tested with $\Delta\mathbf{u}^N$ can be controlled. This implies

$$
\begin{aligned}
&\|\boldsymbol{\nabla}\mathbf{u}^N\|^2_{L^\infty(I,L^2(\Omega))}+\big\|(\widetilde{D}\mathbf{u}^N)^{\frac{p}{2}}\ln(\widetilde{D}\mathbf{u}^N)\big\|^2_{L^2(I,L^2(\Omega))}\\
&+\big\|\boldsymbol{\nabla}\big((\widetilde{D}\mathbf{u}^N)^{\frac{p}{2}}\big)\big\|^2_{L^2(I,L^2(\Omega))}+c\,\big\|\mathcal{I}_\Phi{}^A(\mathbf{u}^N)\big\|_{L^1(I)}+\tfrac{\varepsilon}{2}\|\Delta\pi^N\|^2_{L^2(I,L^2(\Omega))}\le C.
\end{aligned}
$$

It remains to justify the third a priori estimate that was deduced testing the time derivative of the Galerkin system with $\partial_t\mathbf{u}^N \in X_N^2$. In order to do this we need to control

$$
\begin{aligned}
&\Big\langle\partial_t\big((\mathbf{u}^N\cdot\boldsymbol{\nabla})\mathbf{u}^N+\tfrac12(\operatorname{div}\mathbf{u}^N)\mathbf{u}^N\big),\partial_t\mathbf{u}^N\Big\rangle\\
&\quad=\langle(((\partial_t\mathbf{u}^N)\cdot\boldsymbol{\nabla})\mathbf{u}^N,\partial_t\mathbf{u}^N\rangle+\langle(\mathbf{u}^N\cdot\boldsymbol{\nabla})(\partial_t\mathbf{u}^N),\partial_t\mathbf{u}^N\rangle\\
&\qquad+\tfrac12\langle(\operatorname{div}\mathbf{u}^N)(\partial_t\mathbf{u}^N),\partial_t\mathbf{u}^N\rangle+\tfrac12\langle(\operatorname{div}\partial_t\mathbf{u}^N)\mathbf{u}^N,\partial_t\mathbf{u}^N\rangle\\
&\quad=\langle(((\partial_t\mathbf{u}^N)\cdot\boldsymbol{\nabla})\mathbf{u}^N,\partial_t\mathbf{u}^N\rangle+\tfrac12\langle(\operatorname{div}\partial_t\mathbf{u}^N)\mathbf{u}^N,\partial_t\mathbf{u}^N\rangle.
\end{aligned}
$$

The first term can be estimated by $C\,\|\nabla \mathbf{u}^N\|_2 \|\partial_t \mathbf{u}^N\|_4^2$, which can be controlled exactly as in lemma 4.8. For the second term we have

$$\begin{aligned}\big|\langle(\operatorname{div}\partial_t \mathbf{u}^N)\mathbf{u}^N, \partial_t \mathbf{u}^N\rangle\big| &= \big|\langle(\varepsilon\partial_t\Delta\pi^N)\mathbf{u}^N, \partial_t\mathbf{u}^N\rangle\big| \\ &\le \int_\Omega |\varepsilon\partial_t\nabla\pi^N||\nabla\mathbf{u}^N||\partial_t\mathbf{u}^N|\,dx + \int_\Omega |\varepsilon\partial_t\nabla\pi^N||\mathbf{u}^N||\partial_t\nabla\mathbf{u}^N|\,dx \\ &\le \|\varepsilon\partial_t\nabla\pi^N\|_4\|\nabla\mathbf{u}^N\|_2\|\partial_t\mathbf{u}^N\|_4 + \int_\Omega |\varepsilon\partial_t\nabla\pi^N||\mathbf{u}^N||\partial_t\nabla\mathbf{u}^N|\,dx \\ &\le C\,\|\partial_t\mathbf{u}^N\|_4\|\nabla\mathbf{u}^N\|_2\|\partial_t\mathbf{u}^N\|_4 + \int_\Omega |\varepsilon\partial_t\nabla\pi^N||\mathbf{u}^N||\partial_t\nabla\mathbf{u}^N|\,dx.\end{aligned}$$

The first term can once again be estimated as in lemma 4.8. The second term can be controlled analogously to (see above)

$$\int_\Omega |\varepsilon\nabla^2\pi^N||\mathbf{u}^N||\Delta\mathbf{u}^N|\,dx,$$

if we undertake the following substitution within the calculations

$$\begin{aligned}\varepsilon\nabla^2\pi^N &\mapsto \varepsilon\partial_t\nabla\pi^N, \\ \Delta\mathbf{u}^N &\mapsto \partial_t\nabla\mathbf{u}^N, \\ \mathcal{I}_\Phi{}^A(\mathbf{u}^N) &\mapsto \mathcal{J}_\Phi{}^A(\mathbf{u}^N).\end{aligned}$$

Overall we have proven the last missing a priori estimate:

$$\|\partial_t\mathbf{u}^N\|^2_{L^\infty(I,L^2(\Omega))} + \big\|\mathcal{J}_\Phi{}^A(\mathbf{u}^N)\big\|_{L^1(I)} \le C.$$

With all three a priori estimates as tools, we can now conclude the proof as in lemma 4.8. □

4. Shear Dependent Stokes Flow — $C^{1,\alpha}(I \times \Omega)$ Solutions

We will now get to one of the main results of this chapter. We will show that the the systems (4.1), (4.2), and (4.3) have strong solutions $\mathbf{u}, \pi$ with $\mathbf{u} \in C^{1,\alpha}(I \times \Omega)$. More precisely

THEOREM 4.10. *Let $p : \Omega \to (1, 2]$ be uniformly Lipschitz continuous, i.e. $p \in W^{1,\infty}(I \times \Omega)$, with $\frac{3}{2} < p_\infty \le p_0 \le 2$. Let $\mathbf{S}$ be induced by a p–potential F, Φ, which additionally satisfies (3.27). Further let $\mathbf{u}_0 \in W_0^{2,2}(\Omega)$, $\operatorname{div}(\mathbf{S}(\mathbf{D}\mathbf{u}_0)) \in W^{\beta,2}(\Omega)$, $\mathbf{f} \in C(I, W_0^{1,2}(\Omega))$, $\partial_t\mathbf{f} \in L^2(I, W^{-1,s}(\Omega))$, and $\partial_t\mathbf{f} \in L^s(I, L_0^s(\Omega))$ for some $s > 2$, $\beta > 0$. Then there exist constants $\varepsilon_0 > 0$, $q > 2$ and $\alpha > 0$, such that the system (4.2), resp. (4.3), has for all ε with $0 \le \varepsilon \le \varepsilon_0$ a strong solution $\mathbf{u}, \pi$, which satisfies $\mathbf{u} \in C^{1,\alpha}(I \times \Omega) \cap L^\infty(I, W^{2,q}(\Omega))$ and $\pi \in L^\infty(I, W^{1,2}(\Omega))$ with norms bounded independently of ε.*

To prove this result we will proceed similar to Kaplický, Málek and Stará (see [**KMS97b**]), who proved the same result under the conditions $\varepsilon = 0$, $\mathbf{u}_0 = 0$, and p is neither space nor time dependent but is a constant satisfying $\frac{4}{3} < p \le 2$.

In contrast to [**KMS97b**] we have included the possibility of non–zero initial data, since this is quite crucial for the application to numerical analysis. Moreover the result above also covers the pressure stabilized setting. Thereby we carefully pay attention that the norms involved do not depend on ε as long as $0 \leq \varepsilon \leq \varepsilon_0$. So the theorem above provides all the necessary informations regarding regularity to establish a numerical error analysis for a pressure stabilized time discretization.

The idea of the proof is the following: Instead of (4.2), resp. (4.3), we examine their A–approximations. According to the last section these systems have a strong solution $\mathbf{u}^A$, π^A. From the additional a priori estimates we will deduce that the convective term can in all calculations be included in the force term $\mathbf{f}$. This enables us to apply the results of section 2 in order to prove higher regularity for $\mathbf{u}^A$ and π^A. This information unfortunately depends on the value of A and therefore seems not handy for the limit $A \to \infty$. The key idea is that the A–approximation only modifies the system where $|\mathbf{Du}| \geq A$. So by deducing an estimate of the form

$$\|\widetilde{D}\mathbf{u}^A\|_{L^\infty(I\times\Omega)} \leq CA^\nu$$

for some $0 \leq \nu < 1$ (independently of ε with $0 \leq \varepsilon \leq \varepsilon_0$), we see that for sufficiently large A, i.e. $A \geq A_0$ there holds

$$\|\mathbf{Du}^A\|_{L^\infty(I\times\Omega)} < A.$$

But this means that the A–approximated system agrees for $A \geq A_0$ with the original system, so $\mathbf{u}^{A_0}, \pi^{A_0}$ is a solution to the original system (4.1). Since $\mathbf{u}^{A_0} \in C(I, W^{2,q}(\Omega))$ and $\mathbf{u}^{A_0} \in C^{1,\alpha}(I \times \Omega)$ for some $q > 2$ and $\alpha > 0$, this proves the theorem.

Note that Kaplický, Málek and Stará [**KMS97b**] have used a different approximation, namely the $\boldsymbol{\lambda}$–approximation with $\boldsymbol{\lambda} = (\lambda_1, 2)$. Since the $\boldsymbol{\lambda}$–approximation differs from the original system for all $\lambda_1 > 0$, they have to proceed differently to conclude from the regularity of the approximated system to the regularity of the original system. They use that the terms $\lambda_1^{-\frac{1}{2}}$ and $\|\mathbf{Du}^{\boldsymbol{\lambda}}\|_\infty$ are somehow interchangeable within the estimates (see (3.33)). So they show that the estimates depending on λ_1 (see the estimates below and replace A by $\lambda_1^{-\frac{1}{2}}$ due to remark (3.9)) still hold true if $\lambda_1^{-\frac{1}{2}}$ is replaced by $\|\mathbf{u}^\lambda\|_{L^\infty(I\times\Omega)}$. This leads to a chain of inequalities not depending on λ_1, which estimate $\|\widetilde{D}\mathbf{u}^{\boldsymbol{\lambda}}\|_\infty$ by itself, but with a power less than one:

$$\|\widetilde{D}\mathbf{u}^{\boldsymbol{\lambda}}\|_\infty \leq \cdots \leq C + C\,\|\widetilde{D}\mathbf{u}^{\boldsymbol{\lambda}}\|_\infty^\gamma$$

with $0 \leq \gamma < 1$. The finiteness of $\|\widetilde{D}\mathbf{u}^{\boldsymbol{\lambda}}\|_\infty$ then implies the uniform boundedness of $\|\widetilde{D}\mathbf{u}^{\boldsymbol{\lambda}}\|_\infty$ with respect to λ_1. From this information it follows that it is possible to once again apply the result of section 2 to the $\boldsymbol{\lambda}$–approximated system with γ_1 and γ_2 now independent of λ_1. As a result Kaplický, Málek and Stará can extract enough regularity from $\mathbf{u}^{\boldsymbol{\lambda}}$, namely $C(I, W^{2,q}(\Omega))$ for some $q > 2$, which is stable under the limit $\lambda_1 \to 0^+$ and which implies $C^{1,\alpha}(I \times \Omega)$ regularity of the limit $\mathbf{u}$. This is the solution of the original problem with the desired regularity.

When we compare the conclusions for the $\boldsymbol{\lambda}$–approximation with the one for the A–approximation it seems that in the end the A–approximation is easier to handle: There is no need for the limit $A \to \infty$, it just suffices to pick an A_0 large enough. This implies that there is no need to deduce a priori estimates uniformly in A, i.e. estimates which are stable under the limit $A \to \infty$. For this reason we use the A–approximation rather than the $\boldsymbol{\lambda}$–approximation.

PROOF OF THEOREM 4.10. Let $\mathbf{u}^A$, π^A be the solutions of (4.2), resp. (4.3). From lemma 4.8, resp. 4.9, we know that

$$\|\mathbf{u}^A\|_{C(I,W^{1,2p_\infty}(\Omega))} \le C.$$

This implies

$$\|\mathbf{u}^A\|_{C(I\times\Omega)} \le C.$$

As a consequence we get

$$\|(\mathbf{u}^A\cdot\boldsymbol{\nabla})\mathbf{u}^A + \tfrac{1}{2}(\operatorname{div}\mathbf{u}^A)\mathbf{u}^A\|_{C(I,L^{2p_\infty}(\Omega))} \le C,$$

resp. (using $\operatorname{div}\mathbf{u}^A = -\varepsilon\Delta\pi^A$)

$$\|((\mathbf{u}^A - \varepsilon\boldsymbol{\nabla}\pi^A)\cdot\boldsymbol{\nabla})\mathbf{u}^A\|_{C(I,L^{2p_\infty}(\Omega))} \le C.$$

In either cases $\mathbf{u}^A$, π^A satisfy

$$\begin{aligned} \partial_t\mathbf{u}^A - \operatorname{div}(\mathbf{S}^A(\mathbf{D}\mathbf{u}^A)) + \boldsymbol{\nabla}\pi^A &= \mathbf{g}^A && \text{on } I\times\Omega,\\ \operatorname{div}\mathbf{u}^A &= \varepsilon\Delta\pi^A && \text{on } I\times\Omega,\\ \mathbf{u}(0) &= \mathbf{u}_0 && \text{on } \Omega, \end{aligned} \tag{4.96}$$

where $\mathbf{g}^A = (\mathbf{u}^A\cdot\boldsymbol{\nabla})\mathbf{u}^A + \frac{1}{2}(\operatorname{div}\mathbf{u}^A)\mathbf{u}^A$, resp. $\mathbf{g}^A = ((\mathbf{u}^A - \varepsilon\boldsymbol{\nabla}\pi^A)\cdot\boldsymbol{\nabla})\mathbf{u}^A$. Furthermore

$$\|\mathbf{g}^A\|_{C(I,L^{2p_\infty}(\Omega))} \le C. \tag{4.97}$$

Taking the ∂_r derivative of (4.96) implies that $(\mathbf{v}_r^A,\rho^A) := (\partial_r\mathbf{u}^A,\partial_r\pi^A)$ is a solution of

$$\begin{aligned} \partial_t\mathbf{v}_r^A - \operatorname{div}\big((\boldsymbol{\nabla}^2_{n\times n}\Phi^A)(\mathbf{D}\mathbf{u}^A)\mathbf{D}\mathbf{v}_r^A\big) + \boldsymbol{\nabla}\rho_r^A &= \partial_r\mathbf{g}^A && \text{on } I\times\Omega,\\ \operatorname{div}\mathbf{v}_r^A &= \varepsilon\Delta\rho_r^A && \text{on } I\times\Omega,\\ \mathbf{v}_r^A(0) &= \partial_r\mathbf{u}_0 && \text{on } \Omega, \end{aligned} \tag{4.98}$$

where $r = 1,2$. On the other hand taking the ∂_t derivative of (4.96) implies that $(\mathbf{w}^A,\mu^A) := (\partial_t\mathbf{u}^A,\partial_t\pi^A)$ is a solution (in the sense of distributions) of

$$\begin{aligned} \partial_t\mathbf{w}^A - \operatorname{div}\big((\boldsymbol{\nabla}^2_{n\times n}\Phi^A)(\mathbf{D}\mathbf{u}^A)\mathbf{D}\mathbf{w}^A\big) + \boldsymbol{\nabla}\mu^A &= \partial_t\mathbf{g}^A && \text{on } I\times\Omega,\\ \operatorname{div}\mathbf{w}^A &= \varepsilon\Delta\mu^A && \text{on } I\times\Omega,\\ \mathbf{w}^A(0) &= \partial_t\mathbf{u}_0 && \text{on } \Omega. \end{aligned} \tag{4.99}$$

We will now show that

$$\begin{aligned} \|\partial_r\mathbf{g}^A\|_{L^{2p_\infty}(I,W^{-1,2p_\infty}(\Omega))} &\le C,\\ \|\partial_t\mathbf{g}^A\|_{L^{2p_\infty}(I,W^{-1,2p_\infty}(\Omega))} &\le C, \end{aligned}$$

which will enable us to apply the theory of section 2 to $\mathbf{v}_r^A$ and $\mathbf{w}^A$. While the estimate of $\partial_r\mathbf{g}^A$ is just a consequence of $\|\mathbf{g}^A\|_{C(I,L^{2p_\infty}(\Omega))} \le C$, we have to give some explanation for $\partial_t\mathbf{g}^A$.

Since $\|\mathcal{J}_\Phi{}^A(\mathbf{u}^A)\|_{L^1(I)} \le C$, we deduce from lemma 3.13 that

$$\|\partial_t\boldsymbol{\nabla}\mathbf{u}^A\|_{L^{p_\infty}(I,L^{p_\infty}(\Omega))} \overset{\text{Korn}}{\le} C\,\|\partial_t\mathbf{D}\mathbf{u}^A\|_{L^{p_\infty}(I,L^{p_\infty}(\Omega))} \le C.$$

Hence with Korn's inequality (for $p_\infty < 2$) and the embedding $W^{1,p_\infty}(\Omega) \hookrightarrow L^{\frac{2p_\infty}{2-p_\infty}}(\Omega)$

$$\|\partial_t\mathbf{u}^A\|_{L^{p_\infty}(I,L^{\frac{2p_\infty}{2-p_\infty}}(\Omega))} \le C. \tag{4.100}$$

Together with the estimate $\|\partial_t \mathbf{u}^A\|_{L^\infty(I,L^2(\Omega))} \le C$ (see (4.58)) this implies

$$\|\partial_t \mathbf{u}^A\|_{L^{2p_\infty}(I,L^{2p_\infty}(\Omega))}.$$

Let $s := (2p_\infty)' = \frac{2p_\infty}{2p_\infty - 1}$, then for all $\|\boldsymbol{\varphi}\|_{L^s(I,W^{1,s}(\Omega))}$

$$\begin{aligned}
&\big|\big\langle \partial_t((\mathbf{u}^A \cdot \boldsymbol{\nabla})\mathbf{u}^A), \boldsymbol{\varphi}\big\rangle\big| + \big|\big\langle \partial_t((\operatorname{div} \mathbf{u}^A)\mathbf{u}^A), \boldsymbol{\varphi}\big\rangle\big| \\
&\quad \le \big|\big\langle ((\partial_t \mathbf{u}^A) \cdot \boldsymbol{\nabla})\mathbf{u}^A, \boldsymbol{\varphi}\big\rangle\big| + \big|\big\langle (\mathbf{u}^A \cdot \boldsymbol{\nabla})(\partial_t \mathbf{u}^A), \boldsymbol{\varphi}\big\rangle\big| \\
&\qquad + \big|\big\langle (\operatorname{div} \partial_t \mathbf{u}^A)\mathbf{u}^A, \boldsymbol{\varphi}\big\rangle\big| + \big|\big\langle (\operatorname{div} \mathbf{u}^A)(\partial_t \mathbf{u}^A), \boldsymbol{\varphi}\big\rangle\big|.
\end{aligned}$$

Using integration by parts on the second and the third term of the right-hand side we get

$$\begin{aligned}
&\big|\big\langle \partial_t((\mathbf{u}^A \cdot \boldsymbol{\nabla})\mathbf{u}^A), \boldsymbol{\varphi}\big\rangle\big| + \big|\big\langle \partial_t((\operatorname{div} \mathbf{u}^A)\mathbf{u}^A), \boldsymbol{\varphi}\big\rangle\big| \\
&\quad \le 3 \int_\Omega |\partial_t \mathbf{u}^A||\boldsymbol{\nabla}\mathbf{u}^A||\boldsymbol{\varphi}|\, dx + 3 \int_\Omega |\partial_t \mathbf{u}^A||\mathbf{u}^A||\boldsymbol{\nabla}\boldsymbol{\varphi}|\, dx \\
&\quad \le 3\, \|\partial_t \mathbf{u}^A\|_{L^\infty(I,L^2(\Omega))} \|\boldsymbol{\nabla}\mathbf{u}^A\|_{L^{2p_\infty}(I,L^{2p_\infty}(\Omega))} \|\boldsymbol{\varphi}\|_{L^s(I,L^{s^*}(\Omega))} \\
&\qquad + 3\, \|\partial_t \mathbf{u}^A\|_{L^\infty(I,L^2(\Omega))} \|\mathbf{u}^A\|_{L^\infty(I\times\Omega)} \|\boldsymbol{\nabla}\boldsymbol{\varphi}\|_{L^s(I,L^s(\Omega))} \\
&\quad \le C.
\end{aligned}$$

This proves

$$\begin{aligned}
\|\partial_t((\mathbf{u}^A \cdot \boldsymbol{\nabla})\mathbf{u}^A)\|_{L^{2p_\infty}(I,W^{-1,2p_\infty}(\Omega))} &\le C, \\
\|\partial_t((\operatorname{div} \mathbf{u}^A)\mathbf{u}^A)\|_{L^{2p_\infty}(I,W^{-1,2p_\infty}(\Omega))} &\le C.
\end{aligned}$$

The term $\|\partial_t(\varepsilon \boldsymbol{\nabla}\pi^A \cdot \boldsymbol{\nabla})\mathbf{u}^A\|_{L^{2p_\infty}(I,W^{-1,2p_\infty}(\Omega))}$ can be estimated in exactly the same way, if we keep in mind that $\varepsilon \Delta \pi^A = \operatorname{div} \mathbf{u}^A$. Thus we have

$$\|\partial_t \mathbf{g}^A\|_{L^{2p_\infty}(I,W^{-1,2p_\infty}(\Omega))} \le C.$$

We have proven so far that the theory of section 2 is applicable to the systems (4.98) and (4.99). Let us be more precise. From (4.99), (3.32), (3.34), lemma 4.4 and remark 4.5 if follows that there exists a constant $\kappa > 0$, such that for all s with $2 \le s \le \min\{2p_\infty, 2 + \kappa A^{p_\infty - 2}\}$ there holds

$$\|\partial_t \mathbf{u}^A\|_{C(I,B_s^{1-\frac{2}{s}}(\Omega))} \le C_{\gamma_2} A^{\frac{3}{2}(2-p_\infty)} \big(\|\partial_t \mathbf{g}^A\|_{L^s(I,W^{-1,s}(\Omega))} + \|(\partial_t \mathbf{u})(0)\|_{B_s^{1-\frac{2}{s}}(\Omega)} \big).$$

We still have to prove that $\|(\partial_t \mathbf{u})(0)\|_{B_s^{1-\frac{2}{s}}(\Omega)}$ is finite. For this let P be the projection onto the space of divergence free functions. Due to the space periodic setting we know that $P : W_0^{k,2}(\Omega) \to W_0^{k,2}(\Omega)$ is continuous for every $k \ge 0$. Since P is self adjoint it follows that $P : W^{-k,2}(\Omega) \to W^{-k,2}(\Omega)$ is also continuous for all $k > 0$. Let $\beta \in (0, \frac{1}{4})$ and let $\boldsymbol{\varphi}$ with $\|\boldsymbol{\varphi}\|_{W^{-\beta,2}(\Omega)} \le 1$ be arbitrary, then

$$\begin{aligned}
|\langle \partial_t \mathbf{u}(0), \boldsymbol{\varphi}\rangle| &= |\langle \partial_t \mathbf{u}(0), P\boldsymbol{\varphi}\rangle| \\
&= |\langle \operatorname{div}(\mathbf{S}(\mathbf{D}\mathbf{u}_0)) + (\mathbf{u}_0 \cdot \boldsymbol{\nabla})\mathbf{u}_0 - \mathbf{f}(0), P\boldsymbol{\varphi}\rangle| \\
&\le \|\operatorname{div}(\mathbf{S}(\mathbf{D}\mathbf{u}_0))\|_{\beta,2} + \|(\mathbf{u}_0 \cdot \boldsymbol{\nabla})\mathbf{u}_0\|_{\beta,2} + \|\mathbf{f}(0)\|_{\beta,2} \\
&\le \|\operatorname{div}(\mathbf{S}(\mathbf{D}\mathbf{u}_0))\|_{\beta,2} + C\, \|\mathbf{u}_0\|_{2,2}^2 + \|\mathbf{f}(0)\|_{\beta,2} \\
&\le C.
\end{aligned}$$

Thus $\|(\partial_t\mathbf{u})(0)\|_{W^{\beta,2}(\Omega)}\leq C$. Since $W^{\beta,2}(\Omega)\to B_s^{1-\frac{2}{s}}(\Omega)$ if s is close enough to 2, there exists a constant s_0 with $2<s_0<2p_\infty$, such that $\|(\partial_t\mathbf{u})(0)\|_{B_s^{1-\frac{2}{s}}(\Omega)}$ is uniformly bounded for $2\leq s\leq\min\{s_0,2+\kappa A^{p_\infty-2}\}$. Overall we have shown that

$$\|\partial_t\mathbf{u}^A\|_{C(I,B_s^{1-\frac{2}{s}}(\Omega))}\leq C\,A^{\frac{3}{2}(2-p_\infty)} \tag{4.101}$$

uniformly for all $2\leq s\leq\min\{s_0,2+\kappa A^{p_\infty-2}\}=:s_1$. For $\theta\in(0,1)$ define $s_\theta:=2+(s_1-2)\theta$, then we deduce from (4.101) and (4.58), i.e.

$$\|\partial_t\mathbf{u}^A\|_{L^\infty(I,L^2(\Omega))}\leq C,$$

by interpolation that

$$\|\partial_t\mathbf{u}^A\|_{C(I,B_{s_\theta}^{1-\frac{2}{s_\theta}}(\Omega))}\leq C\,(A^{\frac{3}{2}(2-p_\infty)})^{\frac{\theta s}{2+\theta s-2\theta}}\leq C\,A^{\frac{3}{2}\theta(2-p_\infty)}. \tag{4.102}$$

Since $B_{s_\theta}^{1-\frac{2}{s_\theta}}(\Omega)\to L^{s_\theta}(\Omega)$ this implies

$$\|\partial_r\partial_t\mathbf{u}^A\|_{C(I,W^{-1,s_\theta}(\Omega))}\leq C\,A^{\frac{3}{2}\theta(2-p_\infty)}. \tag{4.103}$$

for all $0<\theta<1$. This and the definition of s_θ imply

$$\|\partial_r\partial_t\mathbf{u}^A\|_{C(I,W^{-1,s}(\Omega))}\leq C\,A^{\frac{3}{2}\theta(2-p_\infty)} \tag{4.104}$$

for all $2\leq s\leq 2+\theta\,\min\{s_0-2,\kappa A^{p_\infty-2}\}$. From (4.98) we deduce for a.a. $t\in I$

$$\begin{aligned}-\operatorname{div}\big((\boldsymbol{\nabla}^2_{n\times n}\Phi^A)(\mathbf{D}\mathbf{u}^A)\mathbf{D}\mathbf{v}_r^A\big)+\boldsymbol{\nabla}\rho_r^A&=\partial_r\mathbf{g}^A-\partial_r\partial_t\mathbf{u}^A &&\text{on }\Omega,\\ \operatorname{div}\mathbf{v}_r^A&=\varepsilon\Delta\rho_r^A &&\text{on }\Omega.\end{aligned} \tag{4.105}$$

From this, (4.97), (4.103), (3.32), (3.34), and lemma 4.6 we deduce that

$$\|\boldsymbol{\nabla}\mathbf{v}_r^A\|_{L^s(\Omega)}\leq CA^{\frac{3}{2}(1+\theta)(2-p_\infty)}\;A^{\frac{3}{2}\theta(2-p_\infty)}.$$

Thus for a.a. $t\in I$

$$\|\boldsymbol{\nabla}\mathbf{u}^A\|_{W^{1,s}(\Omega)}\leq CA^{\frac{3}{2}(1+\theta)(2-p_\infty)}, \tag{4.106}$$

for all $2\leq s\leq 2+\theta\,\min\{s_0-2,\kappa\|\widetilde{D}\mathbf{u}^A\|_\infty^{p_\infty-2}\}$ (with a possibly decreased κ). Unfortunately this estimate depends on A and is therefore not robust under $A\to\infty$. But at least it proves that $\|\widetilde{D}\mathbf{u}^A\|_\infty$ is finite. As mentioned before starting this proof we will now try to replace the role of A by $\|\widetilde{D}\mathbf{u}^A\|_\infty$. This is possible by using the second version of (3.32) (not involving the constant A) rather than the first version (involving A). Doing so in the derivation of (4.101) we will get the alternative restriction

$$2\leq s\leq 2+\theta\,\min\{s_0-2,\kappa\|\widetilde{D}\mathbf{u}^A\|_\infty^{p_\infty-2}\}$$

instead of

$$2\leq s\leq 2+\theta\,\min\{s_0-2,\kappa A^{p_\infty-2}\}.$$

And (4.106) will transform to

$$\|\boldsymbol{\nabla}\mathbf{u}^A\|_{W^{1,s}(\Omega)}\leq C\|\widetilde{D}\mathbf{u}^A\|_\infty^{\frac{3}{2}(1+\theta)(2-p_\infty)} \tag{4.107}$$

for all $2 \leq s \leq 2 + \theta \min\{s_0 - 2, \kappa \|\widetilde{D}\mathbf{u}^A\|_\infty^{p_\infty - 2}\}$. Using the embedding $W^{1,s}(\Omega) \to C(\overline{\Omega})$ for two space dimensions with its appropriate embedding constant (see for example Ziemer [**Zie89**]) we get

$$\|\nabla \mathbf{u}^A\|_\infty \leq C\,(s-2)^{-\frac{1}{2}} \|\nabla \mathbf{u}^A\|_{W^{1,s}(\Omega)} \leq C\,(s-2)^{-\frac{1}{2}} \|\widetilde{D}\mathbf{u}^A\|_\infty^{\frac{3}{2}(1+\theta)(2-p_\infty)}$$

for all $2 \leq s \leq 2 + \theta \min\{s_0 - 2, \kappa \|\widetilde{D}\mathbf{u}^A\|_\infty^{p_\infty - 2}\}$. This implies

$$\begin{aligned} \|\widetilde{D}\mathbf{u}^A\|_\infty &\leq 1 + C\,\|\nabla \mathbf{u}^A\|_\infty \\ &\leq 1 + C\,(s-2)^{-\frac{1}{2}} \|\widetilde{D}\mathbf{u}^A\|_\infty^{\frac{3}{2}(1+\theta)(2-p_\infty)} \\ &= 1 + C\,\max\Big\{ s_0^{-\frac{1}{2}}, (\kappa \|\widetilde{D}\mathbf{u}^A\|_\infty)^{\frac{2-p_\infty}{2}} \Big\} \|\widetilde{D}\mathbf{u}^A\|_\infty^{\frac{3}{2}(1+\theta)(2-p_\infty)} \\ &\leq 1 + C\, s_0^{-\frac{1}{2}} \|\widetilde{D}\mathbf{u}^A\|_\infty^{\frac{3}{2}(1+\theta)(2-p_\infty)} + C\,\|\widetilde{D}\mathbf{u}^A\|_\infty^{(\frac{3}{2}(1+\theta)+\frac{1}{2})(2-p_\infty)}. \end{aligned}$$

Since $\|\widetilde{D}\mathbf{u}^A\|_\infty$ is finite, this implies for all p_∞ with $1 > (\frac{3}{2}(1+\theta) + \frac{1}{2})(2 - p_\infty)$ and $1 > \frac{3}{2}(1+\theta)(2-p_\infty)$

$$\|\widetilde{D}\mathbf{u}^A\|_\infty \leq C = C(\theta).$$

Since $p_\infty > \frac{3}{2}$ we can find $\theta_0 \in (0,1)$, such that $1 > (\frac{3}{2}(1+\theta_0) + \frac{1}{2})(2 - p_\infty)$ and $1 > \frac{3}{2}(1+\theta_0)(2-p_\infty)$. This implies

$$\|\widetilde{D}\mathbf{u}^A\|_\infty \leq C(\theta_0) = C.$$

Now this and (4.107) imply that there exists $s_2 > 2$ (independent of A) with

$$\|\nabla \mathbf{u}^A\|_{W^{1,s}(\Omega)} \leq C \|\widetilde{D}\mathbf{u}^A\|_\infty^{\frac{3}{2}(1+\theta_0)(2-p_\infty)} \leq C. \tag{4.108}$$

We have finally found a powerful estimate, which is still robust under $A \to \infty$. Especially we can pick a subsequence A_n and a function $\mathbf{u}$, such that

$$\nabla \mathbf{u}^{A_n} \overset{*}{\rightharpoonup} \nabla \mathbf{u} \qquad \text{in } L^\infty(I, W^{1,s_2}(\Omega)).$$

Now we have found a suitable limit function $\mathbf{u}$ and are only left to prove that $\mathbf{u}$ solves the original equations (4.2), resp. (4.3). As we have seen in lemma 4.8, resp. lemma 4.9, the regularity derived for $\mathbf{u}^A$ is sufficient to justify the limit (of a suitable subsequence) of most of the involved terms. The limit that needs some attention is

$$\operatorname{div}(\mathbf{S}^{A_n}(\mathbf{D}\mathbf{u}^{A_n})) \rightharpoonup \operatorname{div}(\mathbf{S}(\mathbf{D}\mathbf{u})) \qquad \text{in } \mathcal{D}(I \times \Omega),$$

which we will prove now: Since $W^{1,s_2}(\Omega) \to L^\infty(\Omega)$ continuously, we know that $\|\mathbf{D}\mathbf{u}^{A_n}\|_{L^\infty(I\times\Omega)}$ is uniformly bounded for all A by a constant A_0. So by definition of the A–approximation we have for all $A \geq A_0$

$$\mathbf{S}^{A_n}(\mathbf{D}\mathbf{u}^{A_n}) = \mathbf{S}(\mathbf{D}\mathbf{u}^{A_n}).$$

This means that we only have to justify the limit

$$\operatorname{div}(\mathbf{S}(\mathbf{D}\mathbf{u}^{A})) \rightharpoonup \operatorname{div}(\mathbf{S}(\mathbf{D}\mathbf{u})) \qquad \text{in } \mathcal{D}(I \times \Omega).$$

But this can be exactly done as in lemma 4.8, resp. lemma 4.9. Overall we have found a solution $\mathbf{u}$ of (4.2), resp. (4.3), which satisfies

$$\begin{aligned} \|\mathbf{u}\|_{L^\infty(I,W^{2,s_2}(\Omega))} &\leq C, \\ \|\partial_t \mathbf{u}\|_{L^{p_\infty}(I,L^{\frac{2p_\infty}{2-p_\infty}}(\Omega))} &\leq C, \end{aligned}$$

where we have used (4.100). Hence by Morrey's embedding theorems there exists $\alpha>0$, such that

$$\|\mathbf{u}\|_{L^\infty(I,C^{1,\alpha}(\Omega))}\le C,$$
$$\|\partial_t\mathbf{u}\|_{L^{p_\infty}(I,L^{\frac{2p_\infty}{2-p_\infty}}(\Omega))}\le C.$$

Lemma 2.2 in [**JS98**] (see appendix) implies that there exists $\alpha_2>0$ with

$$\|\mathbf{u}\|_{C^{1,\alpha_2}(I\times\Omega)}\le C.$$

From this, $\|\mathbf{u}\|_{L^\infty(I,W^{2,s_2}(\Omega))}\le C$, and $\|\partial_t\mathbf{u}\|_{L^\infty(I,L^2(\Omega))}\le C$ it follows easily that all terms of (4.2), resp. (4.3), apart from $\boldsymbol{\nabla}\pi$ are in $L^\infty(I,L^2(\Omega))$. Therefore also $\boldsymbol{\nabla}\pi\in L^\infty(I,L^2(\Omega))$. This proves the theorem. □

We have just proved the existence of a strong solution with $C^{1,\alpha}(I\times\Omega)$ regularity. It is interesting to observe that this solution is unique within the class of weak solutions satisfying

$$\|\mathbf{u}\|_{L^\infty(I,L^2(\Omega))}+\|\mathbf{D}\mathbf{u}\|_{L^{p(\cdot)}(I\times\Omega)}\le C.$$

More precisely

Lemma 4.11. *Let $\mathbf{u},\pi$ be the solution of theorem 4.10 and let $\mathbf{v},\rho$ be another weak solution with*

$$\|\mathbf{v}\|_{C(I,L^2(\Omega))}+\|\mathbf{D}\mathbf{v}\|_{L^{p(\cdot)}(I\times\Omega)}\le C.$$

Then $\mathbf{u}=\mathbf{v}$.

Proof. Note that for p constant and $\varepsilon=0$ this result has been proven in [**KMS97b**]. The proof is in fact straight forward, but nevertheless we will give a short sketch of it, since we want to point out once more the importance of the dual viscosity.

Let $\mathbf{e}:=\mathbf{u}-\mathbf{v}$ and $\eta:=\pi-\rho$, then by chapter 3 section 1

$$\mathbf{S}(\mathbf{D}\mathbf{u})-\mathbf{S}(\mathbf{D}\mathbf{v})=\boldsymbol{\sigma}(\mathbf{D}\mathbf{u},\mathbf{D}\mathbf{v})\mathbf{D}\mathbf{e}.$$

Note that $\boldsymbol{\sigma}(\mathbf{D}\mathbf{u},\mathbf{D}\mathbf{v})$ is well defined, since $\mathbf{D}\mathbf{u},\mathbf{D}\mathbf{v}\in L^{p(\cdot)}(I\times\Omega)$. This implies that $\mathbf{e},\eta$ is the weak solution of

$$\partial_t\mathbf{e}-\operatorname{div}(\boldsymbol{\sigma}(\mathbf{D}\mathbf{u},\mathbf{D}\mathbf{v})\mathbf{D}\mathbf{e})+\boldsymbol{\nabla}\eta=\mathbf{h}, \tag{4.109}$$
$$\operatorname{div}\mathbf{e}=\varepsilon\Delta\eta, \tag{4.110}$$
$$\mathbf{e}(0)=\mathbf{0}, \tag{4.111}$$

where

$$\mathbf{h}=(\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u}+\tfrac12(\operatorname{div}\mathbf{u})\mathbf{u}-(\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v}+\tfrac12(\operatorname{div}\mathbf{v})\mathbf{v},$$

resp.

$$\mathbf{h}=\big((\mathbf{u}-\varepsilon\boldsymbol{\nabla}\pi)\cdot\boldsymbol{\nabla}\big)\mathbf{u}-\big((\mathbf{v}-\varepsilon\boldsymbol{\nabla}\pi)\cdot\boldsymbol{\nabla}\big)\mathbf{v}.$$

We will now estimate $\langle \mathbf{h}, \mathbf{e}\rangle$ and show that $\langle \mathbf{h}, \mathbf{e}\rangle \in L^1(I)$. We begin with the first version of pressure stabilization, i.e. the convective part is given by $(\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u}+\frac{1}{2}(\operatorname{div}\mathbf{u})\mathbf{u}$:

$$\begin{aligned}|\langle \mathbf{h}, \mathbf{e}\rangle| &= \big|\langle (\mathbf{e}\cdot\boldsymbol{\nabla})\mathbf{u} + \tfrac{1}{2}(\operatorname{div}\mathbf{e})\mathbf{u}, \mathbf{e}\rangle \; + \; \langle (\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{e} + \tfrac{1}{2}(\operatorname{div}\mathbf{u})\mathbf{e}, \mathbf{e}\rangle\big| \\ &= |\langle (\mathbf{e}\cdot\boldsymbol{\nabla})\mathbf{u} + \tfrac{1}{2}(\operatorname{div}\mathbf{e})\mathbf{u}, \mathbf{e}\rangle| \\ &\le \|\mathbf{u}\|_{C^{1,\alpha}(I\times\Omega)}\|\mathbf{e}\|_2^2.\end{aligned}$$

This $\langle \mathbf{h}, \mathbf{e}\rangle \in L^1(I)$. Let us now estimate the second version of pressure stabilization, i.e. the convective part is given by $((\mathbf{u}-\varepsilon\boldsymbol{\nabla}\pi)\cdot\boldsymbol{\nabla})\mathbf{u}$:

$$\begin{aligned}|\langle \mathbf{h}, \mathbf{e}\rangle| &= \big|\langle ((\mathbf{e}-\varepsilon\boldsymbol{\nabla}\eta)\cdot\boldsymbol{\nabla})\mathbf{u}, \mathbf{e}\rangle \; + \; \langle ((\mathbf{u}-\varepsilon\boldsymbol{\nabla}\pi)\cdot\boldsymbol{\nabla})\mathbf{e}, \mathbf{e}\rangle\big| \\ &= |\langle ((\mathbf{e}-\varepsilon\boldsymbol{\nabla}\eta)\cdot\boldsymbol{\nabla})\mathbf{u}, \mathbf{e}\rangle| \\ &\le \|\mathbf{u}\|_{C^{1,\alpha}(I\times\Omega)}\big(\|\mathbf{e}\|_2^2 + \varepsilon\|\boldsymbol{\nabla}\eta\|_2\|\mathbf{e}\|_2\big).\end{aligned}$$

For $\varepsilon = 0$ this proves that $|\langle \mathbf{h}, \mathbf{e}\rangle| \in L^1(I)$. If ε then from $\operatorname{div}\mathbf{e} = \varepsilon\Delta\eta$ we know that $\varepsilon\|\boldsymbol{\nabla}\eta\|_2 \le C\|\mathbf{e}\|_2$, so

$$|\langle \mathbf{h}, \mathbf{e}\rangle| \le C\,\|\mathbf{u}\|_{C^{1,\alpha}(I\times\Omega)}\|\mathbf{e}\|_2^2.$$

Overall we have shown, that for both versions of pressure stabilization and $\varepsilon \ge 0$ there holds $|\langle \mathbf{h}, \mathbf{e}\rangle| \in L^1(I)$ and

$$|\langle \mathbf{h}, \mathbf{e}\rangle| \le C\,\|\mathbf{u}\|_{C^{1,\alpha}(I\times\Omega)}\|\mathbf{e}\|_2^2. \tag{4.112}$$

Let us assume that $\varepsilon = 0$, then $\mathbf{e}$ is an admissible test function for (4.109). (Since $\operatorname{div}\mathbf{e} = 0$ we can omit the pressure.) So for $\varepsilon = 0$ we get

$$\tfrac{1}{2}d_t\|\mathbf{e}\|_2^2 + \big\langle \boldsymbol{\sigma}(\mathbf{Du},\mathbf{Dv})\mathbf{De},\mathbf{De}\big\rangle = \langle \mathbf{h}, \mathbf{e}\rangle \overset{(4.112)}{\le} C\,\|\mathbf{u}\|_{C^{1,\alpha}(I\times\Omega)}\|\mathbf{e}\|_2^2.$$

An application of Gronwall's inequality proves $\mathbf{e} = 0$.

Let us asume that $\varepsilon > 0$. Then due to $\operatorname{div}\mathbf{e} = \varepsilon\Delta\eta$ and $\mathbf{e} \in C(I, L^2(\Omega))$ there holds $\pi \in C(I, W_0^{1,2}(\Omega))$. Thus η is an admissible test function for (4.110) and $\mathbf{e}$ is an admissible test function for (4.109). Summing up the resulting equations implies

$$\tfrac{1}{2}d_t\|\mathbf{e}\|_2^2 + \big\langle \boldsymbol{\sigma}(\mathbf{Du},\mathbf{Dv})\mathbf{De},\mathbf{De}\big\rangle + \varepsilon\|\boldsymbol{\nabla}\eta\|_2^2 = \langle \mathbf{h}, \mathbf{e}\rangle \overset{(4.112)}{\le} C\,\|\mathbf{u}\|_{C^{1,\alpha}(I\times\Omega)}\|\mathbf{e}\|_2^2.$$

An application of Gronwall's inequality proves $\mathbf{e} = 0$. □

Referring to this lemma we will sometimes only speak of *the* solution instead of the weak or the strong solution.

5. Error Estimates and the Dual Problem

Let us recall our original intention. In the beginning we introduced the pressure stabilization (two versions) in order to approximately solve the system (4.1). The main advantage hereby is that to solve the pressure stabilized systems we do not need to use divergence free functions. Actually this kind of stabilization is quite popular within the field of numerics and is for example used in the Van–Kahn, the Chorin, and the Chorin–Uzawa scheme (see [**Pro97**]). The approximated systems are indeed much handier to solve numerically, since we got rid of the constraint $\operatorname{div}\mathbf{u} = 0$. On the other hand the approximated solutions will not be exact and will differ from the real solution. So it is certainly very important to know of which order of magnitude the error due to the pressure stabilization is.

Let for example $\mathbf{v}, \rho$ be the solution of the non–stabilized system and $\mathbf{u}, \pi$ be the solution of the stabilized system. Further let $\mathbf{e} := \mathbf{u} - \mathbf{v}$ and $\eta := \pi - \rho$ denote the error. Then it is important to know how $\|\mathbf{e}\|_X$ depends on ε for some Bochner space X. In the context of the Navier–Stokes equations (2D and 3D) this question has been answered: The error due to the stabilization is of order ε (for special choices of X). Certainly the norm used for measurement depends on the smoothness of the data (and the domain).

Since we are studying this stabilization with the purpose of future practical application to numerics, we will have some restrictions on the norms $\|\cdot\|_X$ used. Especially we are interested in lower order elements for the space discretization. In such a setting it is only meaningful to measure the error in norms which are related to the natural energy norms. This is due to the fact that lower order elements do not allow $-\Delta\mathbf{e}$ as a meaningful test function in the derivation of error estimates. This will restrict us to the following expressions measuring the error:

$$\|\mathbf{e}\|_{L^\infty(I,L^2(\Omega))} \quad \text{and} \quad \Big(\int_0^T \langle \boldsymbol{\sigma}(\mathbf{Du}, \mathbf{Dv})\mathbf{De}, \mathbf{De}\rangle \, dt\Big)^{\frac{1}{2}}.$$

In the Navier–Stokes case this corresponds to

$$\|\mathbf{e}\|_{L^\infty(I,L^2(\Omega))} \quad \text{and} \quad \|\boldsymbol{\nabla}\mathbf{e}\|_{L^2(I,L^2(\Omega))}.$$

So for example A. Prohl proved in [**Pro97**] for the Navier–Stokes equations that the error due to pressure stabilization in combination with a time discretization (Van–Kahn, Chorin and Chorin–Uzawa) satisfies

$$\begin{aligned} \|\mathbf{e}\|_{l^\infty(I,L^2(\Omega))} &\leq C\,(\varepsilon + k), \\ \|\boldsymbol{\nabla}\mathbf{e}\|_{l^2(I,L^2(\Omega))} &\leq C\,(\varepsilon^{\frac{1}{2}} + k), \end{aligned}$$

where $l^2(I)$, resp. $l^\infty(I)$, is the discretized version of $L^2(I)$, resp. $L^\infty(I)$ with time steps of size k. Note that the problems of time discretization and pressure stabilization decouple in the analysis of the error. The point is that the order of the error with respect to ε will immediately transfer to the time discretization. Therefore it suffices to examine the the error due to the pressure stabilization without any discretization in time or space. The calculations of A. Prohl are therefore based on the following estimates (for the time continuous problem)

$$\begin{aligned} \|\mathbf{e}\|_{L^\infty(I,L^2(\Omega))} &\leq C\,\varepsilon, \\ \|\boldsymbol{\nabla}\mathbf{e}\|_{L^2(I,L^2(\Omega))} &\leq C\,\varepsilon^{\frac{1}{2}}. \end{aligned} \tag{4.113}$$

Note that it is possible for the Navier–Stokes equation to improve the $\varepsilon^{\frac{1}{2}}$ in the second estimate to ε but this needs higher regularity of $\mathbf{u}$, especially information about $\boldsymbol{\nabla}^3\mathbf{u}$. We certainly need smooth data to obtain this. But in our case, where the potential depends on an exponent p, the situation is worse. We would need second derivatives of Φ with respect to x, which would require higher smoothness of p than $W^{1,\infty}(I \times \Omega)$. Since we definitely want to restrict ourselves to the case $p \in W^{1,\infty}(I \times \Omega)$, we are restricted to information on $\boldsymbol{\nabla}^2\mathbf{u}$. Note that in this restricted setting the estimates (4.113) are optimal for the Navier–Stokes equation (2D and 3D). The aim of this section is to prove that in $2D$ these optimal estimates also hold true for the generalized case of a p–potential. In order to prove this result we have to make one more assumption on the p–potential, which is satisfied for our example potentials of section 3 of chapter 3 as long as $p \in W^{1,\infty}(I \times \Omega)$.

ASSUMPTION 4.12. *Let F be the p–potential used, then we assume that there exists a constant $C > 0$, such that for all $t \in I$, $x \in \Omega$ and $R \geq 0$*

$$F'''(t, x, R) \leq C. \tag{4.114}$$

Under this assumption there holds

THEOREM 4.13. *Let p, $\mathbf{S}$, $\mathbf{f}$, $\mathbf{u}_0$ be as in theorem 4.10, let assumption 4.12 hold, and let $\mathbf{v}, \rho$ be the solution of (4.1) (in the sense of theorem 4.10). Then for all solutions $\mathbf{u}, \pi$ of (4.2) (in the sense of theorem 4.10), resp. (4.3), with $0 < \varepsilon \leq \varepsilon_0$ there holds*

$$\|\mathbf{e}\|_{L^\infty(I,L^2(\Omega))} \leq C\,\varepsilon, \tag{4.115}$$

$$\|\boldsymbol{\nabla}\mathbf{e}\|_{L^2(I,L^2(\Omega))} \leq C\,\varepsilon^{\frac{1}{2}}, \tag{4.116}$$

where $\mathbf{e} = \mathbf{u} - \mathbf{v}$ and the constants C do not depend on ε.

Before proving this theorem we will provide a weaker result.

LEMMA 4.14. *Under the conditions of theorem 4.13 there holds*

$$\|\mathbf{e}\|_{L^\infty(I,L^2(\Omega))} \leq C\,\varepsilon^{\frac{1}{2}}, \tag{4.117}$$

$$\|\boldsymbol{\nabla}\mathbf{e}\|_{L^2(I,L^2(\Omega))} \leq C\,\varepsilon^{\frac{1}{2}}, \tag{4.118}$$

for all $0 < \varepsilon \leq \varepsilon_0$.

PROOF. Since $\mathbf{v}, \rho$ is the solution of (4.1) and $\mathbf{u}, \pi$ is the solution of (4.2), resp. (4.3), the error $\mathbf{e} = \mathbf{u} - \mathbf{v}$, $\eta = \pi - \rho$ solves

$$\partial_t \mathbf{e} - \operatorname{div}(\boldsymbol{\sigma}(\mathbf{Du}, \mathbf{Dv})\mathbf{De}) + \boldsymbol{\nabla}\eta = \mathbf{h}, \tag{4.119}$$

$$\operatorname{div}\mathbf{e} - \varepsilon\Delta\eta = \varepsilon\Delta\rho, \tag{4.120}$$

$$\mathbf{e}(0) = 0, \tag{4.121}$$

where

$$\mathbf{h} = (\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{u} + \tfrac{1}{2}(\operatorname{div}\mathbf{u})\mathbf{u} - (\mathbf{v} \cdot \boldsymbol{\nabla})\mathbf{v} - \tfrac{1}{2}(\operatorname{div}\mathbf{v})\mathbf{v},$$

resp.

$$\mathbf{h} = \big((\mathbf{u} - \varepsilon\boldsymbol{\nabla}\pi) \cdot \boldsymbol{\nabla}\big)\mathbf{u} - \big(\mathbf{v} \cdot \boldsymbol{\nabla}\big)\mathbf{v}.$$

Using $\mathbf{e}$ as a test function for (4.119) and η as a test function for (4.119) and summing up implies

$$\tfrac{1}{2}d_t\|\mathbf{e}\|_2^2 + \big\langle \boldsymbol{\sigma}(\mathbf{Du}, \mathbf{Dv})\mathbf{De}, \mathbf{De}\big\rangle + \varepsilon\|\boldsymbol{\nabla}\eta\|_2^2 = \langle \mathbf{h}, \mathbf{e}\rangle - \varepsilon\langle \boldsymbol{\nabla}\rho, \boldsymbol{\nabla}\eta\rangle.$$

As in lemma 4.11 this implies

$$\begin{aligned}\tfrac{1}{2}d_t\|\mathbf{e}\|_2^2 + \big\langle \boldsymbol{\sigma}(\mathbf{Du}, \mathbf{Dv})\mathbf{De}, \mathbf{De}\big\rangle + \varepsilon\|\boldsymbol{\nabla}\eta\|_2^2 \\ \leq C\,\|\mathbf{u}\|_{C^{1,\alpha}(I\times\Omega)}\|\mathbf{e}\|_2^2 + \varepsilon\langle \boldsymbol{\nabla}\rho, \boldsymbol{\nabla}\eta\rangle.\end{aligned}$$

Hence by $\|\mathbf{u}\|_{C^{1,\alpha}(I\times\Omega)} \le C$ and Young's inequality

$$\tfrac{1}{2}d_t\|\mathbf{e}\|_2^2 + \big\langle \boldsymbol{\sigma}(\mathbf{Du},\mathbf{Dv})\mathbf{De},\mathbf{De}\big\rangle + \tfrac{\varepsilon}{2}\|\boldsymbol{\nabla}\eta\|_2^2 \le C\,\|\mathbf{u}\|_{C^{1,\alpha}(I\times\Omega)}\|\mathbf{e}\|_2^2 + \tfrac{\varepsilon}{2}\|\boldsymbol{\nabla}\rho\|_2^2.$$

Since $\|\boldsymbol{\nabla}\rho\|_{L^2(I,L^2(\Omega))} \le C$ by theorem 4.10, an application of Gronwall's lemma gives

$$\|\mathbf{e}\|^2_{L^\infty(I,L^2(\Omega))} + \|\boldsymbol{\nabla}\mathbf{e}\|^2_{L^2(I,L^2(\Omega))} + \varepsilon\|\boldsymbol{\nabla}\eta\|^2_{L^2(I,L^2(\Omega))} \le C\,\varepsilon.$$

This proves the lemma. □

This lemma gives a first estimate of the error with respect to a power of ε. Its proof is simply based on the energy estimate of the equation of the error. In order to proof the optimal result, namely theorem 4.13, we need a more subtle method, which is based on the dual problem of the error equation. But this method is only successful if the dual problem has a strong solution. We therefore need the following result:

Lemma 4.15. *Let p, $\mathbf{S}$, $\mathbf{f}$, $\mathbf{u}_0$ be as in theorem 4.10 and let $\mathbf{v},\rho$ be the solution of (4.1) and $\mathbf{u},\pi$ be the solution of (4.2), resp. (4.3), for some $0<\varepsilon\le\varepsilon_0$. Then the backward problem*

$$\begin{aligned} \partial_t\mathbf{w} + \operatorname{div}(\boldsymbol{\sigma}(\mathbf{Du},\mathbf{Dv})\mathbf{Dw}) + \boldsymbol{\nabla}\xi + \mathbf{H}(\mathbf{w}) &= \mathbf{g} \quad \text{on } I\times\Omega,\\ \operatorname{div}\mathbf{w} &= 0 \quad \text{on } I\times\Omega,\\ \mathbf{w}(T) &= 0, \end{aligned} \tag{4.122}$$

where

$$\mathbf{H}(\mathbf{w}) = (\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{w} - \mathbf{w}(\boldsymbol{\nabla}\mathbf{u}) - \tfrac{1}{2}(\operatorname{div}\mathbf{u})\mathbf{w} + \tfrac{1}{2}\boldsymbol{\nabla}(\mathbf{u}\cdot\mathbf{w}), \tag{4.123}$$

resp.

$$\mathbf{H}(\mathbf{w}) = ((\mathbf{u}-\varepsilon\boldsymbol{\nabla}\pi)\cdot\boldsymbol{\nabla})\mathbf{w} - \mathbf{w}(\boldsymbol{\nabla}(\mathbf{u}-\varepsilon\boldsymbol{\nabla}\pi)), \tag{4.124}$$

and $(\mathbf{w}(\boldsymbol{\nabla}\mathbf{u}))_j = \sum_{k=1}^2 w_k\partial_j u_k$ and $\mathbf{g}\in L^2(I,L^2_0(\Omega))$ has a weak solution $\mathbf{w},\xi$. If additionally $\|\mathbf{g}\|_{C^{0,\alpha}(I\times\Omega)} \le K$, then

$$\|\mathbf{w}\|_{C^{1,\beta}(I\times\Omega)} \le C, \tag{4.125}$$

$$\|\mathbf{w}\|_{L^2(I,W_0^{2,2}(\Omega))} \le C\|\mathbf{g}\|_{L^2(I,L^2(\Omega))}, \tag{4.126}$$

$$\|\boldsymbol{\nabla}\xi\|_{L^2(I,L^2_0(\Omega))} \le C\|\mathbf{g}\|_{L^2(I,L^2_0(\Omega))} \tag{4.127}$$

for some $\beta>0$. The constants and β can be chosen independently of $0\le\varepsilon\le\varepsilon_0$.

Proof. From theorem 4.10 we know that

$$\begin{aligned} \|\mathbf{u}\|_{C^{1,\alpha}(I\times\Omega)} + \|\mathbf{u}\|_{L^\infty(I,W^{2,q}(\Omega))} &\le C,\\ \|\pi\|_{L^\infty(I,W^{1,2}(\Omega))} &\le C,\\ \|\mathbf{v}\|_{C^{1,\alpha}(I\times\Omega)} + \|\mathbf{v}\|_{L^\infty(I,W^{2,q}(\Omega))} &\le C,\\ \|\rho\|_{L^\infty(I,W^{1,2}(\Omega))} &\le C, \end{aligned}$$

where the constants do not depend on ε. This implies the existence of $\mu_1,\mu_2>0$ independent of ε with $0\le\varepsilon\le\varepsilon_0$, such that

$$\mu_1|\mathbf{B}^{\text{sym}}|^2 \le \sum_{jklm}\sigma_{jk,lm}(\mathbf{Du},\mathbf{Dv})B_{jk}B_{lm} \le \mu_2|\mathbf{B}^{\text{sym}}|^2. \tag{4.128}$$

From (4.128) and $\|\mathbf{u}\|_{C^{1,\alpha}(I\times\Omega)} \le C$ we deduce (note the linearity of the system) that there exists a unique weak solution, such that

$$\|\mathbf{w}\|_{L^2(I,W_0^{1,2}(\Omega))} \le C,$$
$$\|\mathbf{w}\|_{L^\infty(I,L_0^2(\Omega))} \le C,$$

with constants independent of $0 \le \varepsilon \le \varepsilon_0$. Further, from the continuity of $\boldsymbol{\nabla}^2_{n\times n}\Phi$ and the Hölder continuity of $\boldsymbol{\nabla}\mathbf{u}$ and $\boldsymbol{\lambda}\mathbf{v}$ we deduce that $\boldsymbol{\sigma}$ is Hölder continuous, i.e. in $C^{0,\alpha}(I\times\Omega)$, where the Hölder constant does not depend on ε. Hence the dual problem can be seen as a generalized, instationary Stokes problem with a linear perturbation $\mathbf{H}(\mathbf{w})$. If $\mathbf{u},\pi$ is a solution of (4.2), i.e. $\mathbf{H}$ has the form (4.123), then we see from $\|\mathbf{u}\|_{C^{1,\alpha}(I\times\Omega)} \le C$ that

$$|\mathbf{H}(\mathbf{w})| \le C\,(|\boldsymbol{\nabla}\mathbf{w}| + |\mathbf{w}|) \qquad \text{on } \Omega.$$

If $\mathbf{u},\pi$ is solution of (4.3) (with $\varepsilon > 0$), i.e. $\mathbf{H}$ has the form (4.124), then $\varepsilon\boldsymbol{\nabla}\pi$ solves

$$\varepsilon\boldsymbol{\nabla}\pi = \boldsymbol{\nabla}\Delta^{-1}\operatorname{div}\mathbf{u}.$$

Thus $\|\mathbf{u}\|_{C^{1,\alpha}(I\times\Omega)} \le C$ implies $\|\varepsilon\boldsymbol{\nabla}\pi\|_{C^{1,\alpha}(I\times\Omega)} \le C$. Therefore also in this case

$$|\mathbf{H}(\mathbf{w})| \le C\,(|\boldsymbol{\nabla}\mathbf{w}| + |\mathbf{w}|) \qquad \text{on } \Omega.$$

Overall we see that independently of the dicretization of the convective term (see system (4.2) and (4.3)) $\mathbf{H}(\mathbf{w})$ is a linear perturbation with

$$|\mathbf{H}(\mathbf{w})| \le C\,(|\boldsymbol{\nabla}\mathbf{w}| + |\mathbf{w}|).$$

So we have to deal with a parabolic system in divergence form with Hölder continuous, uniformly elliptic coefficients $\boldsymbol{\sigma}(\mathbf{Du},\mathbf{Dv})$ with pressure and a weak linear perturbation $\mathbf{H}(\mathbf{w})$. As a result there exists a unique weak solution $\mathbf{w},\xi$ of (4.122). But since $\|\mathbf{f}\|_{C^{0,\alpha}}(I\times\Omega) \le C$, we can deduce more: The general theory about such systems implies that $\mathbf{w}$ has also Hölder continuous derivatives $\boldsymbol{\nabla}\mathbf{w}$. (This result is proven as follows: Freeze the coefficients locally to deduce the desired Hölder continuity of $\boldsymbol{\nabla}\overline{\mathbf{w}}$, where $\overline{\mathbf{w}}$ is the solution of the "frozen" system. This implies the existence of estimates of $\boldsymbol{\nabla}\overline{\mathbf{w}}$ in terms of parabolic Campanato spaces. Since the coefficients $\boldsymbol{\sigma}(\mathbf{Du},\mathbf{Dv})$ are Hölder continuous, these estimates can by comparison with the unfrozen system be transferred from $\boldsymbol{\nabla}\overline{\mathbf{w}}$ to $\boldsymbol{\nabla}\mathbf{w}$. This in turn implies the $C^{0,\beta}(I\times\Omega)$ regularity of $\boldsymbol{\nabla}\mathbf{w}$. In all these steps the pressure is handled by projecting the localized test functions via the theorem of Bogovskiĭ to the space of divergence free functions, while preserving the norms. We refer to W. Schlag [**Sch96**] who adapted Campanato's technique from elliptic systems to parabolic systems and to M. Giaquinta and G. Modica [**GM82**] who transfered Campanato's technique to the stationary Stokes system.) Thus we have

$$\|\mathbf{w}\|_{C^{1,\beta}(I\times\Omega)} \le C.$$

for some $\beta > 0$ independently of ε.

It remains to prove (4.126) and (4.127). We will do so by choosing the test function $\Delta\mathbf{w}$. The calculations are only "formal", but since the system is linear in $\mathbf{w}$ the calculations are easily justified by mollifying the equation and the test function $\Delta\mathbf{w}$ beforehand by some (Friedrich's) mollifier φ_δ and passing $\delta \to 0^+$. The most interesting term when testing with $\Delta\mathbf{w}$ is obviously the main part, i.e.

$$\begin{aligned}
&\big\langle \operatorname{div}(\boldsymbol{\sigma}(\mathbf{Du}, \mathbf{Dv})\mathbf{Dw}), \Delta\mathbf{w}\big\rangle \\
&\quad = \sum_{jklmn} \big\langle \partial_j(\sigma_{jk,lm}(\mathbf{Du}, \mathbf{Dv}) D_{lm}\mathbf{w}), \partial_n^2 w_k\big\rangle \\
&\quad = \sum_{jklmn} \big\langle \partial_n(\sigma_{jk,lm}(\mathbf{Du}, \mathbf{Dv}) D_{lm}\mathbf{w}), \partial_n\partial_j w_k\big\rangle \\
&\quad = \sum_{jklmn} \big\langle (\partial_n\sigma_{jk,lm})(\mathbf{Du}, \mathbf{Dv}) D_{lm}\mathbf{w}, D_{jk}(\partial_n\mathbf{w})\big\rangle \\
&\qquad + \sum_{jklmnrs} \big\langle (\partial_{rs}\sigma_{jk,lm})(\mathbf{Du}, \mathbf{Dv})(\partial_n D_{rs}\mathbf{e}) D_{lm}\mathbf{w}, D_{jk}(\partial_n\mathbf{w})\big\rangle \\
&\qquad + \sum_{jklmn} \big\langle \sigma_{jk,lm}(\mathbf{Du}, \mathbf{Dv}) D_{lm}(\partial_n\mathbf{w}), D_{jk}(\partial_n\mathbf{w})\big\rangle \\
&\quad =: K_1 + K_2 + K_3.
\end{aligned}$$

From assumption 4.12, inequality (3.28), the representation of $\boldsymbol{\nabla}^2_{n\times n}\Phi$ of remark 3.2, the identity

$$\begin{aligned}
(\partial_r\sigma_{jk,lm})(\mathbf{Du}, \mathbf{Dv}) &= \int_0^1 (\partial_r\partial_{jk}\partial_{lm}\Phi)([\mathbf{C}, \mathbf{B}]_z)\, dz, \\
(\partial_{rs}\sigma_{jk,lm})(\mathbf{Du}, \mathbf{Dv}) &= \int_0^1 (\partial_{rs}\partial_{jk}\partial_{lm}\Phi)([\mathbf{C}, \mathbf{B}]_z)\, dz,
\end{aligned}$$

and $\mathbf{u}, \mathbf{v} \in C^{1,\alpha}(I \times \Omega)$ we deduce that

$$\begin{aligned}
|(\partial_r\sigma_{jk,lm})(\mathbf{Du}, \mathbf{Dv})| &\leq C, \\
\big|(\partial_{rs}\sigma_{jk,lm})(\mathbf{Du}, \mathbf{Dv})\big| &\leq C.
\end{aligned}$$

Therefore we can estimate K_1 and K_2 by

$$\begin{aligned}
K_1 &\leq C\, \|\boldsymbol{\nabla}\mathbf{w}\|_2 \|\boldsymbol{\nabla}^2\mathbf{w}\|_2 \\
&\leq C_\delta\, \|\boldsymbol{\nabla}\mathbf{w}\|_2^2 + \delta \|\boldsymbol{\nabla}^2\mathbf{w}\|_2^2. \\
K_2 &\leq C\, \|\boldsymbol{\nabla}\mathbf{w}\|_{C^{0,\beta}(I\times\Omega)} \|\boldsymbol{\nabla}^2\mathbf{e}\|_2 \|\boldsymbol{\nabla}^2\mathbf{w}\|_2 \\
&\leq C\, \|\boldsymbol{\nabla}^2\mathbf{e}\|_2 \|\boldsymbol{\nabla}^2\mathbf{w}\|_2 \\
&\leq C_\delta\, \|\boldsymbol{\nabla}^2\mathbf{e}\|_2^2 + \delta \|\boldsymbol{\nabla}^2\mathbf{w}\|_2^2.
\end{aligned}$$

Further (4.128) and lemma 8.5 imply

$$K_3 \geq \mu_1 \sum_r \|\mathbf{D}(\partial_r\mathbf{w})\|_2^2 \geq \frac{\mu_1}{4} \|\boldsymbol{\nabla}^2\mathbf{w}\|_2^2.$$

Thus the terms K_1 and K_2 are of lower order compared to the information we get from K_3. Overall if we test the system (4.122) with $\Delta\mathbf{w}$ and integrate over the time, we get

$$\|\mathbf{w}\|^2_{L^\infty(I,W_0^{1,2}(\Omega))} + \|\mathbf{w}\|^2_{L^2(I,W_0^{2,2}(\Omega))} \le C\|\mathbf{g}\|^2_{L^2(I,L_0^2(\Omega))}.$$

This estimate implies that all terms in system (4.122) but the pressure term $\boldsymbol{\nabla}\pi$ are in $L^2(I, L^2(\Omega))$ with norm bounded by $C\,\|\mathbf{g}\|_{L^2(I,L_0^2(\Omega))}$. So the norm estimate of De Rahm implies that

$$\|\boldsymbol{\nabla}\xi\|_{L^2(I,L_0^2(\Omega))} \le C\|\mathbf{g}\|_{L^2(I,L^2(\Omega))}.$$

This completes the proof. □

Let us remark that it is also possible to derive (4.126) and (4.127) "directly" (i.e. without the use of Campanato spaces) by using theorem 1.1 of [**Sol00**]. Therein, V.A. Solonnikov considers the instationary, generalized Stokes system in the half space

$$\begin{aligned} \partial_t + \mathcal{A}_0\big(\tfrac{\partial}{\partial x}\big)\mathbf{u} + \boldsymbol{\nabla}\pi &= \mathbf{f},\\ \operatorname{div}\mathbf{u} &= 0, \end{aligned}$$

where $\mathcal{A}_0\big(\frac{\partial}{\partial x}\big)$ is a matrix–valued elliptic homogenuous second–order differential operator with real coefficients.

Let us get back to the proof of theorem 4.13.

PROOF OF THEOREM 4.13. Due to lemma 4.14 it remains to prove 4.115. Let $\mathbf{w}$ be the solution of the dual problem in lemma 4.15, then there holds

$$\begin{aligned} \partial_t\mathbf{w} + \operatorname{div}(\boldsymbol{\sigma}(\mathbf{Du},\mathbf{Dv})\mathbf{Dw}) + \boldsymbol{\nabla}\xi + \mathbf{H}(\mathbf{w}) &= \mathbf{g},\\ \operatorname{div}\mathbf{w} &= 0,\\ \mathbf{w}(T) &= 0, \end{aligned} \tag{4.129}$$

where

$$\mathbf{H}(\mathbf{w}) = (\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{w} - \mathbf{w}(\boldsymbol{\nabla}\mathbf{u}) - \tfrac{1}{2}(\operatorname{div}\mathbf{u})\mathbf{w} + \tfrac{1}{2}\boldsymbol{\nabla}(\mathbf{u}\cdot\mathbf{w}),$$

resp.

$$\mathbf{H}(\mathbf{w}) = ((\mathbf{u}-\varepsilon\boldsymbol{\nabla}\pi)\cdot\boldsymbol{\nabla})\mathbf{w} - \mathbf{w}(\boldsymbol{\nabla}(\mathbf{u}-\varepsilon\boldsymbol{\nabla}\pi)).$$

Further the error $\mathbf{e}, \eta$ solve

$$\begin{aligned} \partial_t\mathbf{e} - \operatorname{div}(\boldsymbol{\sigma}(\mathbf{Du},\mathbf{Dv})\mathbf{De}) + \boldsymbol{\nabla}\eta - \mathbf{h} &= \mathbf{0},\\ \operatorname{div}\mathbf{e} - \varepsilon\Delta\eta &= -\varepsilon\Delta\rho,\\ \mathbf{e}(0) &= 0, \end{aligned} \tag{4.130}$$

where

$$\mathbf{h} = (\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u} + \tfrac{1}{2}(\operatorname{div}\mathbf{u})\mathbf{u} - (\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v} - \tfrac{1}{2}(\operatorname{div}\mathbf{v})\mathbf{v},$$

resp.

$$\mathbf{h} = \big((\mathbf{u}-\varepsilon\boldsymbol{\nabla}\pi)\cdot\boldsymbol{\nabla}\big)\mathbf{u} - \big((\mathbf{v}-\varepsilon\boldsymbol{\nabla}\rho)\cdot\boldsymbol{\nabla}\big)\mathbf{v}.$$

In the next step we want to use $\mathbf{w}$ as a test function for system (4.130) and $\mathbf{e}$ as a test function for system (4.129). The results will be added and integrated over time. Now it becomes clear why the dual problem has the special form as chosen: Most of the terms will be cancelled out after integration over space and time. For example $\mathbf{H}(\mathbf{w})$ is chosen such that

$$\langle \mathbf{h}, \mathbf{w} \rangle + \langle \mathbf{H}(\mathbf{w}), \mathbf{e} \rangle = 0,$$

resp.

$$\langle \mathbf{h}, \mathbf{w} \rangle + \langle \mathbf{H}(\mathbf{w}), \mathbf{e} \rangle = -\varepsilon (\boldsymbol{\nabla} \pi \cdot \boldsymbol{\nabla}) \mathbf{u}.$$

So $\langle \mathbf{h}, \mathbf{w} \rangle + \langle \mathbf{H}(\mathbf{w}), \mathbf{e} \rangle$ is either zero or of order ε. Additionally we have

$$\big\langle \boldsymbol{\sigma}(\mathbf{Du}, \mathbf{Dv}) \mathbf{De}, \mathbf{Dw} \big\rangle - \big\langle \boldsymbol{\sigma}(\mathbf{Du}, \mathbf{Dv}) \mathbf{Dw}, \mathbf{De} \big\rangle = 0.$$

Thus we get (if $\mathbf{u}, \pi$ is a solution of (4.2))

$$\begin{aligned}
\langle \mathbf{g}, \mathbf{e} \rangle &= \big\langle \text{l.h.s. of system (4.130)}, \mathbf{w} \big\rangle + \big\langle \text{l.h.s. of system (4.129)}, \mathbf{e} \big\rangle \\
&= \langle \partial_t \mathbf{e}, \mathbf{w} \rangle + \langle \partial_t \mathbf{w}, \mathbf{e} \rangle \\
&\quad + \big\langle \boldsymbol{\sigma}(\mathbf{Du}, \mathbf{Dv}) \mathbf{De}, \mathbf{Dw} \big\rangle - \big\langle \boldsymbol{\sigma}(\mathbf{Du}, \mathbf{Dv}) \mathbf{Dw}, \mathbf{De} \big\rangle \\
&\quad + \langle \boldsymbol{\nabla} \eta, \mathbf{w} \rangle + \langle \boldsymbol{\nabla} \xi, \mathbf{e} \rangle \\
&\quad + \langle \mathbf{h}, \mathbf{w} \rangle + \langle \mathbf{H}(\mathbf{w}), \mathbf{e} \rangle \\
&= \langle \partial_t \mathbf{e}, \mathbf{w} \rangle + \langle \partial_t \mathbf{w}, \mathbf{e} \rangle + \langle \boldsymbol{\nabla} \xi, \mathbf{e} \rangle \\
&= \partial_t \langle \mathbf{e}, \mathbf{w} \rangle - \langle \xi, \operatorname{div} \mathbf{e} \rangle \\
&= \partial_t \langle \mathbf{e}, \mathbf{w} \rangle - \langle \xi, \varepsilon \Delta \pi \rangle \\
&= \partial_t \langle \mathbf{e}, \mathbf{w} \rangle + \varepsilon \langle \boldsymbol{\nabla} \xi, \boldsymbol{\nabla} \pi \rangle.
\end{aligned}$$

If resp. $\mathbf{u}, \pi$ is a solution of (4.3), we get

$$\langle \mathbf{g}, \mathbf{e} \rangle = \partial_t \langle \mathbf{e}, \mathbf{w} \rangle + \varepsilon \langle \boldsymbol{\nabla} \xi, \boldsymbol{\nabla} \pi \rangle - \varepsilon \langle (\boldsymbol{\nabla} \pi \cdot \boldsymbol{\nabla}) \mathbf{u}, \mathbf{w} \rangle.$$

Integration over time implies

$$\int_I \langle \mathbf{g}, \mathbf{e} \rangle \, dt = \mathbf{e}(T) \underbrace{\mathbf{w}(T)}_{=0} - \underbrace{\mathbf{e}(0)}_{=0} \mathbf{w}(0) + \varepsilon \int_I \langle \boldsymbol{\nabla} \xi, \boldsymbol{\nabla} \pi \rangle \, dt,$$

resp.

$$\int_I \langle \mathbf{g}, \mathbf{e} \rangle \, dt = \varepsilon \int_I \langle \boldsymbol{\nabla} \xi, \boldsymbol{\nabla} \pi \rangle \, dt - \varepsilon \int_I \langle (\boldsymbol{\nabla} \pi \cdot \boldsymbol{\nabla}) \mathbf{u}, \mathbf{w} \rangle \, dt$$

Thus if $\mathbf{u}, \pi$ is a solution of (4.2), then

$$\begin{aligned}
\int_I \langle \mathbf{g}, \mathbf{e} \rangle \, dt &\leq \varepsilon \Big| \int_I \langle \boldsymbol{\nabla} \xi, \boldsymbol{\nabla} \pi \rangle \, dt \Big| \\
&\leq \varepsilon \| \boldsymbol{\nabla} \xi \|_{L^2(I, L^2_0(\Omega))} \| \boldsymbol{\nabla} \pi \|_{L^2(I, L^2_0(\Omega))}.
\end{aligned}$$

The regularity of π, i.e. $\| \boldsymbol{\nabla} \pi \|_{L^2(I, L^2_0(\Omega))} \leq C$, and lemma 4.15 imply

$$\int_I \langle \mathbf{g}, \mathbf{e} \rangle \, dt \leq \varepsilon \, C \, \| \mathbf{g} \|_{L^2(I, L^2_0(\Omega))}. \tag{4.131}$$

If resp. $\mathbf{u}, \pi$ is a solution of (4.3), we get

$$\int_I \langle \mathbf{g}, \mathbf{e} \rangle \, dt \le \varepsilon\, C\, \|\boldsymbol{\nabla} \xi\|_{L^2(I, L^2_0(\Omega))} \\ + \varepsilon\, C\, \|\boldsymbol{\nabla} \pi\|_{L^2(I, L^2_0(\Omega))} \|\boldsymbol{\nabla} \mathbf{u}\|_{C^{0,\alpha}(I\times\Omega)} \|\mathbf{w}\|_{L^2(I, L^2(\Omega))}.$$

The regularity of $\mathbf{u}$ and π and lemma 4.15 also imply in this case

$$\int_I \langle \mathbf{g}, \mathbf{e} \rangle \, dt \le \varepsilon\, C\, \|\mathbf{g}\|_{L^2(I, L^2_0(\Omega))}.$$

Up to now $\mathbf{g}$ has been an arbitrary function from $L^2(I, L^2_0(\Omega))$ with $\|\mathbf{g}\|_{C^{0,\alpha}(I\times\Omega)} \le C$. We will now fix $\mathbf{g} := \mathbf{e}$ to gain information from (4.131). This choice is possible since $\|\mathbf{u}\|_{C^{0,\alpha}(I\times\Omega)} + \|\mathbf{v}\|_{C^{0,\alpha}(I\times\Omega)} \le C$ independently of $0 \le \varepsilon \le \varepsilon_0$. Thus

$$\|\mathbf{e}\|^2_{L^2(I, L^2_0(\Omega))} \le \varepsilon\, C\, \|\mathbf{e}\|_{L^2(I, L^2_0(\Omega))}.$$

So

$$\|\mathbf{e}\|_{L^2(I, L^2_0(\Omega))} \le \varepsilon\, C.$$

This proves the theorem. □

Let us make a final remark on the three dimensional case: We have seen in theorem 4.13 that for two space dimensions we get optimal error estimates in terms of ε. These optimal estimates have been derived via the existence of a strong solution to the suitable dual problem. While it is not difficult to prove that the dual problem has a weak solution in the 3D-case, we do not know how to prove the existence of strong solutions. The problem is that when using $\Delta\mathbf{w}$ as a test function, we have to handle terms of the form

$$\int_\Omega \int_0^1 |\boldsymbol{\nabla}^3 \Phi([\mathbf{Du}, \mathbf{Dv}]_z)| |\mathbf{Dw}| |\boldsymbol{\nabla}^2 \mathbf{e}| |\boldsymbol{\nabla}^2 \mathbf{w}| \, dz\, dx.$$

There seems to be no way to control this term without knowledge of the boundedness either of $\mathbf{Dw}$ or of $\boldsymbol{\nabla}^2\mathbf{e}$. But neither of them is at hand in the three dimensional case. In the 2D-case we have used the Hölder continuity of $\mathbf{Du}$ and $\mathbf{Dv}$ to derive Hölder continuity of $\mathbf{Dw}$ via the theory of parabolic systems in divergence form with Hölder continuous coefficients. But since there exists no $C^{1,\alpha}(I \times \Omega)$ theory in the 3D-case we cannot argument in the same way. Therefore we do not see a way to prove the existence of strong solutions for the dual problem in the 3D-case.

Nevertheless at least some error estimates in terms of ε also hold true in the 3D-case. Following on the line of lemma 4.14 (we rather absorb $\mathbf{e}$ into $\langle \boldsymbol{\sigma}(\mathbf{Du}, \mathbf{Dv})\mathbf{De}, \mathbf{De}\rangle$ to avoid the $C^{1,\alpha}(I \times \Omega)$ norm for $\mathbf{u}$), it is easily possible to transfer the results of lemma 4.14 to the 3D-case. This provides the following estimates

$$\|\mathbf{e}\|_{L^\infty(I, L^2(\Omega))} \le C\, \varepsilon^{\frac{1}{2}}, \tag{4.132}$$

$$\|\boldsymbol{\nabla}\mathbf{e}\|_{L^2(I, L^2(\Omega))} \le C\, \varepsilon^{\frac{1}{2}}. \tag{4.133}$$

These estimates are not optimal as in the $2D$-case but still give some control on the error. This at least shows that the pressure stabilization is a possible tool also for three space dimensions.

CHAPTER 5

3D Flow

1. Introduction

In this chapter we will study the instationary p-Navier–Stokes problem in three space dimensions, i.e.

$$\begin{aligned} \partial_t \mathbf{u} - \operatorname{div}(\mathbf{S}(\mathbf{D}\mathbf{u})) + (\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u} + \boldsymbol{\nabla}\pi &= \mathbf{f}, \quad \text{on } I\times\Omega, \\ \operatorname{div}\mathbf{u} &= 0, \quad \text{on } I\times\Omega, \\ \mathbf{u}(0) &= \mathbf{u}_0 \quad \text{on } \Omega. \end{aligned} \tag{5.1}$$

We will use the same notation as in chapter 4, except that Ω denotes the three dimensional torus. We assume that the extra stress $\mathbf{S}$ is induced by a time and space dependent p–potential F and Φ, as we have defined in chapter 3. Further we assume that the exponent p is uniformly Lipschitz continuous, i.e. $p \in W^{1,\infty}(I\times\Omega)$. As in chapter 4, we compensate the missing boundary conditions by restricting the solutions to ones with mean value zero. This ensures that the Poincaré inequality remains valid.

Again we are looking for strong solutions, but other than in the two–dimensional case we will consider only small times, i.e. T is sufficiently small. Note that even in the Navier–Stokes case, i.e. $-\operatorname{div}(\mathbf{S}(\mathbf{D}\mathbf{u}))$ is replaced by $-\Delta\mathbf{u}$, it is not known if for smooth but large data there exists a strong solution for large times. This is even one of the yet unsolved and high priced *Millennium Problems* (Refer to Clay Mathematics Institute).

If $p_\infty > 2$, excluding the classical Navier–Stokes problem, the situation is a little different. So far it has been proven by J. Málek, J. Nečas, M. Rokyta, and M. Růžička in [**MNRR96**] that for p constant with $p > \frac{11}{5}$ there exists a unique strong solution of system (5.1), where Ω denotes the three dimensional torus. This result has been extended by M. Růžička in [**Růž00**] to the case of space and time dependent exponent $p \in C^1(I\times\Omega)$ with

$$\tfrac{9}{4} \le p_\infty \le p_0 \le \tfrac{3(3-p_\infty)}{2(5-2p_\infty)}.$$

Note that M. Růžička examines the case of bounded domains with $C^{3,1}$ boundary. The necessary upper bound of p_0 in terms of p_∞ is due to the not yet fully developed theory of the generalized Sobolev spaces $W^{k,p(\cdot)}(\Omega)$. Some techniques that have been used still require the use of the classical Sobolev spaces, for example embeddings of the type $W^{1,p(\cdot)}(\Omega) \to W^{1,p_\infty}(\Omega)$ reduce information and will later enforce upper bounds for p_0. But it is probable that with a refined theory on generalized Sobolev spaces this additional requirement can be dropped.

In the space periodic case M. Růžička has proven in [**Růž99**] that there exists a strong solution to (5.1) as long as $p \in C^1(I\times\Omega)$ and

$$\tfrac{11}{5} < p_\infty \le p_0 < p_\infty + \tfrac{4}{3}.$$

Again the extra condition $p_0 < p_\infty + 1$ is due to the use of classical Sobolev spaces. This result is quite satisfactory in the sense that it generalizes the lower bound $\frac{11}{5}$ from the case of constant exponent p to the space and time dependent exponent. But unfortunately the lower bound $\frac{11}{5}$ is too big for real fluids.

One way to obtain lower bounds for p_∞ is the use of weak solutions. It has been shown in [**MNRR96**] for p constant, $p > \frac{9}{5}$, and in [**Růž99**] for $p \in C^1(I \times \Omega)$, $\frac{9}{5} < p_\infty \leq p_0 < p_\infty + 1$, that there exists a weak solution of (5.1) for large times. The problem is that it is not known if this solution is unique. Furthermore the weak regularity does not provide a solid basis for the numerical analysis (see chapter 6).

This finally leads us to the search of strong solutions with at least short time existence. Again there is a result to be found in [**MNRR96**]. For p constant with $p > \frac{5}{3}$ Málek, Nečas, Rokyta, and Růžička prove short time existence of strong solutions with

$$\begin{aligned} \|\boldsymbol{\nabla}\mathbf{u}\|_{L^\infty(I,L^2(\Omega))} + \|\partial_t \mathbf{u}\|_{L^2(I,L^2(\Omega))} &\leq C, \\ \|\mathcal{I}_\Phi(\mathbf{u})\|_{L^1(I)} &\leq C. \end{aligned}$$

See also [**PR01**] by Prohl and Růžička, where it has been shown by the same method that this solution further satisfies

$$\|\partial_t \mathbf{u}\|_{L^\infty(I,L^2(\Omega))} + \|\mathcal{J}_\Phi(\mathbf{u})\|_{L^1(I)} \leq C.$$

Due to lemma 3.12 this implies

$$\|(\widetilde{D}\mathbf{u})^{\frac{p}{2}}\|_{L^2(I,W^{1,2}(\Omega))} + \|\partial_t\big((\widetilde{D}\mathbf{u})^{\frac{p}{2}}\big)\|_{L^2(I,L^2(\Omega))} \leq C. \tag{5.2}$$

The theory of traces of parabolic spaces (see also appendix) implies

$$\|(\widetilde{D}\mathbf{u})^{\frac{p}{2}}\|_{C(I,L^3(\Omega))} \leq C\,\|(\widetilde{D}\mathbf{u})^{\frac{p}{2}}\|_{C(I,W^{\frac{1}{2},2}(\Omega))} \leq C. \tag{5.3}$$

In this chapter we will extend the result of Málek, Nečas, Rokyta, and Růžička with respect to several aspects, smoothing the way to numerical analysis of electrorheological fluids. First of all we will extend the result for p constant to the case of a space and time dependent exponent $p \in W^{1,\infty}(\Omega)$. The basic principle of electrorheological fluids lies in the dependence of the extra stress on the electrical field (in the used model via the exponent $p = p(|\mathbf{E}|)$, where $\mathbf{E}$ is the electrical field). Therefore admitting a space and time dependent exponent p is of absolute necessity. Secondly we will lessen the lower bound for p_∞ from $\frac{5}{3}$ to $\frac{7}{5}$ under the condition $p_0 \leq 2$. We will do so by evaluating more a priori estimates than in [**MNRR96**] before passing to the limit of the Galerkin system and by applying a local version of Gronwall's lemma (see lemma 8.7). This new lower bound provides a big step towards the exponents p of real fluids. That is to say from the view of application it is desired to cover a preferable large range of exponents p smaller than 2, for example a physically interesting range is $[1.3, 2]$. The new proven range $[1.4, 2]$ for the exponent p is therefore of strong interest with respect to applications. Last but not least we will improve the regularity of the short time strong solution by the additional estimate

$$\|\mathcal{I}_\Phi(\mathbf{u})\|_{L^{\frac{5p_\infty-6}{2-p_\infty}}(I)}.$$

From this we can deduce by lemma 3.12 and the theory of traces of parabolic spaces (see also appendix), that

$$\|(\widetilde{D}\mathbf{u})^{\frac{p}{2}}\|_{C(I',L^{\frac{12(p_\infty-1)}{p_\infty},\frac{4(p_\infty-1)}{2-p_\infty}})} \leq C \qquad \text{(Lorentz space)},$$

where $L^{q,s}$ denotes the Lorentz space. We further show that within the class of strong solutions, which meet the same regularity as the proven solution, we additionally have uniqueness. This is also a new result and has not been known yet.

2. Special Energies

In chapter 3 section 8 we have derived estimates for $\mathcal{I}_\Phi(\mathbf{u})$ and $\mathcal{J}_\Phi(\mathbf{u})$ for arbitrary dimension. Now in the three dimensional case, we can establish more specialized estimates. For the sake of simplicity we will now state some requirements on p and Φ that will be valid throughout this chapter.

Assumption 5.1. *Let Ω denote the three dimensional torus and $I := [0, T]$ a time interval. We assume that Φ is a space and time dependent p-potential with $p \in W^{1,\infty}(I\times\Omega)$ and $1 < p_\infty \leq p_0 \leq 2$. Note that p will be fixed in our considerations. We will not make use of the constant $\|p\|_{W^{1,\infty}(I\times\Omega)}$ explicitly but rather absorb it into the generic constant C. We do the same with the elliptic constants γ_1 and γ_2. Only when p_∞ is involved, we will state it explicitly.*

Lemma 5.2. *For all (sufficiently smooth) $\mathbf{u}$ there holds*

$$\gamma_1 |\widetilde{D}\mathbf{u}|^{\frac{1}{3}}_{3p(\cdot)} \leq C\left(\mathcal{I}_\Phi(\mathbf{u}) + \int_\Omega |\widetilde{D}\mathbf{u}|^p \ln^2(\widetilde{D}\mathbf{u})\,dx + |\widetilde{D}\mathbf{u}|_{p(\cdot)}\right). \tag{5.4}$$

Proof. By lemma 3.12 and the embedding $W^{1,2}(\Omega) \hookrightarrow L^6(\Omega)$ there holds

$$\begin{aligned}\gamma_1 |\widetilde{D}\mathbf{u}|^{\frac{1}{3}}_{3p(\cdot)} &= \gamma_1 \|(\widetilde{D}\mathbf{u})^{\frac{p}{2}}\|_6^2 \leq C\,\gamma_1(\|\boldsymbol{\nabla}((\widetilde{D}\mathbf{u})^{\frac{p}{2}})\|_2^2 + \|(\widetilde{D}\mathbf{u})^{\frac{p}{2}}\|_2^2)\\ &\leq C\left(\mathcal{I}_\Phi(\mathbf{u}) + \|\boldsymbol{\nabla} p\|_\infty^2 \int_\Omega |\widetilde{D}\mathbf{u}|^p \ln^2(\widetilde{D}\mathbf{u})\,dx\right) + C\,\gamma_1 |\widetilde{D}\mathbf{u}|_{p(\cdot)}.\end{aligned}$$

This proves the lemma. □

Lemma 5.3. *For all (sufficiently smooth) $\mathbf{u}$ with $\langle \mathbf{u}, 1\rangle = 0$ there holds*

$$\|\mathbf{u}\|^{p_\infty}_{2,\frac{3p_\infty}{p_\infty+1}} \leq C\,(\mathcal{I}_\Phi(\mathbf{u}) + 1), \tag{5.5}$$

$$\|\partial_t\mathbf{u}\|^{p_\infty}_{1,\frac{3p_\infty}{p_\infty+1}} \leq C\,\mathcal{J}_\Phi(\mathbf{u})^{\frac{p_\infty}{2}}\left(\mathcal{I}_\Phi(\mathbf{u}) + 1\right)^{\frac{2-p_\infty}{2}} \tag{5.6}$$

$$\leq C(\mathcal{J}_\Phi(\mathbf{u}) + \mathcal{I}_\Phi(\mathbf{u}) + 1). \tag{5.7}$$

Proof. From lemma 3.14 we deduce

$$\begin{aligned}\|\boldsymbol{\nabla}\mathbf{D}\mathbf{u}\|_{\frac{3p_\infty}{p_\infty+1}} &\leq C\,\mathcal{I}_\Phi(\mathbf{u})^{\frac{1}{2}} \|(\widetilde{D}\mathbf{u})^{\frac{2-p}{2}}\|_{\frac{6p_\infty}{2-p_\infty}}\\ &\leq C\,\mathcal{I}_\Phi(\mathbf{u})^{\frac{1}{2}} \|\widetilde{D}\mathbf{u}\|^{\frac{2-p_\infty}{2}}_{3p_\infty}\\ &\leq C\,\mathcal{I}_\Phi(\mathbf{u})^{\frac{1}{2}}\left(1 + \|\mathbf{D}\mathbf{u}\|_{3p_\infty}\right)^{\frac{2-p_\infty}{2}}\\ &\leq C\,\mathcal{I}_\Phi(\mathbf{u})^{\frac{1}{2}}\left(1 + C\,\|\boldsymbol{\nabla}\mathbf{D}\mathbf{u}\|_{\frac{3p_\infty}{p_\infty+1}}\right)^{\frac{2-p_\infty}{2}}, \quad \text{since } \langle \mathbf{u}, 1\rangle = 0.\end{aligned}$$

This implies

$$\|\nabla \mathbf{D}\mathbf{u}\|^{p_\infty}_{\frac{3p_\infty}{p_\infty+1}} \le C\,(\mathcal{I}_\Phi(\mathbf{u})+1).$$

Since $|\nabla \mathbf{D}\mathbf{u}| \le 2\,|\nabla^2\mathbf{u}|$ (see lemma 8.5) and $\langle \mathbf{u}, 1\rangle = 0$, we get

$$\|\mathbf{u}\|^{p_\infty}_{2,\frac{3p_\infty}{p_\infty+1}} \le C\,(\mathcal{I}_\Phi(\mathbf{u})+1).$$

Analogously we can use lemma 3.14 to get

$$\begin{aligned}
\|\partial_t \mathbf{D}\mathbf{u}\|_{\frac{3p_\infty}{p_\infty+1}} &\le C\,\mathcal{J}_\Phi(\mathbf{u})^{\frac12}\|(\widetilde{D}\mathbf{u})^{\frac{2-p}{2}}\|_{\frac{6p_\infty}{2-p_\infty}} \\
&\le C\,\mathcal{J}_\Phi(\mathbf{u})^{\frac12}\|\widetilde{D}\mathbf{u}\|^{\frac{2-p_\infty}{2}}_{3p_\infty} \\
&\le C\,\mathcal{J}_\Phi(\mathbf{u})^{\frac12}\big(1+\|\mathbf{D}\mathbf{u}\|_{3p_\infty}\big)^{\frac{2-p_\infty}{2}} \\
&\le C\,\mathcal{J}_\Phi(\mathbf{u})^{\frac12}\big(1+C\,\|\nabla\mathbf{D}\mathbf{u}\|_{\frac{3p_\infty}{p_\infty+1}}\big)^{\frac{2-p_\infty}{2}}, \qquad \text{since } \langle \mathbf{u},1\rangle = 0 \\
&\overset{(5.5)}{\le} C\,\mathcal{J}_\Phi(\mathbf{u})^{\frac12}\big(1+C\,(\mathcal{I}_\Phi(\mathbf{u})+1)^{\frac{1}{p_\infty}}\big)^{\frac{2-p_\infty}{2}} \\
&\le C\,\mathcal{J}_\Phi(\mathbf{u})^{\frac12}\big(1+\mathcal{I}_\Phi(\mathbf{u})\big)^{\frac{2-p_\infty}{2p_\infty}}.
\end{aligned}$$

Now $\langle \mathbf{u}, 1\rangle = 0$ and Korn's inequality imply

$$\|\partial_t\mathbf{u}\|_{1,\frac{3p_\infty}{p_\infty+1}} \le C\,\|\partial_t\mathbf{D}\mathbf{u}\|_{\frac{3p_\infty}{p_\infty+1}} \le C\,\mathcal{J}_\Phi(\mathbf{u})^{\frac12}\big(1+\mathcal{I}_\Phi(\mathbf{u})\big)^{\frac{2-p_\infty}{2p_\infty}},$$

which proves (5.6). The rest is an application of Young's inequality. □

3. The case $p_\infty > \frac{3}{2}$

In order to solve the system (5.1) we need some regularity for the data. So let us assume for the rest of the chapter that the data $\mathbf{f}$ and $\mathbf{u}_0$ fulfill:

$$\|\mathbf{f}\|_{L^\infty(I,W^{1,2}(\Omega))} + \|\partial_t\mathbf{f}\|_{L^2(I,L^2(\Omega))} + \|\mathbf{u}_0\|_{W^{2,2}_{\operatorname{div}}(\Omega)} \le C. \tag{5.8}$$

We will prove the existence of strong solutions by means of Galerkin approximation. So let $\{\boldsymbol{\omega}^r\}$ denote the set consisting of eigenvectors of the Stokes operator $\mathbb{S}$. Let λ_r be the corresponding eigenvalues and $X_N = \operatorname{span}\{\boldsymbol{\omega}^1,\dots,\boldsymbol{\omega}^N\}$. Let us recall that we are only looking for (space periodic) solutions with mean value zero. So the $\boldsymbol{\omega}^r$ all fulfill $\langle\boldsymbol{\omega}^r,1\rangle = 0$. Define $P^N\mathbf{u} = \sum_{r=1}^N \langle \mathbf{u},\boldsymbol{\omega}^r\rangle, \boldsymbol{\omega}^r$. Then

$$\lambda_r\langle \mathbf{u}^N,\boldsymbol{\omega}^r\rangle = \langle \mathbf{u}^N,\mathbb{S}\boldsymbol{\omega}^r\rangle = \langle \nabla\mathbf{u}^N,\nabla\boldsymbol{\omega}^r\rangle \tag{5.9}$$

and the $P^N : W^{s,2} \to X_N$ are uniformly continuous for all $0 \le s \le 2$. (See [MNRR96] for a proof.)

Let us define $\mathbf{u}^N(t,x) = \sum_{r=1}^N c_r^N(t)\boldsymbol{\omega}(x)$ and $\mathbf{f}^N = P^N\mathbf{f}$, where the coefficients $c_r^N(t)$ solve the Galerkin system (for all $1 \le r \le N$)

$$\begin{aligned}
\big\langle \partial_t\mathbf{u}^N,\boldsymbol{\omega}^r\big\rangle + \langle \mathbf{S}(\mathbf{D}\mathbf{u}^N),\mathbf{D}\boldsymbol{\omega}^r\rangle + \langle(\mathbf{u}^N\cdot\nabla)\mathbf{u}^N,\boldsymbol{\omega}^r\rangle &= \langle \mathbf{f}^N,\boldsymbol{\omega}^r\rangle, \\
\mathbf{u}^N(0) &= P^N\mathbf{u}_0.
\end{aligned} \tag{5.10}$$

Since the matrix $\langle\boldsymbol{\omega}_j,\boldsymbol{\omega}_k\rangle$ with $j,k = 1,\dots,N$ is positive definite, this can be rewritten as a system of ordinary differential equations. This in turn fulfills the Carathéodory conditions and is therefore solvable locally in time, i.e. on a small time interval $I^* = [0,T^*)$. Furthermore, since $\mathbf{f} \in L^\infty(I^*,W^{1,2}(\Omega))$ and $\partial_t\mathbf{f} \in L^2(I^*,L^2(\Omega))$, there holds $\mathbf{f}^N = P^N\mathbf{f} \in L^\infty(I^*, W^{2,2}(\Omega))$ and $\partial_t\mathbf{f}^N = P^N(\partial_t\mathbf{f}) \in L^2(I^*, L^2(\Omega))$. This implies $c_r^N, \partial_t c_r^N, \partial_t^2 c_r^N \in L^2(I^*)$. Thus $\mathbf{u}^N, \partial_t\mathbf{u}^N, \partial_t^2\mathbf{u}^N \in L^2(I^*, X_N)$. (Note that the norms may depebd on N). To ensure solvability for large times at least for this finite dimensional problem we have to establish a first a priori estimate.

Since $\mathbf{u}^N, \partial_t\mathbf{u}^N, \partial_t^2\mathbf{u}^N \in L^2(I^*, X_N)$, we can test the Galerkin system (5.10) with $\mathbf{u}^N$ and get

$$\tfrac{1}{2}d_t\|\mathbf{u}^N\|_2^2 + \langle \mathbf{S}(\mathbf{D}\mathbf{u}^N), \mathbf{D}\mathbf{u}^N\rangle = \langle \mathbf{f}^N, \mathbf{u}^N\rangle.$$

Note that $\langle(\mathbf{u}^N\cdot\boldsymbol{\nabla})\mathbf{u}^N, \mathbf{u}^N\rangle = 0$ due to $\operatorname{div}\mathbf{u}^N = 0$. The coercivity of $\mathbf{S}$ implies

$$\tfrac{1}{2}d_t\|\mathbf{u}^N\|_2^2 + |\mathbf{D}\mathbf{u}^N|_{p(\cdot)} \le \|\mathbf{f}^N\|_2^2 + \|\mathbf{u}^N\|_2^2.$$

By Gronwall's lemma and $\|\mathbf{f}\|_{L^2(I,L^2(\Omega))} \le C$

$$\tfrac{1}{2}\max_{I^*}\|\mathbf{u}^N\|_2^2 + \int_{I^*}\int_\Omega |\mathbf{D}\mathbf{u}^N|^p\,dx\,dt \le C. \tag{5.11}$$

This implies

$$\|c_r^N\|_{L^\infty(I^*)} \le C, \qquad 1 \le r \le N.$$

As a consequence we can iterate Carathéodory's theorem to push the solvability of the Galerkin system (5.10) up to any fixed time interval $I = [0, T)$ as inequality (5.11) remains valid for I^* replaced by I. (Compare with the proof lemma 4.1.) Hence independently of N there holds

$$\|\mathbf{u}^N\|_{L^\infty(I,L^2(\Omega))} + \|\mathbf{D}\mathbf{u}^N\|_{L^{p(\cdot)}(I\times\Omega)} \le C. \tag{5.12}$$

We got the first a priori estimate by using $\mathbf{u}^N$ as a test function. To derive our second a priori estimate we want to use $\mathbb{S}\mathbf{u}^N$ as a test function. The special choice of base functions $\boldsymbol{\omega}^r$ ensures that we do not leave X_N, the space of admissible test functions: More explicitly we multiply the r-th equation of the Galerkin system (5.10) by $\lambda_r c_r^N$, use the definition of the $\boldsymbol{\omega}^r$, λ^r, and sum up over $r = 1, \dots, N$ to obtain

$$\langle\partial_t\mathbf{u}^N, \mathbb{S}\mathbf{u}^N\rangle - \langle\mathbf{S}(\mathbf{D}\mathbf{u}^N), \mathbf{D}\mathbb{S}\mathbf{u}^N\rangle - \langle(\mathbf{u}^N\cdot\boldsymbol{\nabla})\mathbf{u}^N, \mathbb{S}\mathbf{u}^N\rangle = \langle\boldsymbol{\nabla}\mathbf{f}^N, \boldsymbol{\nabla}\mathbf{u}^N\rangle.$$

Due to the space periodicity $\mathbb{S} = -\Delta$, so

$$\tfrac{1}{2}d_t\|\boldsymbol{\nabla}\mathbf{u}^N\|_2^2 - \langle\mathbf{S}(\mathbf{D}\mathbf{u}^N), \mathbf{D}\Delta\mathbf{u}^N\rangle - \langle(\mathbf{u}^N\cdot\boldsymbol{\nabla})\mathbf{u}^N, \Delta\mathbf{u}^N\rangle = \langle\boldsymbol{\nabla}\mathbf{f}^N, \boldsymbol{\nabla}\mathbf{u}^N\rangle.$$

Let us simplify the second and third term on the left-hand side:

$$
\begin{aligned}
\langle(\mathbf{u}^N\cdot\boldsymbol{\nabla})\mathbf{u}^N,-\Delta\mathbf{u}^N\rangle &= \sum_{ijk}\langle -\mathbf{u}_i^N\partial_i\mathbf{u}_j^N,\partial_k^2\mathbf{u}_j^N\rangle\\
&= \sum_{ijk}\langle\partial_k\mathbf{u}_i^N\partial_i\mathbf{u}_j^N,\partial_k\mathbf{u}_j^N\rangle + \sum_{ijk}\langle\mathbf{u}_i^N,\partial_i(\tfrac12(\partial_k\mathbf{u}_j^N)^2)\rangle\\
&= \sum_{ijk}\langle\partial_k\mathbf{u}_i^N\partial_i\mathbf{u}_j^N,\partial_k\mathbf{u}_j^N\rangle - \sum_{ijk}\langle\partial_i\mathbf{u}_i^N,(\tfrac12(\partial_k\mathbf{u}_j^N)^2)\rangle\\
&= \sum_{ijk}\langle\partial_k\mathbf{u}_i^N\partial_i\mathbf{u}_j^N,\partial_k\mathbf{u}_j^N\rangle, \qquad \text{since } \operatorname{div}\mathbf{u}^N=0,\\
\langle\mathbf{S}(\mathbf{D}\mathbf{u}^N),-\mathbf{D}\Delta\mathbf{u}^N\rangle &= \sum_r\langle\partial_r(\mathbf{S}(\mathbf{D}\mathbf{u}^N)),\partial_r\mathbf{D}\mathbf{u}^N\rangle\\
&= \sum_{rkl}\big\langle\partial_r\big((\partial_{kl}\Phi)(\mathbf{D}\mathbf{u}^N)\big),\partial_r D_{kl}\mathbf{u}^N\big\rangle\\
&= \sum_{rijkl}\big\langle(\partial_{ij}\partial_{kl}\Phi)(\mathbf{D}\mathbf{u}^N)\partial_r D_{ij}\mathbf{u}^N,\partial_r D_{kl}\mathbf{u}^N\big\rangle\\
&\quad+\sum_{rkl}\big\langle(\partial_r\partial_{kl}\Phi)(\mathbf{D}\mathbf{u}^N),\partial_r D_{kl}\mathbf{u}^N\big\rangle\\
&\overset{(3.15)}{\geq} \mathcal{I}_\Phi(\mathbf{u}^N) - C\int_\Omega(\widetilde{D}\mathbf{u}^N)^{p-1}\ln(\widetilde{D}\mathbf{u}^N)|\boldsymbol{\nabla}\mathbf{D}\mathbf{u}^N|\,dx,
\end{aligned}
$$

where we have used $\|\boldsymbol{\nabla}p\|_\infty\leq C$. Using (3.49) we get

$$
\langle\mathbf{S}(\mathbf{D}\mathbf{u}^N),-\mathbf{D}\Delta\mathbf{u}^N\rangle \geq \tfrac12\mathcal{I}_\Phi(\mathbf{u}^N) - C\int_\Omega(\widetilde{D}\mathbf{u}^N)^p\ln^2(\widetilde{D}\mathbf{u}^N)\,dx.
$$

This gives

$$
\begin{aligned}
\tfrac12 d_t\|\boldsymbol{\nabla}\mathbf{u}^N\|_2^2+\tfrac12\mathcal{I}_\Phi(\mathbf{u}^N) &\leq \|\boldsymbol{\nabla}\mathbf{u}^N\|_3^3 + |\langle\boldsymbol{\nabla}\mathbf{f}^N,\boldsymbol{\nabla}\mathbf{u}^N\rangle|\\
&\quad + C\int_\Omega(\widetilde{D}\mathbf{u}^N)^p\ln^2(\widetilde{D}\mathbf{u}^N)\,dx\\
&\leq C\,(1+\|\boldsymbol{\nabla}\mathbf{u}^N\|_3^3)+|\langle\boldsymbol{\nabla}\mathbf{f}^N,\boldsymbol{\nabla}\mathbf{u}^N\rangle|,
\end{aligned}
\tag{5.13}
$$

where we used $p_0\leq 2$. If $p_\infty>\frac{11}{5}$ one can show that $\|\boldsymbol{\nabla}\mathbf{u}^N\|_3^3\leq C_\varepsilon|\boldsymbol{\nabla}\mathbf{u}^N|_{p(\cdot)}\|\boldsymbol{\nabla}\mathbf{u}^N\|_2^2+\varepsilon\mathcal{I}_\Phi(\mathbf{u}^N)$ (see [**MNRR96**]), which enables us to apply Gronwall's inequality after absorbing $\varepsilon\mathcal{I}_\Phi(\mathbf{u}^N)$ on the left-hand side. This would give us a global estimate. (Note that we have $\||\mathbf{u}^N|_{p(\cdot)}\|_{L^1(I)}\leq C$, since $|\mathbf{u}^N|_{p(\cdot),I\times\Omega}\leq C$, both independently of N.) If $p_\infty>\frac53$, it is still possible to deduce $\|\boldsymbol{\nabla}\mathbf{u}^N\|_3^3\leq C_\varepsilon|\boldsymbol{\nabla}\mathbf{u}^N|_{p(\cdot)}\|\boldsymbol{\nabla}\mathbf{u}^N\|_2^R+\varepsilon\mathcal{I}_\Phi(\mathbf{u}^N)$ for some constant $1<R<\infty$ and thereafter to absorb $\varepsilon\mathcal{I}_\Phi(\mathbf{u}^N)$ on the left-hand side and apply a local version of Gronwall's inequality (see lemma 8.7). Instead of using Gronwall it is also possible to divide the inequality by $(1+\|\boldsymbol{\nabla}\mathbf{u}^N\|_2)^R$ as was done in [**MNRR96**]) and derive the same local estimates. This in turn implies enough regularity for $\mathbf{u}^N$ to justify all the later testing of the Galerkin system with "$\partial_t\mathbf{u}^N$" and "$\partial_t\mathbf{u}^N\partial_t$".

Nevertheless we will not make use of these facts, since we are also interested in smaller values of p than $\frac{5}{3}$. What we do is, we test immediately with "$\partial_t \mathbf{u}^N \partial_t$" to get in addition to (5.13) another estimate. Then we will use the resulting two estimates at the same time to derive quite strong a priori estimates for $\mathbf{u}^N$ for values up to $p_\infty > \frac{3}{2}$ in this section and up to $p > \frac{7}{5}$ in the next section.

Let us take the time derivative of the Galerkin system (5.10):

$$\langle \partial_t^2 \mathbf{u}^N, \boldsymbol{\omega}^r \rangle + \langle \partial_t \mathbf{S}(\mathbf{D}(\mathbf{u}^N)), \mathbf{D}\boldsymbol{\omega}^r \rangle + \langle \partial_t((\mathbf{u}^N \cdot \boldsymbol{\nabla})\mathbf{u}^N), \boldsymbol{\omega}^r \rangle = \langle \partial_t \mathbf{f}^N, \boldsymbol{\omega}^r \rangle,$$

for $1 \le r \le N$. Since $\mathbf{u}^N \in W^{2,2}(I, X_n)$, this makes sense and we can even test with $\partial_t \mathbf{u}^N \in W^{1,2}(I, X_n)$:

$$\begin{aligned} \tfrac{1}{2} d_t \|\partial_t \mathbf{u}^N\|_2^2 + \langle \partial_t(\mathbf{S}(\mathbf{D}(\mathbf{u}^N))), \partial_t \mathbf{D}\mathbf{u}^N \rangle \\ + \langle \partial_t((\mathbf{u}^N \cdot \boldsymbol{\nabla})\mathbf{u}^N), \partial_t \mathbf{u}^N \rangle = \langle \partial_t \mathbf{f}^N, \partial_t \mathbf{u}^N \rangle. \end{aligned}$$

Once again the second term on the left-hand side has a sign, namely

$$\begin{aligned} \langle \partial_t(\mathbf{S}(\mathbf{D}\mathbf{u}^N)), \partial_t \mathbf{D}\mathbf{u}^N \rangle &= \sum_{ikl} \langle \partial_t\big((\partial_{kl}\Phi)(\mathbf{D}\mathbf{u}^N)\big), \partial_t D_{kl}\mathbf{u}^N \rangle \\ &= \sum_{ikl} \langle (\partial_{ij}\partial_{kl}\Phi)(\mathbf{D}\mathbf{u}^N)\partial_t D_{ij}\mathbf{u}^N, \partial_t D_{kl}\mathbf{u}^N \rangle \\ &\quad + \sum_{ikl} \langle (\partial_t \partial_{kl}\Phi)(\mathbf{D}\mathbf{u}^N), \partial_t D_{kl}\mathbf{u}^N \rangle \\ &\overset{(3.15)}{\ge} \mathcal{J}_\Phi(\mathbf{u}^N) - C \int_\Omega (\widetilde{D}\mathbf{u}^N)^{p-1} \ln(\widetilde{D}\mathbf{u}^N) |\partial_t \mathbf{D}\mathbf{u}^N| \, dx \\ &\overset{(3.50)}{\ge} \tfrac{1}{2}\mathcal{J}_\Phi(\mathbf{u}^N) - C \int_\Omega (\widetilde{D}\mathbf{u}^N)^p \ln^2(\widetilde{D}\mathbf{u}^N) \, dx. \end{aligned}$$

This yields

$$\begin{aligned} d_t \|\partial_t \mathbf{u}^N\|_2^2 + \mathcal{J}_\Phi(\mathbf{u}) &\le C \big(|\langle \partial_t((\mathbf{u}^N \cdot \boldsymbol{\nabla})\mathbf{u}^N), \partial_t \mathbf{u}^N \rangle| + |\langle \partial_t \mathbf{f}^N, \partial_t \mathbf{u}^N \rangle| \big) \\ &\quad + C \int_\Omega (\widetilde{D}\mathbf{u}^N)^p \ln^2(\widetilde{D}\mathbf{u}^N) \, dx, \end{aligned} \tag{5.14}$$

the second inequality to start from. Let us restate the two important inequalities (5.13) and (5.14):

$$d_t\big(\|\boldsymbol{\nabla}\mathbf{u}^N\|_2^2\big) + \mathcal{I}_\Phi(\mathbf{u}^N) \le C \big(1 + \|\boldsymbol{\nabla}\mathbf{u}^N\|_3^3 + |\langle \boldsymbol{\nabla}\mathbf{f}^N, \boldsymbol{\nabla}\mathbf{u}^N \rangle|\big), \tag{5.15}$$

$$\begin{aligned} d_t\big(\|\partial_t \mathbf{u}^N\|_2^2\big) + \mathcal{J}_\Phi(\mathbf{u}^N) &\le C \big(|\langle \partial_t((\mathbf{u}^N \cdot \boldsymbol{\nabla})\mathbf{u}^N), \partial_t \mathbf{u}^N \rangle| + |\langle \partial_t \mathbf{f}^N, \partial_t \mathbf{u}^N \rangle| \big) \\ &\quad + C \int_\Omega (\widetilde{D}\mathbf{u}^N)^p \ln^2(\widetilde{D}\mathbf{u}^N) \, dx. \end{aligned} \tag{5.16}$$

By a first view we have gained nothing. We have to control one more bad term, namely $|\langle \partial_t((\mathbf{u}^N \cdot \boldsymbol{\nabla})\mathbf{u}^N), \partial_t \mathbf{u}^N \rangle|$, but we only got more information about the time derivative of $\mathbf{u}^N$. But the critical term $\|\boldsymbol{\nabla}\mathbf{u}^N\|_3^3$, which gave the lower bound for p_∞ has no time derivatives. The next lemma shows that $\mathcal{J}_\Phi(\mathbf{u}^N)$ reveals indeed more information.

Lemma 5.4. *Let $1 < q < \infty$, then*

$$
\begin{aligned}
d_t(\|\widetilde{D}\mathbf{u}\|_q^q) &\le q\, C\, \mathcal{J}_\Phi(\mathbf{u})^{\frac{1}{2}} \big(|\widetilde{D}\mathbf{u}|_{2q-p(\cdot)}\big)^{\frac{1}{2}} \\
&\le \varepsilon \mathcal{J}_\Phi(\mathbf{u}) + C_\varepsilon |\widetilde{D}\mathbf{u}|_{2q-p(\cdot)}.
\end{aligned}
\tag{5.17}
$$

Proof. Note that

$$\partial_t\big((\widetilde{D}\mathbf{u})^q\big) = q(\widetilde{D}\mathbf{u})^{q-2}(D_{jk}\mathbf{u})\,(\partial_t D_{jk}\mathbf{u}).$$

Hence

$$
\begin{aligned}
d_t(\|\widetilde{D}\mathbf{u}\|_q^q) &\le q \int_\Omega (\widetilde{D}\mathbf{u})^{q-1} |\partial_t \mathbf{D}\mathbf{u}|\, dx \\
&= q \int_\Omega (\widetilde{D}\mathbf{u})^{\frac{p-2}{2}} |\partial_t \mathbf{D}\mathbf{u}|\, (\widetilde{D}\mathbf{u})^{q-\frac{p}{2}}\, dx \\
&\le q\, C\, \mathcal{J}_\Phi(\mathbf{u})^{\frac{1}{2}} \big(|\widetilde{D}\mathbf{u}|_{2q-p(\cdot)}\big)^{\frac{1}{2}},
\end{aligned}
$$

where $|\widetilde{D}\mathbf{u}|_{2q-p(\cdot)} = \int_\Omega (\widetilde{D}\mathbf{u})^{2q-p}\, dx$ even if $2q - p_0 < 1$. The rest is an implication of Young's inequality. □

This lemma enables us to produce $d_t(\|\nabla\mathbf{u}^N\|_q^q)$ on the left-hand side of (5.16) if we add $C\,|\widetilde{D}\mathbf{u}^N|_{2q-p(\cdot)}$ to the right-hand side. Note that we have now three critical terms to control, which are

$$\|\nabla\mathbf{u}^N\|_3^3, \qquad |\langle \partial_t((\mathbf{u}^N\cdot\nabla)\mathbf{u}^N), \partial_t\mathbf{u}^N\rangle|, \qquad C\,|\widetilde{D}\mathbf{u}^N|_{2q-p(\cdot)}.$$

The first and the second one will be easier to estimate for large q, but the third one for small q. The problem now is to find the optimal choice for q. We start by examining which values of q are needed for the first and the second term. In the view of lemma 8.7, which we want to apply later, we will be able to control arbitrary powers of $\|\nabla\mathbf{u}^N\|_q^q$ and $\|\partial_t\mathbf{u}^N\|_2^2$. Note that we will skip the index N of $\mathbf{u}^N$ to keep the notations simple.

Lemma 5.5. *Let $q > \frac{9-3p_\infty}{2}$, then there exists a constant $R_1 > 1$, such that*

$$\|\nabla\mathbf{u}\|_3^3 \le C_\varepsilon \|\nabla\mathbf{u}\|_q^{R_1} + \varepsilon \mathcal{I}_\Phi(\mathbf{u}) + \varepsilon.$$

Proof. If $q \ge 3$, then there is nothing to prove. Therefore assume that $q < 3$. We can interpolate $L^3(\Omega) = [L^q(\Omega), L^{3p_\infty}(\Omega)]_\theta$ with

$$\frac{1}{3} = \frac{(1-\theta)}{q} + \frac{\theta}{3p_\infty} \quad\Rightarrow\quad \theta = \frac{(3-q)p_\infty}{3p_\infty - q}, \quad 1-\theta = \frac{q(p_\infty - 1)}{3p_\infty - q}.$$

Therefore

$$\|\nabla\mathbf{u}\|_3^3 \le \|\nabla\mathbf{u}\|_q^{3(1-\theta)} \|\nabla\mathbf{u}\|_{3p_\infty}^{3\theta}.$$

If $3\theta < p_\infty$, there exists an $\delta > 1$ such that

$$
\begin{aligned}
\|\nabla\mathbf{u}\|_3^3 &\le C_\varepsilon \|\nabla\mathbf{u}\|_q^{3(1-\theta)\delta'} + \varepsilon \|\nabla\mathbf{u}\|_{3p_\infty}^{p_\infty} \\
&\le C_\varepsilon \|\nabla\mathbf{u}\|_q^{3(1-\theta)\delta'} + C\,\varepsilon \|\nabla\mathbf{u}\|_{2,\frac{3p_\infty}{p_\infty+1}}^{p_\infty} \\
&\le C_\varepsilon \|\nabla\mathbf{u}\|_q^{3(1-\theta)\delta'} + \varepsilon C\,(\mathcal{I}_\Phi(\mathbf{u}) + 1 + \|\nabla\mathbf{u}\|_3^3),
\end{aligned}
$$

where we have used (5.5). So

$$\|\nabla \mathbf{u}\|_3^3 \le C_{\varepsilon_2} \|\nabla \mathbf{u}\|_q^{3(1-\theta)\delta'} + \varepsilon_2 \mathcal{I}_\Phi(\mathbf{u}) + \varepsilon_2.$$

We still have to verify $3\theta < p_\infty$, but this is equivalent to

$$\frac{3(3-q)p_\infty}{3p_\infty - q} < p_\infty \quad \Leftrightarrow \quad \frac{9-3p_\infty}{2} < q,$$

which holds due to the assumptions on q. □

LEMMA 5.6. *Let $q > \frac{9-3p_\infty}{2}$, then there exist constants $R_2, R_3 > 1$ such that*

$$|\langle(\partial_t \mathbf{u} \cdot \nabla)\mathbf{u}, \partial_t \mathbf{u}\rangle| \le \varepsilon \mathcal{J}_\Phi(\mathbf{u}) + C_\varepsilon\big(\|\partial_t \mathbf{u}\|_2^{R_2} + \|\widetilde{D}\mathbf{u}\|_q^{R_3} + 1\big).$$

PROOF. Note that lemma 3.14 ($q \mapsto \frac{2q}{2-p_\infty+q}$) implies

$$\begin{aligned} \|\partial_t \mathbf{D}\mathbf{u}\|_{\frac{2q}{2-p_\infty+q}} &\le C\, \mathcal{J}_\Phi(\mathbf{u})^{\frac{1}{2}} \|(\widetilde{D}\mathbf{u})^{\frac{2-p}{2}}\|_{\frac{2q}{2-p_\infty}} \\ &\le C\, \mathcal{J}_\Phi(\mathbf{u})^{\frac{1}{2}} \|\widetilde{D}\mathbf{u}\|_q^{\frac{2-p_\infty}{2}}. \end{aligned} \tag{5.18}$$

Furthermore $W^{1,\frac{2q}{2-p_\infty+q}}(\Omega) \hookrightarrow L^{\frac{6q}{6-3p_\infty+q}}(\Omega)$. Since $\frac{9-3p_\infty}{2} < q$ is equivalent to $\frac{2q}{q-1} < \frac{6q}{6-3p_\infty+q}$ we can use the interpolation

$$L^{\frac{2q}{q-1}}(\Omega) = [L^2(\Omega), L^{\frac{6q}{6-3p_\infty+q}}]_\theta.$$

Hence

$$\begin{aligned} |\langle(\partial_t \mathbf{u} \cdot \nabla)\mathbf{u}, \partial_t \mathbf{u}\rangle| &\le \|\partial_t \mathbf{u}\|^2_{\frac{2q}{q-1}} \|\nabla \mathbf{u}\|_q \\ &\le C\, \|\partial_t \mathbf{u}\|_2^{2(1-\theta)} \|\partial_t \mathbf{u}\|^{2\theta}_{\frac{6q}{6-3p_\infty+q}} \|\nabla \mathbf{u}\|_q \\ &\le C\, \|\partial_t \mathbf{u}\|_2^{2(1-\theta)} \|\partial_t \nabla \mathbf{u}\|^{2\theta}_{\frac{2q}{2-p_\infty+q}} \|\nabla \mathbf{u}\|_q \\ &\overset{(5.18)}{\le} C\, \|\partial_t \mathbf{u}\|_2^{2(1-\theta)} \Big(\mathcal{J}_\Phi(\mathbf{u})^{\frac{1}{2}} \|\widetilde{D}\mathbf{u}\|_q^{\frac{2-p_\infty}{2}}\Big)^{2\theta} \|\nabla \mathbf{u}\|_q \\ &\le \varepsilon \mathcal{J}_\Phi(\mathbf{u}) + C_\varepsilon\big(\|\partial_t \mathbf{u}\|_2^{R_2} + \|\nabla \mathbf{u}\|_q^{R_3} + 1\big). \end{aligned}$$

□

It is indeed interesting that both terms $|\langle\partial_t((\mathbf{u}^N \cdot \nabla)\mathbf{u}^N), \partial_t \mathbf{u}^N\rangle|$ and $\|\nabla \mathbf{u}^N\|_3^3$ require the same bound for q, which is $q > \frac{9-3p_\infty}{2}$. Now we have to find the upper bound for q, which restricts the control of $|\nabla \mathbf{u}^N|_{2q-p(\cdot)}$. Unfortunately this requires extensive calculations, so we will postpone this to the next section. Since the calculations for $p_\infty > \frac{3}{2}$ are a lot simpler, we will finish this section by outlining how to proceed in this simpler case.

So let us assume for the rest of this section that $p_\infty > \frac{3}{2}$. Set $q := \frac{3+p_\infty}{2}$, then $2q - p(\cdot) \le 2q - p_\infty = 3$, so

$$|\nabla \mathbf{u}^N|_{2q-p(\cdot)} \le C\,(\|\nabla \mathbf{u}\|_3^3 + 1).$$

That means that $|\nabla \mathbf{u}^N|_{2q-p(\cdot)}$ can be controlled if $\|\nabla \mathbf{u}^N\|_3^3$ can be controlled. But the choice of q and $p_\infty > \frac{3}{2}$ ensures that $q > \frac{9-3p_\infty}{2}$. Hence by lemma 5.5, lemma 5.6

and the above calculations we get

$$\begin{aligned} &d_t\big(\|\nabla\mathbf{u}^N\|_2^2\big)+\mathcal{I}_\Phi(\mathbf{u}^N)\\ &\qquad\le C\,\big(1+\|\nabla\mathbf{u}^N\|_2^{R_1}+|\langle\nabla\mathbf{f}^N,\nabla\mathbf{u}^N\rangle|\big),\\ &d_t\big(\|\partial_t\mathbf{u}^N\|_2^2\big)+d_t\big(\|\nabla\mathbf{u}^N\|_q^q\big)+\mathcal{J}_\Phi(\mathbf{u}^N)\\ &\qquad\le C\,\|\partial_t\mathbf{u}^N\|_2^{R_2}+C\,\|\nabla\mathbf{u}^N\|_q^{R_3}\\ &\qquad\quad+C\,|\langle\partial_t\mathbf{f}^N,\partial_t\mathbf{u}^N\rangle|+C\int_\Omega(\widetilde{D}\mathbf{u}^N)^p\ln^2(\widetilde{D}\mathbf{u}^N)\,dx. \end{aligned}$$

The remaining terms involving $\mathbf{f}^N$ are easy to control:

$$\begin{aligned} |\langle\nabla\mathbf{f}^N,\nabla\mathbf{u}^N\rangle| &\le \|P^N\mathbf{f}\|_{1,2}\|\nabla\mathbf{u}^N\|_2\le C\,\|\mathbf{f}\|_{1,2}\|\nabla\mathbf{u}^N\|_2\\ &\le C\,\|\mathbf{f}\|_{1,2}^2+C\,\|\nabla\mathbf{u}^N\|_2^2,\\ |\langle\partial_t\mathbf{f}^N,\partial_t\mathbf{u}^N\rangle| &\le \|P^N(\partial_t\mathbf{f})\|_2\|\partial_t\mathbf{u}^N\|_2\le C\,\|\partial_t\mathbf{f}\|_2\|\nabla\mathbf{u}^N\|_2\\ &\le C\,\|\partial_t\mathbf{f}\|_2^2+C\,\|\nabla\mathbf{u}^N\|_2^2. \end{aligned}$$

Even the term with $\ln(\widetilde{D}\mathbf{u}^N)$ makes no difficulties, since $q>2$:

$$\begin{aligned} \int_\Omega(\widetilde{D}\mathbf{u}^N)^p\ln^2(\widetilde{D}\mathbf{u}^N)\,dx &\le \|\widetilde{D}\mathbf{u}^N\|_q^q+C\\ &\le C\,\|\mathbf{D}\mathbf{u}^N\|_q^q+C\\ &\overset{\text{Korn}}{\le} C\,\|\nabla\mathbf{u}^N\|_q^q+C. \end{aligned}$$

Overall

$$\begin{aligned} &d_t\big(\|\nabla\mathbf{u}^N\|_2^2\big)+d_t\big(\|\partial_t\mathbf{u}^N\|_2^2\big)+d_t\big(\|\nabla\mathbf{u}^N\|_q^q\big)+\mathcal{I}_\Phi(\mathbf{u}^N)+\mathcal{J}_\Phi(\mathbf{u}^N)\\ &\le C\,\big(1+\|\nabla\mathbf{u}^N\|_2^{\max\{R_1,R_3,q\}}+\|\partial_t\mathbf{u}^N\|_2^{\max\{R_1,2\}}+\|\mathbf{f}\|_{1,2}^2+\|\partial_t\mathbf{f}\|_2^2\big). \end{aligned}$$

Now lemma 8.7 ensures that for small times, i.e. T is small, we get boundedness of the following expressions (uniformly in N):

$$\begin{aligned} &\|\partial_t\mathbf{u}^N\|_{L^\infty(I',L^2(\Omega))}^2, && \|\partial_t\mathbf{u}^N\|_{L^\infty(I',L^2(\Omega))}^2, && \|\nabla\mathbf{u}^N\|_{L^\infty(I',L^q(\Omega))}^q,\\ &\|\mathcal{I}_\Phi(\mathbf{u}^N)\|_{L^1(I')}, && \|\mathcal{J}_\Phi(\mathbf{u}^N)\|_{L^1(I')}. \end{aligned}$$

Later in section 5 will see that these a priori estimates are sufficient to pass to the limit $N\to\infty$ to get a solution $\mathbf{u}$ of our original problem (5.1). But beforehand we will show in the next section how to derive similar a priori estimates in the more general case $\frac{7}{5}<p_\infty\le 2$. Certainly the next section in more general than this section, but we wanted to point out explicitly how a simplified method works for $p_\infty>\frac{3}{2}$.

4. The case $p_\infty>\frac{7}{5}$

If p_∞ is smaller than $\frac{3}{2}$, we have to do more subtle calculations. We cannot just add (5.15) and (5.16) in order to get control of $|\widetilde{D}\mathbf{u}^N|_{2q-p(\cdot)}$. Recall that we need $q>\frac{9-3p_\infty}{2}$ for $\|\nabla\mathbf{u}^N\|_3^3$ and $|\langle(\partial_t\mathbf{u}\cdot\nabla)\mathbf{u},\partial_t\mathbf{u}\rangle|$. But this implies $(2q-p)_0>9-3p_\infty-p_\infty>3$. So $|\widetilde{D}\mathbf{u}^N|_{2q-p(\cdot)}$ is worse than $\|\nabla\mathbf{u}^N\|_3^3$. Since $|\widetilde{D}\mathbf{u}^N|_{2q-p(\cdot)}$ grows with respect to q a lot faster than $\|\nabla\mathbf{u}^N\|_q^q$, the term $|\widetilde{D}\mathbf{u}^N|_{2q-p(\cdot)}$ requires a preferably small choice of q. But since we cannot control $\|\nabla \mathbf{u}^N\|_3^3$ for $q = \frac{9-3p_\infty}{2}$, we certainly cannot control the worse term $|\widetilde{D}\mathbf{u}^N|_{2q-p(\cdot)}$ for $q = \frac{9-3p_\infty}{2}$ and thus for no $q \geq \frac{9-3p_\infty}{2}$. Hence we must proceed in a different way.

The central idea is that we have not made use of the term $d_t\|\nabla \mathbf{u}^N\|_2^2$. Since it contains less information than $d_t\|\nabla \mathbf{u}^N\|_q^q$, there is no need to extract information out of it. So we try to transfer $d_t\|\nabla \mathbf{u}^N\|_2^2$ in its original form $\langle \partial_t \mathbf{u}^N, -\Delta \mathbf{u}^N\rangle$ to the right-hand side of (5.15). This gives

$$\mathcal{I}_\Phi(\mathbf{u}^N) \leq C\left(1 + \|\nabla \mathbf{u}^N\|_3^3 + |\langle \nabla \mathbf{f}^N, \nabla \mathbf{u}^N\rangle| + |\langle \partial_t \mathbf{u}^N, -\Delta \mathbf{u}^N\rangle|\right), \tag{5.19}$$

The disadvantage is that we have to control one extra term, but the advantage is that we can raise this inequality to the r-th power. This gives, as long as we can control the right-hand side, information on $\mathcal{I}_\Phi(\mathbf{u}^N)^r$, which can be used to control $|\widetilde{D}\mathbf{u}|_{2q-p(\cdot)}$ for higher values of q.

Before we calculate the maximal allowed r and the resulting q we will reduce (5.19) to a more suitable form: Lemma 5.5 implies that for $q > \frac{9-3p_\infty}{2}$ there holds

$$\mathcal{I}_\Phi(\mathbf{u}^N) \leq C\left(1 + \|\nabla \mathbf{u}^N\|_q^{R_1} + |\langle \nabla \mathbf{f}^N, \nabla \mathbf{u}^N\rangle| + |\langle \partial_t \mathbf{u}^N, -\Delta \mathbf{u}^N\rangle|\right).$$

Since

$$\|\mathbf{f}^N\|_{L^\infty(I,W^{1,2}(\Omega))} = \|P^N \mathbf{f}\|_{L^\infty(I,W^{1,2}(\Omega))} \leq C\,\|\mathbf{f}\|_{L^\infty(I,W^{1,2}(\Omega))} \leq C,$$

this reduces to

$$\mathcal{I}_\Phi(\mathbf{u}^N) \leq C\left(1 + \|\nabla \mathbf{u}^N\|_q^{R_1} + |\langle \partial_t \mathbf{u}^N, -\Delta \mathbf{u}^N\rangle|\right). \tag{5.20}$$

Since we can control arbitrary powers of $\|\nabla \mathbf{u}^N\|_q$ by the local Gronwall's lemma 8.7, we see that the convective and the force terms do not raise difficulties for $q > \frac{9-3p_\infty}{2}$, even if we raise the inequality to the r-th power. The following lemma gives control of the remaining term $|\langle \partial_t \mathbf{u}^N, -\Delta \mathbf{u}^N\rangle|$.

Lemma 5.7. *For $p_\infty > \frac{7}{5}$ there holds*

$$|\langle \partial_t \mathbf{u}, \Delta \mathbf{u}\rangle| \leq C\,\|\partial_t \mathbf{u}\|_2^{\frac{4(p_\infty-1)}{3p_\infty-2}}\,\mathcal{J}_\Phi(\mathbf{u})^{\frac{2-p_\infty}{2(3p_\infty-2)}}\,(\mathcal{I}_\Phi(\mathbf{u})+1)^{\frac{p_\infty+2}{2(3p_\infty-2)}}.$$

Proof. With the help of lemma 5.3 we conclude

$$\begin{aligned}
|\langle \partial_t \mathbf{u}, \Delta \mathbf{u}\rangle| &\leq \|\partial_t \mathbf{u}\|_{\frac{3p_\infty}{2p_\infty-1}}\,\|\mathbf{u}\|_{2,\frac{3p_\infty}{p_\infty+1}}\\
&\leq C\,\|\partial_t \mathbf{u}\|_{\frac{3p_\infty}{2p_\infty-1}}\,(\mathcal{I}_\Phi(\mathbf{u})+1)^{\frac{1}{p_\infty}}\\
&\leq \|\partial_t \mathbf{u}\|_2^{1-\theta}\|\partial_t \mathbf{u}\|_{1,\frac{3p_\infty}{1+p_\infty}}^{\theta}\,(\mathcal{I}_\Phi(\mathbf{u})+1)^{\frac{1}{p_\infty}}\\
&\leq \|\partial_t \mathbf{u}\|_2^{1-\theta}\left(\mathcal{J}_\Phi(\mathbf{u})^{\frac{1}{2}}(\mathcal{I}_\Phi(\mathbf{u})+1)^{\frac{2-p_\infty}{2p_\infty}}\right)^{\theta}(\mathcal{I}_\Phi(\mathbf{u})+1)^{\frac{1}{p_\infty}}
\end{aligned}$$

with

$$\frac{2p_\infty-1}{3p_\infty} = \frac{1-\theta}{2} + \frac{\theta}{3p_\infty}.$$

Therefore $\theta = \frac{2-p_\infty}{3p_\infty-2}$ and $1-\theta = \frac{4p_\infty-4}{3p_\infty-2}$ and $\frac{2-p_\infty}{2p_\infty}\cdot\theta + \frac{1}{p_\infty} = \frac{p_\infty+2}{2(3p_\infty-2)}$. This proves the lemma. $\square$

This lemma and (5.20) imply

$$\mathcal{I}_\Phi(\mathbf{u}^N) \le C\,\Big(1 + \|\boldsymbol{\nabla}\mathbf{u}^N\|_q^{R_1} + \|\partial_t\mathbf{u}^N\|_2^{\frac{4(p_\infty-1)}{3p_\infty-2}}\,\mathcal{J}_\Phi(\mathbf{u}^N)^{\frac{2-p_\infty}{2(3p_\infty-2)}}\big(\mathcal{I}_\Phi(\mathbf{u}^N)+1\big)^{\frac{p_\infty+2}{2(3p_\infty-2)}}\Big).$$

Thus

$$\mathcal{I}_\Phi(\mathbf{u}^N) \le C\,\Big(1 + \|\boldsymbol{\nabla}\mathbf{u}^N\|_q^{R_1} + \|\partial_t\mathbf{u}^N\|_2^{\frac{8(p_\infty-1)}{5p_\infty-6}}\,\mathcal{J}_\Phi(\mathbf{u}^N)^{\frac{2-p_\infty}{5p_\infty-6}}\Big). \tag{5.21}$$

We are finally at the point where we can raise the inequality to the r-th power

$$\mathcal{I}_\Phi(\mathbf{u}^N)^r \le C\,2^{r-1}\Big(1 + \|\boldsymbol{\nabla}\mathbf{u}^N\|_q^{rR_1} + \|\partial_t\mathbf{u}^N\|_2^{r\frac{8(p_\infty-1)}{5p_\infty-6}}\,\mathcal{J}_\Phi(\mathbf{u}^N)^{r\frac{2-p_\infty}{5p_\infty-6}}\Big).$$

As long as $r < \frac{5p_\infty-6}{2-p_\infty}$, the last term can be broken up into a large power of $\|\partial_t\mathbf{u}^N\|_2^2$ and $\mathcal{J}_\Phi(\mathbf{u}^N)$. We summarize

Lemma 5.8. *Let $p_\infty > \frac{7}{5}$, $q > \frac{9-3p_\infty}{2}$ and $1 \le r < \frac{5p_\infty-6}{2-p_\infty}$, then there exist constants R_4, R_5, such that*

$$\mathcal{I}_\Phi(\mathbf{u}^N)^r \le C_\varepsilon\big(1 + \|\boldsymbol{\nabla}\mathbf{u}^N\|_q^{R_4} + \|\partial_t\mathbf{u}^N\|_2^{R_5}\big) + \varepsilon\mathcal{J}_\Phi(\mathbf{u}^N).$$

As in the case $p_\infty > \frac{3}{2}$ we calculate from (5.16)

$$\begin{aligned} d_t\big(\|\partial_t\mathbf{u}^N\|_2^2\big) &+ \mathcal{J}_\Phi(\mathbf{u}^N) \\ &\le C\,\big(|\langle\partial_t((\mathbf{u}^N\cdot\boldsymbol{\nabla})\mathbf{u}^N), \partial_t\mathbf{u}^N\rangle| + |\langle\partial_t\mathbf{f}^N, \partial_t\mathbf{u}^N\rangle|\big) \\ &\quad + C\int_\Omega(\widetilde{D}\mathbf{u}^N)^p\ln^2(\widetilde{D}\mathbf{u}^N)\,dx \\ &\le C\,\big(1 + |\langle\partial_t((\mathbf{u}^N\cdot\boldsymbol{\nabla})\mathbf{u}^N), \partial_t\mathbf{u}^N\rangle| + \|\boldsymbol{\nabla}\mathbf{u}^N\|_q^q + |\langle\partial_t\mathbf{f}^N, \partial_t\mathbf{u}^N\rangle|\big). \end{aligned}$$

So lemma 5.6 implies for $q > \frac{9-3p_\infty}{2}$

$$d_t\big(\|\partial_t\mathbf{u}^N\|_2^2\big) + \mathcal{J}_\Phi(\mathbf{u}^N) \le C\,\big(1 + \|\partial_t\mathbf{u}^N\|_2^{R_2} + \|\widetilde{D}\mathbf{u}^N\|_q^{R_3} + \|\partial_t\mathbf{f}\|_2^2\big),$$

where we have used $\|\partial_t\mathbf{f}^N\|_2 = \|P^N(\partial_t\mathbf{f})\|_2 \le C\,\|\partial_t\mathbf{f}^N\|_2$. Hence by lemma 5.4 and lemma 5.8 for $q > \frac{9-3p_\infty}{2}$ and $r < \frac{5p_\infty-6}{2-p_\infty}$

$$\begin{aligned} \mathcal{I}_\Phi(\mathbf{u}^N)^r &\le C_\varepsilon\big(1 + \|\boldsymbol{\nabla}\mathbf{u}^N\|_q^{R_4} + \|\partial_t\mathbf{u}^N\|_2^{R_5}\big) + \varepsilon\mathcal{J}_\Phi(\mathbf{u}^N), \\ d_t\big(\|\partial_t\mathbf{u}^N\|_2^2\big) &+ d_t\big(\|\partial_t\mathbf{u}^N\|_q^q\big) + \mathcal{J}_\Phi(\mathbf{u}^N) \le \\ &C\,\big(1 + \|\partial_t\mathbf{u}^N\|_2^{R_2} + \|\widetilde{D}\mathbf{u}^N\|_q^{R_3} + |\widetilde{D}\mathbf{u}^N|_{2q-p(\cdot)} + \|\partial_t\mathbf{f}\|_2^2\big). \end{aligned} \tag{5.22}$$

Different from the case $p_\infty > \frac{3}{2}$ we can use $\mathcal{I}_\Phi(\mathbf{u}^N)^r$ to control $|\widetilde{D}\mathbf{u}^N|_{2q-p(\cdot)}$.

Lemma 5.9. *Let $\frac{7}{5} < p_\infty < q < \min\{\frac{3p_\infty(r+1)}{3+r}, 2p_\infty\}$, then there exists a constant $R_6 > 1$, such that*

$$|\widetilde{D}\mathbf{u}|_{2q-p(\cdot)} \le C_\varepsilon\|\widetilde{D}\mathbf{u}\|_q^{R_6} + \varepsilon\big(\mathcal{I}_\Phi(\mathbf{u})^r + 1\big).$$

Proof. From the assumptions we know that $p_\infty < q < 2p_\infty$, which implies $q < 2q - p_\infty < 3p_\infty$. Hence we can interpolate $L^{2q-p_\infty}(\Omega) = [L^q(\Omega), L^{3p_\infty}(\Omega)]_\theta$. The constant θ can be obtained by $\frac{1}{2q-p_\infty} = \frac{1-\theta}{q} + \frac{\theta}{3p_\infty}$. This yields $\theta = \frac{3p_\infty(q-p_\infty)}{(2q-p_\infty)(3p_\infty-q)}$ and $1-\theta = \frac{2q(2p_\infty-q)}{(2q-p_\infty)(3p_\infty-q)}$. Now we can estimate

$$\begin{aligned} |\widetilde{D}\mathbf{u}|_{2q-p(\cdot)} \le \|\widetilde{D}\mathbf{u}\|_{2q-p_\infty}^{2q-p_\infty} &\le \|\widetilde{D}\mathbf{u}\|_q^{(1-\theta)(2q-p_\infty)} \|\widetilde{D}\mathbf{u}\|_{3p_\infty}^{\theta(2q-p_\infty)} \\ &\overset{(5.5)}{\le} \|\widetilde{D}\mathbf{u}\|_q^{(1-\theta)(2q-p_\infty)} \big(\mathcal{I}_\Phi(\mathbf{u})+1\big)^{\frac{\theta(2q-p_\infty)}{p_\infty}}. \end{aligned}$$

If $\frac{\theta(2q-p_\infty)}{p_\infty} < r$, then we can use Young with $\delta := \frac{p_\infty r}{\theta(2q-p_\infty)} > 1$ to obtain the desired result

$$\|\widetilde{D}\mathbf{u}\|_{2q-p_\infty}^{2q-p_\infty} \le C_\varepsilon \|\widetilde{D}\mathbf{u}\|_q^{(1-\theta)(2q-p_\infty)\delta'} + \varepsilon\big(\mathcal{I}_\Phi(\mathbf{u})^r + 1\big).$$

Still we have to verify the condition $\frac{\theta(2q-p_\infty)}{p_\infty} < r$. For this note that

$$\frac{\theta(2q-p_\infty)}{p_\infty} < r \quad \Leftrightarrow \quad \frac{3(q-p_\infty)}{3p_\infty - q} < r \quad \Leftrightarrow \quad q < \frac{3p_\infty(r+1)}{3+r},$$

which holds due to the assumptions on q. □

This lemma and (5.22) imply

$$\begin{aligned} d_t\big(\|\partial_t\mathbf{u}^N\|_2^2\big) + d_t\big(\|\partial_t\mathbf{u}^N\|_q^q\big) + \mathcal{J}_\Phi(\mathbf{u}^N) + \mathcal{I}_\Phi(\mathbf{u}^N)^r \\ \le C\left(1 + \|\partial_t\mathbf{u}^N\|_2^{\max\{R_2,R_5\}} + \|\widetilde{D}\mathbf{u}^N\|_q^{\max\{R_3,R_4,R_6\}}\right) \end{aligned} \tag{5.23}$$

as long as

$$\max\left\{p_\infty, \tfrac{9-3p_\infty}{2}\right\} < q < \min\left\{\tfrac{3p_\infty(r+1)}{3+r}, 2p_\infty\right\} \quad \text{and} \quad r < \tfrac{5p_\infty-6}{2-p_\infty}.$$

Since $\frac{3p_\infty(r+1)}{3+r}$ is increasing in r we can always find a suitable r if p_∞, q fulfill

$$\max\left\{p_\infty, \tfrac{9-3p_\infty}{2}\right\} < q < \min\left\{6(p_\infty-1), 2p_\infty\right\}.$$

The existence of a suitable q is in turn equivalent to $p_\infty > \frac{7}{5}$. Hence we have shown that for all $p_\infty > \frac{7}{5}$ we can find suitable r and q, such that (5.23) is valid. For example we can choose $q = \frac{12}{5}$ and $r = \frac{5}{3}$. Before we can apply lemma 8.7 to ensure uniform estimates for $\mathbf{u}^N$, we have to take a look at the initial data, namely $\|(\nabla\mathbf{u}^N)(0)\|_q$ and $\|(\partial_t\mathbf{u}^N)(0)\|_2$. The first one is easily bounded by

$$\begin{aligned} \|(\nabla\mathbf{u}^N)(0)\|_q = \|\nabla P^N\mathbf{u}_0\|_q &\le C\,\|P^N\mathbf{u}_0\|_{1,2p_\infty} \\ &\le C\,\|P^N\mathbf{u}_0\|_{2,2} \le C\,\|\mathbf{u}_0\|_{2,2} \le C. \end{aligned}$$

To bound $(\partial_t\mathbf{u}^N)(0)$ let $\boldsymbol{\varphi} \in L^2(\Omega)$ with $\|\varphi\|_2 \le 1$, then

$$\begin{aligned} |\langle\partial_t\mathbf{u}^N, \varphi\rangle| &= |\langle\partial_t\mathbf{u}_0^N(0), P^N\varphi\rangle| \\ &= |\langle\operatorname{div}\mathbf{S}(\mathbf{D}\mathbf{u}_0^N)) + (\mathbf{u}_0^N\cdot\nabla)\mathbf{u}_0^N - \mathbf{f}^N(0), P^N\varphi\rangle| \\ &\le \|\operatorname{div}\mathbf{S}(\mathbf{D}\mathbf{u}_0^N)\|_2 + \|(\mathbf{u}_0^N\cdot\nabla)\mathbf{u}_0^N\|_2 + \|\mathbf{f}^N(0)\|_2 \\ &\le \|\nabla\mathbf{S}(\mathbf{D}\mathbf{u}_0^N)\|_2 + C\,\|\mathbf{u}_0\|_{2,2}^2 + \|\mathbf{f}^N(0)\|_2 \\ &\le C\,\|(\widetilde{D}\mathbf{u}_0^N)^{p-2}\nabla\mathbf{D}\mathbf{u}_0^N\|_2 + C\,\|\mathbf{u}_0\|_{2,2}^2 + \|\mathbf{f}^N(0)\|_2 \\ &\le C\,\|\nabla\mathbf{D}\mathbf{u}_0^N\|_2 + C\,\|\mathbf{u}_0\|_{2,2}^2 + \|\mathbf{f}^N(0)\|_2 \\ &\le C\,(\|\mathbf{u}_0\|_{2,2} + \|\mathbf{u}_0\|_{2,2}^2 + \|\mathbf{f}(0)\|_2) \le C. \end{aligned}$$

Here we have used that $\mathbf{f} \in L^\infty(I, W^{1,2}(\Omega))$ and that $\partial_t \mathbf{f} \in L^2(I, L^2(\Omega))$ implies $\mathbf{f} \in C(I, L^2(\Omega))$. Thus $\|(\partial_t \mathbf{u}^N)(0)\|_2 \le C$. So we can apply lemma 8.7 to (5.23) and get for small times $I' = [0, T']$

$$\|\partial_t \mathbf{u}^N\|_{L^\infty(I',L^2(\Omega))} + \|\nabla \mathbf{u}^N\|_{L^\infty(I',L^q(\Omega))} \tag{5.24}$$

$$+\|\mathcal{I}_\Phi(\mathbf{u}^N)\|_{L^r(I')} + \|\mathcal{J}_\Phi(\mathbf{u}^N)\|_{L^1(I')} \le C. \tag{5.25}$$

We use (5.21) to get rid of the r–dependence:

$$\|\mathcal{I}_\Phi(\mathbf{u}^N)\|_{L^{\frac{5p_\infty-6}{2-p_\infty}}(I')} \le C, \tag{5.26}$$

where $\frac{5p_\infty-6}{2-p_\infty} = \infty$ if $p_\infty = 2$.

In the next section we will show that these a priori estimates are by far enough to pass to the limit $N \to \infty$.

5. Strong Solutions

THEOREM 5.10. *Let* $\frac{7}{5} < p_\infty \le p_0 \le 2$ *and*

$$\|\mathbf{f}\|_{L^\infty(I,W^{1,2}(\Omega))} + \|\partial_t \mathbf{f}\|_{L^2(I,L^2(\Omega))} + \|\mathbf{u}_0\|_{W^{2,2}_{\mathrm{div}}(\Omega)} \le K.$$

Then there exists a constant $T' = T'(K)$ *with* $0 < T' < T$*, such that the system* (5.1) *has a strong solution* $\mathbf{u}$ *on* $I' = [0, T']$*. Further*

$$\begin{aligned} &\|\partial_t \mathbf{u}\|_{L^\infty(I',L^2(\Omega))} + \|\mathbf{u}\|_{L^\infty(I',W^{1,\frac{12}{5}}(\Omega))} \\ &+ \|\mathcal{J}_\Phi(\mathbf{u})\|_{L^1(I')} + \|\mathcal{I}_\Phi(\mathbf{u})\|_{L^{\frac{5p_\infty-6}{2-p_\infty}}(I')} \le C. \end{aligned} \tag{5.27}$$

PROOF. In sections 3 and 4 we have proven the existence of approximative solutions $\mathbf{u}^N$, which solve (5.10) and satisfy

$$\begin{aligned} &\|\partial_t \mathbf{u}^N\|_{L^\infty(I',L^2(\Omega))} + \|\nabla \mathbf{u}^N\|_{L^\infty(I',L^q(\Omega))} \\ &+\|\mathcal{I}_\Phi(\mathbf{u}^N)\|_{L^{\frac{5p_\infty-6}{2-p_\infty}}(I')} + \|\mathcal{J}_\Phi(\mathbf{u}^N)\|_{L^1(I')} \le C. \end{aligned} \tag{5.28}$$

Since $q = \frac{12}{5}$ and $r = \frac{5}{3}$ was an admissible choice within the derivation of the a priori estimates, we can assume $q \ge \frac{12}{5}$. Estimate 5.28 especially implies $\|\mathcal{I}_\Phi(\mathbf{u}^N)\|_{L^1(I')} \le C$, so by lemma 3.13 $|\nabla^2 \mathbf{u}^N|_{p(\cdot),I'\times\Omega} \le C$. Thus $\|\nabla^2 \mathbf{u}^N\|_{p(\cdot),I'\times\Omega} \le C$ and therefore $\|\mathbf{u}^N\|_{L^{p_\infty}(I',W^{2,p_\infty}(\Omega))} \le C$, since $\langle \mathbf{u}^N, 1\rangle = 0$. Overall we can pick a subsequence (still denoted by $\mathbf{u}^N$) with

$$\mathbf{u}^N \rightharpoonup \mathbf{u} \qquad \text{in } L^{p_\infty}(I', W^{2,p_\infty}(\Omega)), \tag{5.29}$$

$$\mathbf{u}^N \overset{*}{\rightharpoonup} \mathbf{u} \qquad \text{in } L^\infty(I', W^{1,\frac{12}{5}}(\Omega)), \tag{5.30}$$

$$\partial_t \mathbf{u}^N \overset{*}{\rightharpoonup} \partial_t \mathbf{u} \qquad \text{in } L^\infty(I', L^2(\Omega)), \tag{5.31}$$

where we have used that the weak limit of distributions on $I \times \Omega$ is unique. Since $W^{2,p_\infty}(\Omega) \hookrightarrow\hookrightarrow W^{1,2}(\Omega)$ for $p_\infty > \frac{7}{5}$, the lemma of Aubin–Lions (see appendix) implies the existence of a subsequence, such that

$$\nabla \mathbf{u}^N \to \nabla \mathbf{u} \qquad \text{in } L^2(I' \times \Omega). \tag{5.32}$$

As a consequence we get convergence of the convective term

$$(\mathbf{u}^N \cdot \nabla)\mathbf{u}^N \to (\mathbf{u} \cdot \nabla)\mathbf{u} \qquad \text{in } L^{\frac{4}{3}}(I' \times \Omega). \tag{5.33}$$

Observe that

$$\|\mathbf{S}(\mathbf{D}\mathbf{u}^N)\|_{L^2(I'\times\Omega)} \overset{(3.11)}{\leq} C\,\|(\widetilde{D}\mathbf{u}^N)^{p-1}\|_{L^2(I'\times\Omega)} \leq C\,(1+\|\boldsymbol{\nabla}\mathbf{u}^N\|_{L^2(I'\times\Omega)}) \leq C. \tag{5.34}$$

On the other hand by (5.32) $\mathbf{D}\mathbf{u}^N \to \mathbf{D}\mathbf{u}$ a.e. in $I'\times\Omega$, so

$$\mathbf{S}(\mathbf{D}\mathbf{u}^N) \to \mathbf{S}(\mathbf{D}\mathbf{u}) \qquad \text{a.e. in } I'\times\Omega \tag{5.35}$$

due to the continuity properties of $\mathbf{S}$. Now Vitali's theorem, (5.34), and (5.35) imply

$$\mathbf{S}(\mathbf{D}\mathbf{u}^N) \to \mathbf{S}(\mathbf{D}\mathbf{u}) \qquad \text{a.e. in } L^1(I'\times\Omega). \tag{5.36}$$

Choose $\boldsymbol{\omega}^r$ and $\varphi \in C_0^\infty(I')$, then we can conclude from (5.10), (5.31), (5.33), and (5.36) that

$$\int_{I'} \varphi\Big(\big\langle\partial_t\mathbf{u},\boldsymbol{\omega}^r\big\rangle + \langle\mathbf{S}(\mathbf{D}(\mathbf{u})),\mathbf{D}\boldsymbol{\omega}^r\rangle + \langle(\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u},\boldsymbol{\omega}^r\rangle\Big)\,dt = \int_{I'} \varphi\langle\mathbf{f},\boldsymbol{\omega}^r\rangle dt.$$

Furthermore $\mathbf{u}$ fulfills

$$\|\partial_t\mathbf{u}\|_{L^2(I'\times\Omega)} + \|\mathbf{S}(\mathbf{D}\mathbf{u})\|_{L^1(I'\times\Omega)} + \|(\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u}\|_{L^{\frac{4}{3}}(I'\times\Omega)} \leq C.$$

Since $\{\boldsymbol{\omega}^1,\boldsymbol{\omega}^2,\dots\}$ is dense in $W^{s,2}_{\operatorname{div}}(\Omega)$ and $W^{s,2}_{\operatorname{div}}(\Omega) \hookrightarrow W^{1,\infty}_{\operatorname{div}}(\Omega)$ for $s>\frac{5}{2}$, we deduce that

$$\int_{I'} \varphi\Big(\big\langle\partial_t\mathbf{u},\boldsymbol{\omega}\big\rangle + \langle\mathbf{S}(\mathbf{D}(\mathbf{u})),\mathbf{D}\boldsymbol{\omega}\rangle + \langle(\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u},\boldsymbol{\omega}\rangle\Big)\,dt = \int_{I'} \varphi\langle\mathbf{f},\boldsymbol{\omega}\rangle dt$$

is fulfilled for all $\boldsymbol{\omega} \in W^{s,2}_{\operatorname{div}}(\Omega)$, especially for all $\boldsymbol{\omega}\in\mathcal{V}$. Note that

$$\big\langle\partial_t\mathbf{u},\boldsymbol{\omega}\big\rangle, \langle\mathbf{S}(\mathbf{D}(\mathbf{u})),\mathbf{D}\boldsymbol{\omega}\rangle, \langle(\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u},\boldsymbol{\omega}\rangle, \langle\mathbf{f},\boldsymbol{\omega}\rangle \in L^1(I')$$

so

$$\big\langle\partial_t\mathbf{u},\boldsymbol{\omega}\big\rangle + \langle\mathbf{S}(\mathbf{D}(\mathbf{u})),\mathbf{D}\boldsymbol{\omega}\rangle + \langle(\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u},\boldsymbol{\omega}\rangle = \langle\mathbf{f},\boldsymbol{\omega}\rangle \tag{5.37}$$

for all $\boldsymbol{\omega}\in\mathcal{V}$ and a.e. $t\in I'$. It remains to show that $\mathbf{u}(0)=\mathbf{u}_0$. But this follows from the parabolic embedding

$$\begin{aligned}\|P^N\mathbf{u}_0 - \mathbf{u}(0)\|_2 &= \|\mathbf{u}^N(0)-\mathbf{u}(0)\|_2 \\ &\leq C\,\underbrace{\|\mathbf{u}^N-\mathbf{u}\|^{\frac{1}{2}}_{L^2(I',L^2(\Omega))}}_{\to 0}\,\underbrace{\|\partial_t\mathbf{u}^N-\partial_t\mathbf{u}\|^{\frac{1}{2}}_{L^2(I',L^2(\Omega))}}_{\leq C} \to 0.\end{aligned} \tag{5.38}$$

Since $P^N\mathbf{u}_0 \to \mathbf{u}_0$ in $L^2(\Omega)$ we get $\mathbf{u}(0)=\mathbf{u}_0$. Overall we have shown by (5.37) and (5.38) that $\mathbf{u}$ satisfies (5.1) in the weak sense. It remains to prove the norm estimates for $\mathbf{u}$, $\mathcal{I}_\Phi(\mathbf{u})$ and $\mathcal{J}_\Phi(\mathbf{u})$. First of all from (5.30) and (5.31) there follows

$$\|\partial_t\mathbf{u}\|_{L^\infty(I',L^2(\Omega))} + \|\mathbf{u}\|_{L^\infty(I',W^{1,\frac{12}{5}}(\Omega))} \leq C.$$

Define $H : I\times\Omega\times\mathbb{R}^{d\times d}\times\mathbb{R}^{d\times d}$ by

$$H(t,x,\mathbf{y},\mathbf{z}) := \sum_{jk\alpha\beta}(\partial_{\alpha\beta}\partial_{jk}\Phi)(t,x,\mathbf{y})\,z_{\alpha\beta}\,z_{jk},$$

then

(a) $H\geq 0$,

(b) H is measurable in (t,x) for all $\mathbf{y},\mathbf{z}$,
(c) H is continuous in $\mathbf{z}$ and $\mathbf{y}$ for almost every $(t,x)\in I'\times\Omega$,
(d) H is convex in $\mathbf{z}$ for all $\mathbf{y}$ and almost every $(t,x)\in I'\times\Omega$.

Furthermore

$$\|\mathcal{J}_\Phi{}^A(\mathbf{u}^N)\|_{L^1(I',L^1(\Omega))} = \Big\|H(\mathbf{Du}^N,\partial_t\mathbf{Du}^N)\Big\|_{L^1(I'\times\Omega)}, \tag{5.39}$$

$$\|\mathcal{I}_\Phi{}^A(\mathbf{u}^N)\|_{L^1(I',L^1(\Omega))} = \Big\|\sum_{k=1}^d H(\mathbf{Du}^N,\partial_k\mathbf{Du}^N)\Big\|_{L^1(I'\times\Omega)}. \tag{5.40}$$

Due to lemma 5.3 and $\mathcal{I}_\Phi(\mathbf{u}^N)_{L^1(I)}+\mathcal{J}_\Phi(\mathbf{u}^N)_{L^1(I)}\le C$ we have

$$\|\partial_t\mathbf{u}^N\|_{L^{p_\infty}(I,W^{1,\frac{3p_\infty}{p_\infty+1}}(\Omega))}\le C.$$

Thus we can pass to a subsequence (still denoted by $\mathbf{u}^N$) with

$$\partial_t\boldsymbol{\nabla}\mathbf{u}^N \rightharpoonup \partial_t\boldsymbol{\nabla}\mathbf{u} \qquad \text{in } L^{p_\infty}(I',L^{\frac{3p_\infty}{p_\infty+1}}(\Omega)). \tag{5.41}$$

Note (5.41), (5.29), and (5.32) imply

$$\begin{aligned}
\boldsymbol{\nabla}^2\mathbf{u}^N &\rightharpoonup \boldsymbol{\nabla}^2\mathbf{u} && \text{in } L^1(I'\times\Omega),\\
\partial_t\boldsymbol{\nabla}\mathbf{u}^N &\rightharpoonup \partial_t\boldsymbol{\nabla}\mathbf{u} && \text{in } L^1(I'\times\Omega),\\
\boldsymbol{\nabla}\mathbf{u}^N &\to \boldsymbol{\nabla}\mathbf{u} && \text{in } L^1(I'\times\Omega).
\end{aligned}$$

Thus from the semicontinuity theorem of De Giorgi ([GMS98], pg. 132), (5.28), and (5.39) deduce

$$\|\mathcal{J}_\Phi{}^A(\mathbf{u})\|_{L^1(I')}\le C. \tag{5.42}$$

Furthermore H, $\mathbf{Du}^N$, $\partial_k\mathbf{Du}^N$ fulfill all the requirements of corollary 8.12. Thus we deduce from (5.28) and corollary 8.12

$$\|\mathcal{I}_\Phi(\mathbf{u})\|_{L^{\frac{5p_\infty-6}{2-p_\infty}}(I')}\le C. \tag{5.43}$$

This proves the theorem. □

The next corollary shows what regularity for $\mathbf{u}$ can be deduced from (5.27). This justifies that we call $\mathbf{u}$ a “strong” solution.

Corollary 5.11. *Let $\mathbf{u}$ be the solution of theorem 5.10, then*

$$\begin{aligned}
\mathbf{u} &\in L^{\frac{p_\infty(5p_\infty-6)}{2-p_\infty}}(I',W^{2,\frac{3p_\infty}{p_\infty+1}}(\Omega)),\\
\partial_t^2\mathbf{u} &\in L^2(I',(W^{1,2}_{\operatorname{div}}(\Omega))'),\\
(\widetilde{D}\mathbf{u})^{\frac{p}{2}} &\in C(I',L^{\frac{12(p_\infty-1)}{p_\infty},\frac{4(p_\infty-1)}{2-p_\infty}}) \qquad \textit{(Lorentz space)}.
\end{aligned}$$

For all $1\le s<6(p_\infty-1)$ there holds

$$\mathbf{u}\in C(I',W^{1,s}(\Omega)).$$

Furthermore there exists a pressure π with

$$\boldsymbol{\nabla}\pi\in L^{\frac{2(5p_\infty-6)}{2-p_\infty}}(I',L^2(\Omega))$$

such that

$$\partial_t \mathbf{u} - \operatorname{div}(\mathbf{S}(\mathbf{Du})) + (\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u} + \boldsymbol{\nabla}\pi = \mathbf{f} \tag{5.44}$$

a.e. in $I' \times \Omega$.

PROOF. From (5.27) and lemma 3.12 we deduce that

$$(\widetilde{D}\mathbf{u})^{\frac{p}{2}} \in L^{\frac{2(5p_\infty-6)}{2-p_\infty}}(I, W^{1,2}(\Omega)),$$
$$\partial_t\big((\widetilde{D}\mathbf{u})^{\frac{p}{2}}\big) \in L^2(I, L^2(\Omega)).$$

Thus by theorem 8.21 with $\theta = \frac{2-p_\infty}{4(p_\infty-1)}$ we get

$$\begin{aligned}(\widetilde{D}\mathbf{u})^{\frac{p}{2}} &\in C(I, [W^{1,2}(\Omega), L^2(\Omega)]_{\theta,\frac{1}{\theta}})\\ &= C(I, B^{\frac{5p_\infty-6}{4(p_\infty-1)},2}_{\frac{4(p_\infty-1)}{2-p_\infty}}(\Omega)) \qquad \text{Besov Space}\\ &\hookrightarrow C(I, L^{\frac{12(p_\infty-1)}{p_\infty},\frac{4(p_\infty-1)}{2-p_\infty}}(\Omega)) \quad \text{Lorentz Space.}\end{aligned}$$

For more details regarding Besov spaces and Lorentz spaces see Bergh, Löfström [**BL76**] and Triebel [**Tri78**]. Let $1 \le s < 6(p_\infty - 1)$, then

$$(\widetilde{D}\mathbf{u})^{\frac{p}{2}} \in C(I, L^{\frac{12(p_\infty-1)}{p_\infty},\frac{4(p_\infty-1)}{2-p_\infty}}(\Omega)) \hookrightarrow C(I, L^{\frac{2s}{p_\infty}}(\Omega)).$$

As a consequence $\|(\widetilde{D}\mathbf{u})^{\frac{p}{2}}\|_{\frac{2s}{p_\infty}} \in C(I')$, so

$$\begin{aligned}&\int_\Omega (\widetilde{D}\mathbf{u})^{\frac{ps}{p_\infty}}\,dx \in C(I')\\ \Rightarrow\quad &\big|(\widetilde{D}\mathbf{u})^2\big|_{\frac{p(\cdot)s}{2p_\infty}} \in C(I')\\ \Rightarrow\quad &\big||\mathbf{Du}|^2\big|_{\frac{p(\cdot)s}{2p_\infty}} \in C(I')\\ \Rightarrow\quad &\big|\mathbf{Du}\big|_{\frac{p(\cdot)s}{p_\infty}} \in C(I')\\ \Rightarrow\quad &\big\|\mathbf{Du}\big\|_{\frac{p(\cdot)s}{p_\infty}} \in C(I').\end{aligned}$$

Since $L^{\frac{p(\cdot)s}{p_\infty}}(\Omega)$ is uniformly convex, this implies

$$\mathbf{Du} \in C(I', L^{\frac{p(\cdot)s}{p_\infty}}(\Omega)) \hookrightarrow C(I', L^s(\Omega)).$$

From Korn's inequality we deduce

$$\mathbf{u} \in C(I', W^{1,s}(\Omega)).$$

From $\|(\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u}\|_2 \le \|\mathbf{u}\|_{1,\frac{12}{5}}$ and the choice $s := \frac{12}{5} < 6(p_\infty - 1)$ we deduce

$$(\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u} \in C(I, L^2(\Omega)).$$

From (5.27) and lemma 5.3 we deduce that

$$\|\mathbf{u}\|_{L^{\frac{p_\infty(5p_\infty-6)}{2-p_\infty}}(I, W^{2,\frac{3p_\infty}{p_\infty+1}}(\Omega))} \le C.$$

Further note that

$$|\boldsymbol{\nabla}(\mathbf{S}(\mathbf{Du}))| \le C\,(\widetilde{D}\mathbf{u})^{p-2}|\boldsymbol{\nabla}\mathbf{Du}| + (\widetilde{D}\mathbf{u})^{p-1}\ln(\widetilde{D}\mathbf{u})|\boldsymbol{\nabla}p|,$$

so

$$\|\boldsymbol{\nabla}(\mathbf{S}(\mathbf{Du}))\|_2 \le C\,\mathcal{I}_\Phi(\mathbf{u})^{\frac12} + C\,\|\widetilde{D}\mathbf{u}\|_{\frac{12}{5}}.$$

Thus $\mathbf{S}(\mathbf{Du}) \in L^{\frac{2p_\infty(5p_\infty-6)}{2-p_\infty}}(I, W^{1,2}(\Omega))$. We have shown that all the terms $\partial_t\mathbf{u}$, $-\operatorname{div}(\mathbf{S}(\mathbf{Du}))$, and $(\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u}$ in (5.37) are in $L^{\frac{2(5p_\infty-6)}{2-p_\infty}}(I, L^2(\Omega))$. Thus De Rahm's theorem ensures the existence of a pressure π with $\boldsymbol{\nabla}\pi \in L^{\frac{2(5p_\infty-6)}{2-p_\infty}}(I, L^2(\Omega))$. From

$$|\partial_t(\mathbf{S}(\mathbf{Du}))| \le C\,(\widetilde{D}\mathbf{u})^{p-2}|\partial_t\mathbf{Du}| + (\widetilde{D}\mathbf{u})^{p-1}\ln(\widetilde{D}\mathbf{u})|\partial_t p|$$

and lemma 3.12 we deduce

$$\|\partial_t(\mathbf{S}(\mathbf{Du}))\|_2^2 \le C\,\mathcal{J}_\Phi(\mathbf{u}) + C\,\|\widetilde{D}\mathbf{u}\|_{\frac{12}{5}}^{\frac{12}{5}} + C.$$

This proves the corollary. □

6. Uniqueness

THEOREM 5.12. *Let $\frac75 < p_\infty \le p_0 \le 2$ and let $\mathbf{u}$ and $\mathbf{v}$ be weak solutions of (5.1) with*

$$\mathbf{u},\mathbf{v} \in C(I, W^{1,\frac{12}{5}}(\Omega)).$$

Then $\mathbf{u} = \mathbf{v}$.

PROOF. Let $\mathbf{e} := \mathbf{u} - \mathbf{v}$. We take the difference of the equations of $\mathbf{u}$ and $\mathbf{v}$ and use $\mathbf{e}$ as a test function, then

$$\langle \partial_t\mathbf{e}, \mathbf{e}\rangle + \langle \mathbf{S}(\mathbf{Du}) - \mathbf{S}(\mathbf{Dv}), \mathbf{Du} - \mathbf{Dv}\rangle + \langle (\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u} - (\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v}, \mathbf{e}\rangle = 0.$$

This reduces to

$$\tfrac12 d_t\|\mathbf{e}\|_2^2 + \langle \mathbf{S}(\mathbf{Du}) - \mathbf{S}(\mathbf{Dv}), \mathbf{Du} - \mathbf{Dv}\rangle \le \big|\langle (\mathbf{e}\cdot\boldsymbol{\nabla})\mathbf{u}, \mathbf{e}\rangle\big|. \tag{5.45}$$

Since $p_\infty > \frac75$, there exists $q > \frac85$ with

$$\frac{2-p}{2}\cdot\frac{2q}{2-q} < \frac{12}{5}.$$

Thus

$$\|(\widetilde{D}\mathbf{u})^{\frac{2-p}{2}} + (\widetilde{D}\mathbf{v})^{\frac{2-p}{2}}\|_{\frac{2q}{2-q}} \in L^\infty(I).$$

Lemma 3.15 implies

$$\|\mathbf{De}\|_q \le C\,\big\langle \mathbf{S}(\mathbf{Du}) - \mathbf{S}(\mathbf{Dv}), \mathbf{Du} - \mathbf{Dv}\big\rangle^{\frac12}.$$

Korn's inequality implies $\|\mathbf{e}\|_{\frac{3q}{3-q}} \le C\,\|\mathbf{e}\|_{1,q} \le C\,\|\mathbf{De}\|_q$. So by (5.45)

$$\tfrac12 d_t\|\mathbf{e}\|_2^2 + c\,\|\mathbf{e}\|_{\frac{3q}{3-q}}^2 \le \big|\langle (\mathbf{e}\cdot\boldsymbol{\nabla})\mathbf{u}, \mathbf{e}\rangle\big| \le \|\mathbf{e}\|_{\frac{24}{7}}^2\|\mathbf{u}\|_{1,\frac{12}{5}} \le C\,\|\mathbf{e}\|_{\frac{24}{7}}^2.$$

Since $q > \frac85$ there holds $2 < \frac{3q}{3-q} < \frac{24}{7}$ and $L^{\frac{24}{7}}(\Omega) = [L^2(\Omega), L^{\frac{3q}{3-q}}(\Omega)]_\theta$ with $0 < \theta < 1$. Thus for $\delta > 0$

$$\tfrac12 d_t\|\mathbf{e}\|_2^2 + c\,\|\mathbf{e}\|_{\frac{3q}{3-q}}^2 \le C\,\|\mathbf{e}\|_2^{2(1-\theta)}\|\mathbf{e}\|_{\frac{3q}{3-q}}^{2\theta} \le C_\delta\|\mathbf{e}\|_2^2 + \delta\|\mathbf{e}\|_{\frac{3q}{3-q}}^2.$$

Gronwall's inequality implies $\mathbf{e} = 0$, i.e. $\mathbf{u} = \mathbf{v}$. □

Note that for $p_\infty > \frac{7}{5}$ we have derived for small times the existence of a strong solution $\mathbf{u}$ with $\mathbf{u} \in C(I', W^{1,s}(\Omega))$ for all $1 \leq s < 6(p_\infty - 1)$. Especially this solution satisfies $\mathbf{u} \in C(I', W^{1,\frac{12}{5}}(\Omega))$. The theorem 5.12 above ensures that this solution is unique within the class of strong solutions in $C(I', W^{1,\frac{12}{5}}(\Omega))$. It is interesting to observe that the uniqueness as proven above exactly holds up to the same bound $p_\infty > \frac{7}{5}$ for which we have derived the existence of such solutions.

CHAPTER 6

Time Discretization – Nonlinear Stabilization

1. Introduction

In the last chapter we have studied the instationary p-Navier–Stokes problem in three space dimensions

$$\begin{aligned} \partial_t \mathbf{u} - \operatorname{div}(\mathbf{S}(\mathbf{D}\mathbf{u})) + (\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u} + \boldsymbol{\nabla}\pi = \mathbf{f}, &\quad \text{on } I\times\Omega,\\ \operatorname{div}\mathbf{u} = 0, &\quad \text{on } I\times\Omega,\\ \mathbf{u}(0) = \mathbf{u}_0 &\quad \text{on } \Omega, \end{aligned} \tag{6.1}$$

where Ω is the three dimensional torus and $\mathbf{S}$ is induced by a p–potential. We have studied existence of strong solutions, its regularity and its uniqueness. Based on these results we will now develop a numerical scheme to approximate the system by a time discretized version. Thereafter we will derive error estimates in terms of the time step size k.

The first investigations regarding time discretizations of system (6.1) have been made by A. Prohl and M. Růžička. In [**PR01**] they examine the following time discretized version

$$\begin{aligned} d_t\mathbf{v}^m - \operatorname{div}(\mathbf{S}(\mathbf{D}\mathbf{v}^m)) + (\mathbf{v}^m\cdot\boldsymbol{\nabla})\mathbf{v}^m + \boldsymbol{\nabla}q^m = \mathbf{f}^m, &\quad \text{on } I_k\times\Omega,\\ \operatorname{div}\mathbf{v}^m = 0, &\quad \text{on } I_k\times\Omega,\\ \mathbf{v}(0) = \mathbf{u}_0 &\quad \text{on } \Omega, \end{aligned} \tag{6.2}$$

where d_t is the discrete time derivative with respect to the uniform step size k, i.e. $d_t\mathbf{v}^m = \frac{1}{k}(\mathbf{v}^m - \mathbf{v}^{m-1})$. I_k is the set of time steps within the time interval I, which comes from (6.1), i.e. $I_k = \{0, k, 2k, \ldots\} \cap I$. Further $\mathbf{f}^m := \mathbf{f}(t_m)$, where t_m is the time of the m-th time step, i.e. $t_m := km$. This kind of discretization is well known from the classical Navier–Stokes equations and is usually referred to as the "fully implicit" scheme. This name is due to the way the convection term is handled. "Fully implicit" means that the full convective term is treated on the time level of the new time step. In contrast we refer to a "semi implicit" scheme if the convective term is discretized by $(\mathbf{v}^{m-1}\cdot\boldsymbol{\nabla})\mathbf{v}^m$.

From a theoretical point of view the "fully implicit" discretization is the most basic one. The nonlinear discretization of the convective term is as close as possible to the original system, which is very convenient for the analysis. For this kind of discretization we expect, as in the case of the classical Navier–Stokes equations, optimal error estimates. Indeed it is easy to show that we have optimal control of the error if we assume that there exists an arbitrary smooth solution to the continuous problem (6.1) (for example $\partial_t^2\mathbf{u} \in L^2(I, (W^{1,p}_{\operatorname{div}}(\Omega))')$ suffices). But we will not do so for several reasons. One reason is that arbitrary smooth solutions require arbitrary smooth data $\mathbf{f}$, $\mathbf{u}_0$ and p. In the view of applications this is not realistic and error estimates of this kind would lack the connection to real problems. But most important of all it is not known if the continuous problem actually has arbitrary smooth solutions for smooth data. This is not a problem of the convection or the pressure but of the nonlinear main part $-\operatorname{div}(\mathbf{S}(\mathbf{Du}))$. The nonstandard growth condition of $\mathbf{S}$ in combination with the sole dependence on the symmetric part of the gradient generates technical difficulties that have not been solved yet. So it is not known for space dimension $d \geq 3$ even for p constant, no convection and no constraint $\operatorname{div} \mathbf{u} = 0$, i.e. no pressure, if there exist solutions with $\partial_t^2 \mathbf{u} \in L^2(I, (W^{1,p}_{\operatorname{div}}(\Omega))')$. So as a basement of our considerations we will only assume such regularity of $\mathbf{u}, \pi$ as has been proven in chapter 5 for short time solutions. This basic assumption has also been used by A. Prohl and M. Růžička.

In the study of time discretized problem there arise several questions of fundamental importance. Most important is the unique solvability of the discretized problem. Certainly it is also important to know what kind of solutions (weak or strong) the discretized problem has, because the corresponding regularity is very important to the error analysis. In fact we will see that for different lower bounds of p_∞ we will have different type (weak/strong) of solutions implying different error estimates: Smaller values of p_∞ enforce the concept of weak solutions with worse error estimates. Moreover, strong solutions to the discretized problem are also important as a base for a space–time discretization. In [**PR01**] it is indicated that the space discretization requires strong solutions.

Let us summarize the results of A. Prohl and M. Růžička [**PR01**], who considered system (6.1) for p constant, since this work is intended to be an extension of theirs. Especially we will overcome some of the difficulties which they have encountered. Additionally, we will consider the case of time and space dependent exponent p, which has not yet been investigated before in numerics, but is important for the applicability to the underlying physical problem. A. Prohl and M. Růžička show that the continuous problem has a strong solution and the data satisfies some regularity conditions, which are slightly weaker than the ones assumed for the existence of the continuous problem in chapter 5. They have shown that for $\frac{5}{3} < p \leq 2$ there exists a weak solution $\mathbf{v}^m, q^m$ of (6.1). If $\frac{3+\sqrt{29}}{5} < p \leq 2$ this solution further satisfies

$$\|\mathbf{e}^m\|^2_{l^\infty(I_k, L^2(\Omega))} + \|\mathbf{e}^m\|^2_{l^2(I_k, W^{1,p}(\Omega))} \leq c\, k^{2\alpha(p)} \tag{6.3}$$

with

$$\alpha(p) = \frac{5p-6}{2p},$$

where $\mathbf{e}^m := \mathbf{u}(t_m) - \mathbf{v}^m$ denotes the error. Note that $\frac{3+\sqrt{29}}{5} \approx 1.677$. Furthermore they have proved that for $\frac{9}{5} < p \leq 2$ there exists a strong solution $\mathbf{v}^m, q^m$ (i.e. information about second derivatives), which satisfies the same error estimate. Note that although not mentioned in [**PR01**] the same estimates hold true with $\alpha(p) = \frac{5p-6}{2p}$ replaced by $\alpha(p) = \frac{5p-6}{4(p-1)}$.

It is remarkable that the value of p determines both the existence of weak/strong solutions and the order of convergence. This phenomenon has its origin in the natural energy associated with the nonlinear main part $-\operatorname{div}(\mathbf{S}(\mathbf{Du}))$, namely

$$\int_I \int_\Omega \mathbf{S}(\mathbf{Du}) \cdot \mathbf{Du}\, dx\, dt.$$

The inherent information about $\|\mathbf{Du}\|_{L^{p(\cdot)}(I\times\Omega)}$ becomes less significant with decreasing p, making it more difficult to control the convective term. But here we have to distinguish between two aspects: In order to get any kind of estimates – a priori or error estimates – we have to control the error somehow. This necessity restricts the range of admissible p, especially it determines the lower bound for p: Like $\frac{5}{3}$ for weak solutions, $\frac{3+\sqrt{29}}{5}$ for an error estimate, and $\frac{9}{5}$ for strong solutions in the context of [**PR01**]. The other aspect is that p determines the order of convergence, i.e. the error estimated in terms of the step size k. This effect would even occur in the absence of pressure and convection. It is important to emphasize and distinguish these two aspects in order to get a better comprehension of the problem. This for example implies that in the case of the parabolic p–Laplacian equation (no pressure and no convection) the same error estimates hold true for the full range $1 < p \leq 2$. In this point of view the results of A. Prohl and M. Růžička improve the results of J. W. Barrett and W. B. Liu [**BL93**] and [**BL94**], who proved error estimates for a space and time discretization of the parabolic p–Laplacian equation, where $k^{2\alpha(p)}$ in (6.3) is replaced by $h^p + k$. It must be mentioned, of course, that J. W. Barrett and W. B. Liu do not consider the simplified space periodic case.

The main weakness of system (6.2) as an approximation of (6.1) is that only for a rather small range of p namely $\frac{9}{5} < p \leq 2$ there exist strong solutions of (6.2). But strong solutions seem to be a necessity for an additional space discretization, so this range is of great importance for the computations and simulations. Since the physically relevant case rather involves small values of p, this range and even the one for the existence of weak solutions is too small. A physically reasonable range of p would for example be $(1.3, 2]$. One of the main goals of this work is therefore to extend the range of admissible p providing the existence of weak and strong solutions and error estimates significantly.

Another important aspect of strong solutions are pointwise in time error estimates. While for weak solutions it is only possible to extract the pointwise in time error estimate from its integrated version (6.3), strong solutions also fulfill an additional pointwise in time estimate. This technique will improve the error estimate

$$\|\mathbf{e}\|^2_{l^\infty(I_k,W^{1,p}(\Omega))} \leq c\, k^{2\alpha(p)-1}$$

to

$$\|\mathbf{e}\|^2_{l^\infty(I_k,W^{1,p}(\Omega))} \leq c\, k^{\alpha(p)}.$$

Note that this result is also new and cannot be found in [**PR01**] but also applies to the scheme (6.2) used therein.

The problem in passing from weak to strong solutions for system (6.2) is that the a priori and the error estimate for weak solutions only imply that

$$\|\nabla \mathbf{v}^m\|_{L^\infty(I,W^{1,p}(\Omega))} \leq C. \tag{6.4}$$

Although this information is better than the regularity known for weak solutions of the continuous problem (6.1), it is not sufficient to pass to strong solutions in the time discretized problem. The reason for this is that in the continuous setting we can use a local version of Gronwall's lemma enabling us to estimate higher powers for small times (see appendix), while in the time discretized setting we cannot. But there is a way to overcome this problem. Additionally to (6.4) we know that the error $\mathbf{e}$

measured in the same norm is not only bounded but still has some smallness in terms of a power of k, i.e.

$$\|\nabla \mathbf{e}^m\|_{L^\infty(I,W^{1,p}(\Omega))} \le C k^{2\alpha(p)-1}. \tag{6.5}$$

So if we can extract a priori some higher regularity information for $\mathbf{v}^m$ in possibly negative powers of k, we can use this smallness of the error and the a priori estimates for the continuous problem to extract more information about $\mathbf{v}^m$. Unfortunately there is no higher regularity information for weak solutions of system (6.2). Therefore A. Prohl, M. Růžička, and L. Diening proposed to stabilize the system slighty by $-k^{\alpha(p)}\Delta\mathbf{v}^m$. This way the order of convergence is not reduced and it is possible to extract higher regularity (in negative powers of k) from the a priori estimate of weak solutions. Overall this will enlarge the range of p for which to get strong solutions to approximately $[1.6955, 2]$. This concept is the subject of a publication to appear shortly.

The linearity of the stabilization $k^{\alpha(p)}\Delta\mathbf{v}^m$ seems to be very handy when it comes to computations, since not many costs are involved. But on the other hand the nonlinear main part $-\operatorname{div}(\mathbf{S}(\mathbf{Du}))$ induces a nonlinear problem at every time step. So there is no need to use a linear stabilization, as long as the stabilization does not increase the costs significantly. For example it is also possible to add $-k^{\alpha(p)}\operatorname{div}((\widetilde{D}\mathbf{u})^{q-2}\mathbf{Du})$ for stabilization with a suitable choice of q. Most of the work required to calculate the matrices of a finite element discretization of this term is already included in the finite element discretization of $-\operatorname{div}(\mathbf{S}(\mathbf{Du}))$. So the additional costs are rather low. First considerations have shown that the choice $q = \frac{6(p-1)^2}{p} + 1$ will improve the range of strong solutions to approximately $p \in [1.6, 2]$. This kind of stabilization will also be subject of the publication by A. Prohl, M. Růžička and L. Diening.

But here in this work we will proceed differently. Instead of adding some stabilizing term to the system, we will stabilize the extra stress $\mathbf{S}$ itself. Namely we will replace $\mathbf{S}$ by its A–approximation $\mathbf{S}^A$. Note that the A–approximation modifies $\mathbf{S}$ only for large values of $|\mathbf{Dv}^m|$. So the scheme will only be stabilized where there is the need to. If the computations show that the solution has bounded symmetric gradients, then there will be no stabilization at all. This is a fundamental difference to the approximations $-k^{\alpha(p)}\Delta\mathbf{v}^m$ and $-k^{\alpha(p)}\operatorname{div}((\widetilde{D}\mathbf{v}^m)^{q-2}\mathbf{Dv}^m)$ mentioned above: Imagine that there exists a unique, regular solution $\mathbf{u} \in C^{1,\beta}(I \times \Omega)$ to (6.1) although we can only prove less regularity. Then (for suitable large A corresponding to sufficiently small time steps) the A–approximation does not modify the original system and in addition we get optimal order of convergence, i.e. k^2 in (6.3). On the other hand stabilizing with $-k^{\alpha(p)}\Delta\mathbf{v}^m$ and $-k^{\alpha(p)}\operatorname{div}((\widetilde{D}\mathbf{v}^m)^{q-2}\mathbf{Dv}^m)$ requires that we adjust $\alpha(p)$ and q in advance. If we want to get the widest range for p to guarantee strong solutions, then we have to stabilize exactly with the order of convergence we expect. So if we expect $k^{2\alpha(p)}$ as the order of convergence for (6.3), then we have to use $-k^{\alpha(p)}\Delta\mathbf{v}^m$, i.e. half the exponent for k. But this implies that even for a smooth solution $\mathbf{u}$ the optimal order of the stabilized system reduces from k^2 to $k^{2\alpha(p)}$. So it might be that we accidentally decrease the order of convergence, just because we do not know the optimal regularity of $\mathbf{u}$. This cannot happen with the A–approximation! If on the other hand we use $-k\Delta\mathbf{v}^m$ to prevent this problem, then the range of p for strong solutions derived by the theory would reduce to approximately $[1.763, 2]$. But this is almost no gain compared to $[1.8, 2]$ in the case of no stabilization at all.

Summarized we can say that the A–approximation as suggested below provides the widest improvement in the range of p guaranteeing strong solutions and will not decrease the optimal order of convergence if $\mathbf{u}$ is smooth. In addition we will see that it is quite easy to transfer the error estimates of the fully implicit, A–approximated scheme to the semi implicit, A–approximated scheme. So the nonlinearity due to the convective term can be eliminated.

2. The Scheme

Let $\mathbf{u}, \pi$ be the unique strong solution of (6.1) as described in theorem 5.10 with $\mathbf{u}_0, \mathbf{f}, \mathbf{S}, p$ fulfilling the necessary requirements. Let I be the respective time interval of existence. Then we propose the following scheme as a time discretization of (6.1)

$$\begin{aligned} d_t\mathbf{v}^m - \operatorname{div}(\mathbf{S}^A(\mathbf{D}\mathbf{v}^m)) + (\mathbf{v}^m \cdot \boldsymbol{\nabla})\mathbf{v}^m + \boldsymbol{\nabla} q^m &= \mathbf{f}^m \quad \text{on } I_k \times \Omega, \\ \operatorname{div} \mathbf{v}^m &= 0 \quad \text{on } I_k \times \Omega, \\ \mathbf{v}(0) &= \mathbf{u}_0 \quad \text{on } \Omega \end{aligned} \tag{6.6}$$

with $m = 1, 2, \ldots, M$ and a suitable choice of $A \geq 1$ depending on the step size k. Hereby Ω denotes the three dimensional torus, k the time step size and I_k the set of time steps included in the time interval I of existence, i.e. $I_k := \{0, k, 2k, \ldots\} \cap I = \{0, k, 2k, \ldots, Mk\}$. Further let $\mathbf{f}^m := \mathbf{f}(t_m)$. Note that as in the continuous case (6.1) we compensate the missing boundary condition by only looking at solutions with mean value zero.

To get a better understanding of the problem we will rewrite (6.1) in a time discretized form. Let

$$\begin{aligned} \mathbf{u}^m &:= \mathbf{u}(t_m), & \pi^m &:= \pi(t_m), \\ \mathbf{f}^m &:= \mathbf{f}(t_m), & \mathbf{R}^m &:= d_t(\mathbf{u}(t_m)) - \partial_t \mathbf{u}(t_m), \end{aligned}$$

then $\mathbf{u}^m, \pi^m$ is the solution of

$$\begin{aligned} d_t\mathbf{u}^m - \operatorname{div}(\mathbf{S}(\mathbf{D}\mathbf{u}^m)) + \boldsymbol{\nabla}\pi^m + (\mathbf{u}^m \cdot \boldsymbol{\nabla})\mathbf{u}^m &= \mathbf{f}^m + \mathbf{R}^m \quad \text{on } I_k \times \Omega, \\ \operatorname{div} \mathbf{u}^m &= 0 \quad \text{on } I_k \times \Omega, \\ \mathbf{u}^0 &= \mathbf{u}_0 \quad \text{on } \Omega, \end{aligned} \tag{6.7}$$

with $m = 1, 2, \ldots, M$. In this way we immediately realize the connection of (6.6) and (6.7). The differences consist in replacing $\mathbf{S}$ by $\mathbf{S}^A$ and the error term $\mathbf{R}^m$, which arises from the approximation of $\partial_t \mathbf{u}^m$ by $d_t \mathbf{u}^m$.

We have seen in theorem 5.10 and corollary 5.11 that $\mathbf{u}$ has at least the following regularity

$$\begin{aligned} \|\partial_t \mathbf{u}\|_{L^\infty(I, L^2_0(\Omega))} + \|\partial_t^2 \mathbf{u}\|_{L^2(I, (W^{1,2}_{\operatorname{div}}(\Omega))^*)} &\leq C, \\ \big\|(\widetilde{D}\mathbf{u})^{\frac{p}{2}}\big\|_{C(I', L^{\frac{12(p_\infty-1)}{p_\infty}, \frac{4(p_\infty-1)}{2-p_\infty}}(\Omega))} &\leq C \qquad \text{(Lorentz Space)}, \end{aligned}$$

where C certainly depends on the data. But since the regularity of $\mathbf{u}$ might be better we introduce a parameter $r \geq \frac{4}{p_\infty}$ and require

$$\begin{aligned} \|\partial_t \mathbf{u}\|_{L^\infty(I, L^2_0(\Omega))} + \|\partial_t^2 \mathbf{u}\|_{L^2(I, (W^{1,2}_{\operatorname{div}}(\Omega))^*)} &\leq C, \\ \big\|(\widetilde{D}\mathbf{u})^{\frac{p}{2}}\big\|_{C(I', L^{r,\infty}(\Omega))} &\leq C \qquad \text{(Lorentz Space)}, \end{aligned} \tag{6.8}$$

in order to be more general. Nevertheless we will always show how the error estimates will look for the special case $r = \frac{12(p_\infty - 1)}{p_\infty}$. Since $(\widetilde{D}\mathbf{u})^{\frac{p}{2}} \geq |\widetilde{D}\mathbf{u}|^{\frac{p_\infty}{2}}$ and Lorentz spaces are Banach lattices, we have

$$\big\|\widetilde{D}\mathbf{u}\big\|_{C(I', L^{\frac{rp_\infty}{2},\infty}(\Omega))} \leq C \qquad \text{(Lorentz Space)}.$$

Let us point out once more one of the advantages of (6.6) over the other stabilizations mentioned in section 1: Since A is chosen only depending on k and p_∞, the proposed stabilization (6.6) does neither depend on the expected regularity for $\mathbf{u}$ nor on the expected order of convergence. Therefore the estimates for $r = \frac{12(p_\infty - 1)}{p_\infty}$ provide a minimal(!) order of convergence, but if the regularity of $\mathbf{u}$ is better, we get a better rate of convergence. Especially the approximation that we have chosen does not reduce the theoretically optimal order of convergence.

Before we get to the main results let us introduce some "time discretized" spaces. Let X be some Banach Space, then by $l^q(I_k, X)$ we denote the Bochner Space with discrete measure in time (weighted by the factor k), i.e.

$$\|(\mathbf{u}^m)_{m=0,\dots,M}\|_{l^q(I_k,X)} = \begin{cases} \Big(k \sum\limits_{m=0}^{M} \|\mathbf{u}^m\|_X^q\Big)^{\frac{1}{q}} & \text{for } 1 \leq q < \infty, \\ \sup\limits_{m=0,\dots,M} \|\mathbf{u}^m\|_X & \text{for } q = \infty. \end{cases}$$

Thus $L^q(I, X)$ measures $\mathbf{u}$ "continuously" in time and $l^q(I_k, X)$ measures $\mathbf{u}$ at the discrete time steps $0, k, 2k, \dots$. The factor k in the norm is necessary in order to ensure

$$\|\mathbf{u}(t_m)_{m=0,\dots,M}\|_{l^q(I_k,X)} \to \|\mathbf{u}\|_{L^q(I,X)}$$

for smooth $\mathbf{u}$ and $k \to 0^+$. As a general rule I will always be measured by the Lebesgue measure, while I_k will always be measured by the discrete measure (weighted by the factor k). We further use the notation L^q and l^q in order to point out this difference. For simplicity of notations we will not distinguish between $(\mathbf{u}^m)_{m=0,\dots,M}$ and $\mathbf{u}^m$. For example instead of $\|(\mathbf{u}^m)_{m=0,\dots,M}\|_{l^q(I_k,X)}$ we will write $\|\mathbf{u}^m\|_{l^q(I_k,X)}$.

We will also make use of the generalized Orlicz Space $L^{p(\cdot)}(I_k \times \Omega)$ induced by the modular

$$|\mathbf{D}\mathbf{v}^m|_{L^{p(\cdot)}(I_k\times\Omega)} := k \sum_{m=0}^{M} \int\limits_{\Omega} |\mathbf{D}\mathbf{v}^m|^p \, dx.$$

Note that we use different measures in time and space although we use the capital letter L, which we otherwise reserved for the Lebesgue measure.

THEOREM 6.1. *Let $p \in W^{1,\infty}(\overline{I \times \Omega})$ with $1 < p_\infty \leq p_0 \leq 2$. Further let $\|\mathbf{u}_0\|_{2,2}, \|\mathbf{f}\|_{C(I,L^2(\Omega))} \leq C$ and assume that (6.1) has a solution $\mathbf{u}, \pi$ which satisfies*

$$\begin{aligned} \|\partial_t \mathbf{u}\|_{L^\infty(I,L^2(\Omega))} + \|\partial_t^2 \mathbf{u}\|_{L^2(I,(W^{1,2}_{\operatorname{div}}(\Omega))^*)} &\leq C, \\ \big\|(\widetilde{D}\mathbf{u})^{\frac{p}{2}}\big\|_{C(I,L^{r,\infty}(\Omega))} &\leq C \quad \textit{(Lorentz space)} \end{aligned} \tag{6.9}$$

with $r > 2$. Let $A := k^{\frac{4}{4p_\infty - rp_\infty - 4}}$, then (6.6) has a unique weak solution $\mathbf{v}^m, \pi^m$ which satisfies

$$\|\mathbf{v}^m\|^2_{l^\infty(I_k, L^2(\Omega))} + |\mathbf{D}\mathbf{v}^m|_{L^{p(\cdot)}(I_k\times\Omega)} \le C,$$
$$\|\mathbf{D}\mathbf{v}^m\|^{p_\infty}_{l^{p_\infty}(I_k, L^{p_\infty}(\Omega))} + A^{p_\infty - 2}\|\mathbf{D}\mathbf{v}^m\|^2_{l^2(I_k, L^2(\Omega))} \le C.$$

Note that $A = k^{-\frac{1}{2(p_\infty-1)}}$ for $r = \frac{12(p_\infty-1)}{p_\infty}$.

Note that the existence of the solution $\mathbf{u}$, π implicitly requires higher regularity of $\mathbf{f}$ (see theorem 5.10 and corollary 5.11). But in order to point out which regularity of $\mathbf{f}$ is needed for the error analysis, we only demand this lower regularity of $\mathbf{f}$.

Before we continue with the error estimates for weak solutions it is important to mention the way that we measure the error. Instead of writing down the error estimates in terms of $\|\boldsymbol{\nabla}\mathbf{e}\|_X$ for some Banach space X, we prefer to estimate the natural error, which is given by

$$\big(\boldsymbol{\sigma}^A(\mathbf{D}\mathbf{u}^m, \mathbf{D}\mathbf{v}^m)\mathbf{D}\mathbf{e}^M\big)\cdot\mathbf{D}\mathbf{e}^m = \big(\mathbf{S}^A(\mathbf{D}\mathbf{u}^m) - \mathbf{S}^A(\mathbf{D}\mathbf{v}^m)\big)\cdot\big(\mathbf{D}\mathbf{u}^m - \mathbf{D}\mathbf{v}^m\big).$$

Using the notation

$$\boldsymbol{\sigma}^{A,m}_{\mathbf{u},\mathbf{v}} := \boldsymbol{\sigma}^A(\mathbf{D}\mathbf{u}^m, \mathbf{D}\mathbf{v}^m)$$

this reads

$$(\boldsymbol{\sigma}^{A,m}_{\mathbf{u},\mathbf{v}}\mathbf{D}\mathbf{e}^m)\cdot\mathbf{D}\mathbf{e}^m.$$

We rather use this natural error, since it contains all of the error information. But depending on the regularity of $\mathbf{u}^m, \mathbf{v}^m$ it is also possible to retrieve information about $\mathbf{D}\mathbf{e}^m$ and by Korn's inequality about $\boldsymbol{\nabla}\mathbf{e}^m$. This can be done by the following lemma, which is a direct consequence of lemma 3.15.

Lemma 6.2. *Let $p \in W^{1,\infty}(\Omega)$ with $1 < p_\infty \le p_0 \le 2$. Then for all (sufficiently smooth) $\mathbf{u}$ and $\mathbf{v}$ and for all $1 \le q \le 2$ there holds:*

$$\frac{\|\mathbf{D}\mathbf{e}\|_q^2}{\|(\widetilde{D}\mathbf{u})^{\frac{2-p}{2}} + (\widetilde{D}\mathbf{v})^{\frac{2-p}{2}}\|_{\frac{2q}{2-q}}} \le C\,\big\langle\boldsymbol{\sigma}^A_{\mathbf{u},\mathbf{v}}\mathbf{D}\mathbf{e}, \mathbf{D}\mathbf{e}\big\rangle, \tag{6.10}$$

where $\mathbf{e} = \mathbf{u} - \mathbf{v}$. If $q = 2$, then $\frac{2q}{2-q} := \infty$. Note that this estimate holds true if Ω is replaced by $I_k \times \Omega$, i.e.

$$\frac{\|\mathbf{D}\mathbf{e}^m\|^2_{L^q(I_k\times\Omega)}}{\|(\widetilde{D}\mathbf{u}^m)^{\frac{2-p}{2}} + (\widetilde{D}\mathbf{v}^m)^{\frac{2-p}{2}}\|_{L^{\frac{2q}{2-q}}(I_k\times\Omega)}} \le C\,k\sum_{m=0}^{M}\big\langle\boldsymbol{\sigma}^{A,m}_{\mathbf{u},\mathbf{v}}\mathbf{D}\mathbf{e}^m, \mathbf{D}\mathbf{e}^m\big\rangle \tag{6.11}$$

$$= C\,\big\|\big\langle\boldsymbol{\sigma}^{A,m}_{\mathbf{u},\mathbf{v}}\mathbf{D}\mathbf{e}^m, \mathbf{D}\mathbf{e}^m\big\rangle\big\|_{l^1(I_k)}. \tag{6.12}$$

This lemma especially states that if $\boldsymbol{\nabla}\mathbf{u}^m, \boldsymbol{\nabla}\mathbf{v}^m \in l^\infty(I_k, L^{\frac{(2-p_\infty)q}{2-q}}(\Omega))$, then we can retrieve $\|\boldsymbol{\nabla}\mathbf{e}^m\|^2_{l^\infty(I_k, L^q(\Omega))}$ information about the error. If $\boldsymbol{\nabla}\mathbf{u}^m, \boldsymbol{\nabla}\mathbf{v}^m \in L^{\frac{2q}{2-q}}(I_k\times\Omega)$, then we can retrieve $\|\boldsymbol{\nabla}\mathbf{e}\|^2_{L^q(I_k\times\Omega)}$ information. Note that in the first case we will get better error information with respect to the time, while in the second case we get (due to a slightly better choice of q) better space information. Writing down all possibilities to extract information about $\boldsymbol{\nabla}\mathbf{e}^m$ from the natural error would be quite lengthy and would not provide us with further insight of the problem. Moreover, for

$p_\infty > \frac{6}{5}$ it is possible for weak and strong solutions to choose $q := p_\infty$. Thus we can always extract $\|\nabla \mathbf{e}^m\|^2_{l^{p_\infty}(I_k,L^{p_\infty}(\Omega))}$, resp. $\|\nabla \mathbf{e}^m\|^2_{l^\infty(I_k,L^{p_\infty}(\Omega))}$, information from the natural error. With a slighty better q (close to 2) we would only gain little, since the best information extractable from the natural error (for arbitrary smooth $\mathbf{u}^m$ and $\mathbf{v}^m$) is $\|\nabla \mathbf{e}^m\|^2_{l^2(I_k,L^2(\Omega))}$, resp. $\|\nabla \mathbf{e}^m\|^2_{l^\infty(I_k,L^2(\Omega))}$. For all these reasons we will consider the natural error only.

THEOREM 6.3 (First Error Estimate). *Let $p, A, \mathbf{u}_0, \mathbf{f}, \mathbf{u}, \pi, \mathbf{v}^m$, and q^m be as in theorem 6.1. If $r > \max\{4, \frac{12}{p_\infty(5p_\infty-6)}\}$, then the error $\mathbf{e}^m$ with*

$$\mathbf{e}^m := \mathbf{u}^m - \mathbf{v}^m = \mathbf{u}(t_m) - \mathbf{v}^m$$

satisfies for sufficiently small k

$$\tfrac{1}{2}\|\mathbf{e}^m\|^2_{l^\infty(I_k,L^2(\Omega))} + \big\|\big\langle \boldsymbol{\sigma}^{A,m}_{\mathbf{u},\mathbf{v}} \mathbf{D}\mathbf{e}^m, \mathbf{D}\mathbf{e}^m\big\rangle\big\|_{l^1(I_k)} \le C\, k^{2\alpha(p_\infty)} \tag{6.13}$$

with $\alpha(p_\infty) = \frac{(r-2)p_\infty}{4-4p_\infty+rp_\infty}$. As a consequence

$$\big\|\big\langle \boldsymbol{\sigma}^{A,m}_{\mathbf{u},\mathbf{v}} \mathbf{D}\mathbf{e}^m, \mathbf{D}\mathbf{e}^m\big\rangle\big\|_{l^\infty(I_k)} \le C\, k^{2\alpha(p_\infty)-1}. \tag{6.14}$$

Note that $\alpha(p_\infty) = \frac{5p_\infty-6}{4(p_\infty-1)}$ if $r = \frac{12(p_\infty-1)}{p_\infty}$. In this case condition $r > \max\{4, \frac{12}{p_\infty(5p_\infty-6)}\}$ is equivalent to $p_\infty > \frac{11+\sqrt{21}}{10} \approx 1.55826$.

THEOREM 6.4 (Strong solution). *Let $p, A, r, \mathbf{u}_0, \mathbf{f}, \mathbf{u}, \pi, \mathbf{v}^m$, and q^m be as in theorem 6.1. Additionally assume that*

$$\|\mathbf{f}\|_{C(I,W^{1,2}(\Omega))} + \|\partial_t \mathbf{f}\|_{C(I,L^2(\Omega))} \le C.$$

If $\frac{4}{3} \le p_\infty \le p_0 \le 2$ and $r > \max\{4, \frac{12}{p_\infty(5p_\infty-6)}\}$, then $\mathbf{v}^m$ satisfies

$$\|\mathbf{v}^m\|_{l^\infty(I_k,W^{1,2}(\Omega))} + \|\mathbf{v}^m\|_{l^2(I_k,W^{1,\frac{12}{8-3p_\infty}}(\Omega))} + \|\mathbf{v}^m\|_{l^2(I_k,W^{2,\frac{4}{4-p_\infty}}(\Omega))} \le C,$$
$$\|d_t\mathbf{v}^m\|_{l^\infty(I_k,L^2(\Omega))} + \|d_t\mathbf{v}^m\|_{l^2(I_k,L^{\frac{12}{8-3p_\infty}}(\Omega))} + \|d_t\mathbf{v}^m\|_{l^\infty(I_k,W^{1,\frac{4}{4-p_\infty}}(\Omega))} \le C.$$

Note that for $r = \frac{12(p_\infty-1)}{p_\infty}$ the condition $r > \max\{4, \frac{12}{p_\infty(5p_\infty-6)}\}$ is equivalent to $p_\infty > \frac{11+\sqrt{21}}{10} \approx 1.55826$.

THEOREM 6.5 (Second error estimate). *Let $p, A, r, \mathbf{u}_0, \mathbf{f}, \mathbf{u}, \pi, \mathbf{v}^m$, and q^m be as in theorem 6.4, then for sufficiently small k*

$$\big\|\big\langle \boldsymbol{\sigma}^{A,m}_{\mathbf{u},\mathbf{v}} \mathbf{D}\mathbf{e}^m, \mathbf{D}\mathbf{e}^m\big\rangle\big\|_{l^\infty(I_k)} \le C\, k^{\alpha(p_\infty)}$$

with $\alpha(p_\infty) = \frac{(r-2)p_\infty}{4-4p_\infty+rp_\infty}$. Note that $\alpha(p_\infty) = \frac{5p_\infty-6}{4(p_\infty-1)}$ if $r = \frac{12(p_\infty-1)}{p_\infty}$.

THEOREM 6.6. *Theorem 6.1, 6.3, 6.4, and 6.5 still hold true if $(\mathbf{v}^m \cdot \nabla)\mathbf{v}^m$ is replaced in (6.6) by $(\mathbf{v}^{m-1} \cdot \nabla)\mathbf{v}^m$ and if we additionally require $\mathbf{u} \in C(I \times \Omega)$. If $r > \frac{6}{p_\infty}$, then $\mathbf{D}\mathbf{u} \in C(I, L^3(\Omega))$, since $(\mathbf{D}\mathbf{u})^{\frac{p}{2}} \in C(I, L^{r,\infty}(\Omega))$. This implies $\mathbf{u} \in C(I, W^{1,3+\varepsilon}(\Omega))$ for some $\varepsilon > 0$ using Korn's inequality. Hence $\mathbf{u} \in C(I \times \Omega)$. So if $r = \frac{12(p_\infty-1)}{p_\infty}$ and $p_\infty > \frac{3}{2}$ this additional requirement is fulfilled.*

3. Weak Solutions

At first we show that system (6.6) has under the assumptions of theorem 6.1 a well defined, unique solution in the following sense: There exists a solution $\mathbf{v}^m, q^m$ such that $\mathbf{v}^m \in l^\infty(I_k, W^{2,2}_{\operatorname{div}}(\Omega))$, where the norm may depend on k and m. Let us start with the existence. For this we reinterpret the system as an iterative sequence of stationary problems: Let $\mathbf{v}^{m-1}$ be given, then $\mathbf{v}^m, q^m$ is the solution of

$$\begin{aligned} \tfrac{1}{k}\mathbf{v}^m - \operatorname{div}(\mathbf{S}^A(\mathbf{D}\mathbf{v}^m)) + \boldsymbol{\nabla} q^m + (\mathbf{v}^m \cdot \boldsymbol{\nabla})\mathbf{v}^m &= \mathbf{f}^m + \tfrac{1}{k}\mathbf{v}^{m-1}, \\ \operatorname{div} \mathbf{v}^m &= 0. \end{aligned} \tag{6.15}$$

We will show that $\mathbf{v}^m \in W^{2,2}_{\operatorname{div}}(\Omega)$ for all m. Since $\mathbf{v}^0 = \mathbf{u}_0 \in W^{2,2}_{\operatorname{div}}(\Omega)$ we can prove the claim by induction, so assume that $\mathbf{v}^{m-1} \in W^{2,2}_{\operatorname{div}}(\Omega)$. To prove $\mathbf{v}^m \in W^{2,2}_{\operatorname{div}}(\Omega)$ we have to derive two a priori estimates for $\mathbf{v}^m$ by testing with $\mathbf{v}^m$ and $-\Delta\mathbf{v}^m$. This can be justified by means of Galerkin approximation, so we focus on the a priori estimates. Testing (6.15) with $\mathbf{v}^m$ gives

$$\tfrac{1}{k}\|\mathbf{v}^m\|_2^2 + \langle \mathbf{S}^A(\mathbf{D}\mathbf{v}^m), \mathbf{D}\mathbf{v}^m\rangle = \langle \mathbf{f}^m, \mathbf{v}^m\rangle + \tfrac{1}{k}\langle \mathbf{v}^{m-1}, \mathbf{v}^m\rangle.$$

By (3.41) and Young's inequality

$$\tfrac{1}{k}\|\mathbf{v}^m\|_2^2 + c\,A^{p_\infty-2}\,\|\mathbf{D}\mathbf{v}^m\|_2^2 \le C\left(k\,\|\mathbf{f}\|_2^2 + \tfrac{1}{k}\|\mathbf{v}^{m-1}\|_2^2\right) < \infty.$$

Korn's inequality implies $\mathbf{v}^m \in W^{1,2}(\Omega)$. Testing (6.15) with $-\Delta\mathbf{v}^m$ gives (compare chapter 5)

$$\begin{aligned} \tfrac{1}{k}\|\boldsymbol{\nabla}\mathbf{v}^m\|_2^2 + \mathcal{I}_\Phi{}^A(\mathbf{v}^m) \le |\langle \mathbf{f}^m, \Delta\mathbf{v}^m\rangle| + \tfrac{1}{k}|\langle \boldsymbol{\nabla}\mathbf{v}^{m-1}, \boldsymbol{\nabla}\mathbf{v}^m\rangle| \\ + |\langle (\mathbf{v}^m \cdot \boldsymbol{\nabla})\mathbf{v}^m, \Delta\mathbf{v}^m\rangle|. \end{aligned} \tag{6.16}$$

By (3.59) we have $\mathcal{I}_\Phi{}^A(\mathbf{v}^m) \ge c\,A^{p_\infty-2}\|\mathbf{u}\|_{2,2}^2$. So by Young's inequality

$$\begin{aligned} &\tfrac{1}{k}\|\boldsymbol{\nabla}\mathbf{v}^m\|_2^2 + c\,A^{p_\infty-2}\|\mathbf{v}^m\|_{2,2}^2 \\ &\quad \le \tfrac{C}{k}\|\boldsymbol{\nabla}\mathbf{v}^{m-1}\|_2^2 + CA^{2-p_\infty}\left(\|\mathbf{f}\|_2^2 + \|(\mathbf{v}^m \cdot \boldsymbol{\nabla})\mathbf{v}^m\|_2^2\right) \\ &\quad \le \tfrac{C}{k}\|\boldsymbol{\nabla}\mathbf{v}^{m-1}\|_2^2 + CA^{2-p_\infty}\left(\|\mathbf{f}\|_2^2 + \|\mathbf{v}^m\|_{1,\frac{12}{5}}^4\right) \\ &\quad \le \tfrac{C}{k}\|\boldsymbol{\nabla}\mathbf{v}^{m-1}\|_2^2 + CA^{2-p_\infty}\left(\|\mathbf{f}\|_2^2 + \|\mathbf{v}^m\|_{2,2}^2\right) \\ &\quad \le \tfrac{C}{k}\|\boldsymbol{\nabla}\mathbf{v}^{m-1}\|_2^2 + CA^{2-p_\infty}\left(\|\mathbf{f}\|_2^2 + \|\mathbf{v}^m\|_{1,2}^3\|\mathbf{v}^m\|_{2,2}\right) \\ &\quad \le \tfrac{C}{k}\|\boldsymbol{\nabla}\mathbf{v}^{m-1}\|_2^2 + CA^{2-p_\infty}\|\mathbf{f}\|_2^2 + C_\delta\,A^{2(2-p_\infty)}\|\mathbf{v}^m\|_{1,2}^3 + \delta A^{p_\infty-2}\|\mathbf{v}^m\|_{2,2}^2. \end{aligned}$$

If δ is sufficiently small, then we can absorb the last term on the left-hand side. (Note that we do not subtract ∞, since the steps are justified by means of the Galerkin method.) This proves that $\mathbf{v}^m \in W^{2,2}_{\operatorname{div}}(\Omega)$. Although we are not interested in the pressure yet, let us mention for the sake of completeness that $q^m \in W^{1,2}_0(\Omega)$ for all m. Indeed it follows from (3.42) that all terms of the first equation of (6.15) but $\boldsymbol{\nabla} q^m$ are in $L^2_0(\Omega)$, so by De Rahm it has to be also in $L^2_0(\Omega)$, i.e. $q^m \in W^{1,2}_0(\Omega)$. We remark that with this regularity for $\mathbf{v}^m$ and q^m it is very easy to verify that $\mathbf{v}^m$ is unique: Indeed $\mathbf{v}^m$ and q^m are for every time step the solution of a stationary, uniformly elliptic problem.

Note that in the above estimates additional multiplicative constants appeared for every iteration ($k-1 \mapsto k$). For an increasing number of iterations these constants multiply and grow rapidly. Since k relates to the number of iterations, the global norm $\|\mathbf{v}^m\|_{l^\infty(I_k,W^{2,2}(\Omega))}$ does strongly depend on k. So the regularity we have derived so far is useful to justify(!) all the calculations later, but nothing more. In order to derive error estimates we need global norm estimates which are independent of k. This will be our next focus.

Using $\mathbf{v}^m$ as a test function for (6.6) gives

$$\tfrac{1}{2}d_t\|\mathbf{v}^m\|_2^2 + \langle \mathbf{S}^A(\mathbf{D}\mathbf{v}^m), \mathbf{D}\mathbf{v}^m\rangle \le |\langle \mathbf{f}^m, \mathbf{v}^m\rangle|.$$

With the help of (3.41) we conclude

$$\tfrac{1}{2}d_t\|\mathbf{v}^m\|_2^2 + c\,|\mathbf{D}\mathbf{v}^m|_{p(\cdot)} + c\,A^{p_\infty-2}\|\mathbf{D}\mathbf{v}^m\|_2^2 \le C\|\mathbf{f}^m\|_2\|\mathbf{v}^m\|_2.$$

The use of the discrete Gronwall and Young's inequality implies

$$\|\mathbf{v}^m\|^2_{l^\infty(I_k,L^2(\Omega))} + \tfrac{1}{k}\sum_{m=0}^{M}|\mathbf{D}\mathbf{v}^m|_{p(\cdot)} + A^{p_\infty-2}\|\mathbf{D}\mathbf{v}^m\|^2_{l^2(I_k,L^2(\Omega))} \le C(\mathbf{f}).$$

We see that $\mathbf{v}^m$ is a weak solution of (6.6). This proves theorem (6.1). Later we will see that for a certain range of p_∞'s this weak solution is a unique strong solution.

4. The Error

We will now derive estimates for the error of weak solutions. Let $\mathbf{e}^m := \mathbf{u}^m - \mathbf{v}^m = \mathbf{u}(t_m) - \mathbf{v}^m$ and $\eta^m := \pi^m - q^m$. Further assume that the assumptions of theorem 6.3 are satisfied, then

$$\begin{aligned} d_t\mathbf{e}^m - \operatorname{div}\big(\mathbf{S}(\mathbf{D}\mathbf{u}^m) - \mathbf{S}^A(\mathbf{D}\mathbf{v}^m)\big) + \boldsymbol{\nabla}\eta^m + (\mathbf{u}^m\cdot\boldsymbol{\nabla})\mathbf{u}^m - (\mathbf{v}^m\cdot\boldsymbol{\nabla})\mathbf{v}^m &= \mathbf{R}^m,\\ \operatorname{div}\mathbf{e}^m &= 0,\\ \mathbf{e}^0 &= 0. \end{aligned}$$

Since $\boldsymbol{\sigma}^{A,m}_{\mathbf{u},\mathbf{v}} = \boldsymbol{\sigma}^A(\mathbf{D}\mathbf{u}^m, \mathbf{D}\mathbf{v}^m) = \mathbf{S}^A(\mathbf{D}\mathbf{u}^m) - \mathbf{S}^A(\mathbf{D}\mathbf{v}^m)$, there holds

$$\begin{aligned} d_t\mathbf{e}^m - \operatorname{div}(\boldsymbol{\sigma}^{A,m}_{\mathbf{u},\mathbf{v}}\mathbf{D}\mathbf{e}^m) &= \mathbf{R}^m + \operatorname{div}(\mathbf{S}^A(\mathbf{D}\mathbf{u}^m) - \mathbf{S}(\mathbf{D}\mathbf{u}^m))\\ &\quad - \boldsymbol{\nabla}\eta^m - (\mathbf{v}^m\cdot\boldsymbol{\nabla})\mathbf{e}^m - (\mathbf{e}^m\cdot\boldsymbol{\nabla})\mathbf{u}^m,\\ \operatorname{div}\mathbf{e}^m &= 0,\\ \mathbf{e}^0 &= 0. \end{aligned} \tag{6.17}$$

For the examination of the error it is important to control the right-hand side of (6.17). Especially the term $\mathbf{R}^m$ is of great importance, since it will determine the order of convergence. The following lemma provides all the necessary estimates for $\mathbf{R}^m$.

Lemma 6.7. *Let $p, A, r, \mathbf{u}_0, \mathbf{f}, \mathbf{u}$, and π be as in theorem 6.1. Especially $\mathbf{u}$ satisfies*

$$\|\partial_t\mathbf{u}\|_{L^\infty(I,L^2_0(\Omega))} + \|\partial_t^2\mathbf{u}\|_{L^2(I,(W^{1,2}_{\operatorname{div}}(\Omega))^*)} \le C. \tag{6.18}$$

Then $\partial_t\mathbf{u} \in C_{\text{weak}}(I, L^2_0(\Omega))$ and

$$\|\partial_t\mathbf{u}\|_{C_{\text{weak}}(I,L^2_0(\Omega))} \le \|\partial_t\mathbf{u}\|_{L^\infty(I,L^2_0(\Omega))} \le C, \tag{6.19}$$

i.e. there exists a constant $C > 0$, such that for all $\mathbf{g} \in L^2_0(\Omega)$ the mapping $t \mapsto \langle\partial_t\mathbf{u}(t), \mathbf{g}\rangle$ is continuous and there holds

$$\big\|\langle\partial_t\mathbf{u}, \mathbf{g}\rangle\big\|_{L^\infty(I)} \le C\|\mathbf{g}\|_2\|\partial_t\mathbf{u}\|_{L^\infty(I,L^2_0(\Omega))} \le C\|\mathbf{g}\|_2.$$

As a consequence $(\partial_t \mathbf{u}(t_m))_{m=0,\dots,M}$ *is well defined and*

$$\|\partial_t \mathbf{u}(t_m)\|_{l^\infty(I_k, L^2_0(\Omega))} \le \|\partial_t \mathbf{u}\|_{L^\infty(I, L^2_0(\Omega))},$$
$$\|d_t(\mathbf{u}(t_m))\|_{L^2_0(\Omega)} \le \|\partial_t \mathbf{u}\|_{L^\infty(I, L^2_0(\Omega))}.$$

Furthermore $\mathbf{R}^m = d_t(\mathbf{u}(t_m)) - \partial_t \mathbf{u}(t_m)$ *satisfies*

$$\|\mathbf{R}^m\|_{l^2(I_k,(W^{1,2}_{\mathrm{div}}(\Omega))^*)} \le C\,k, \tag{6.20}$$

$$\|\mathbf{R}^m\|_{l^\infty(I_k, L^2_0(\Omega))} \le 2\,\|\partial_t \mathbf{u}\|_{L^\infty(I, L^2_0(\Omega))} \le C. \tag{6.21}$$

Note that we do not need $\partial_t \mathbf{u} \in C(I, L^2_0(\Omega))$.

Proof. We rewrite $\mathbf{R}^m = d_t(\mathbf{u}(t_m)) - \partial_t \mathbf{u}(t_m)$ as

$$\mathbf{R}^m = \tfrac{1}{k} \int_{t_{m-1}}^{t_m} (s - t_{m-1})(\partial_t^2 \mathbf{u})(s)\,ds.$$

So by assumptions on $\mathbf{u}$

$$\begin{aligned}
\|\mathbf{R}^m\|^2_{l^2(I_k,(W^{1,2}_{\mathrm{div}}(\Omega))^*)} &\le k \sum_{m=0}^{M} \Big\| \tfrac{1}{k} \int_{t_{m-1}}^{t_m} (s - t_{m-1})(\partial_t^2 \mathbf{u})(s)\,ds \Big\|^2_{(W^{1,2}_{\mathrm{div}}(\Omega))^*} \\
&\le k \sum_{m=0}^{M} \tfrac{1}{k} \int_{t_{m-1}}^{t_m} (s - t_{m-1})^2 \big\|(\partial_t^2 \mathbf{u})(s)\big\|^2_{(W^{1,2}_{\mathrm{div}}(\Omega))^*}\,ds \\
&\le k^2 \sum_{m=0}^{M} \int_{t_{m-1}}^{t_m} \big\|(\partial_t^2 \mathbf{u})(s)\big\|^2_{(W^{1,2}_{\mathrm{div}}(\Omega))^*}\,ds \\
&= k^2 \|\partial_t^2 \mathbf{u}\|^2_{L^2(I,(W^{1,2}_{\mathrm{div}}(\Omega))^*)} \\
&\le k^2\,C.
\end{aligned} \tag{6.22}$$

This proves (6.20). Since $\mathbf{u} \in L^\infty(I, L^2_0(\Omega))$ and $\operatorname{div} \mathbf{u} = 0$, there further holds $\mathbf{u} \in L^\infty(I, L^2_{\mathrm{div},0}(\Omega))$.

Additionally $\partial_t(\partial_t \mathbf{u}) \in L^2(I, (W^{1,2}_{\mathrm{div},0}(\Omega))^*)$ and $L^2_{\mathrm{div},0} \hookrightarrow (W^{1,2}_{\mathrm{div},0}(\Omega))^*$ continuously, so $\partial_t \mathbf{u} \in C(I, (W^{1,2}_{\mathrm{div},0}(\Omega))^*)$. Since $L^2_{\mathrm{div},0}(\Omega)$ is a closed subspace of the Hilbert space $L^2_0(\Omega)$, it is also a Hilbert space and its dual can be identified with itself. Thus

$$\partial_t \mathbf{u} \in L^\infty(I, (L^2_{\mathrm{div},0}(\Omega))^*), \qquad \partial_t \mathbf{u} \in C(I, (W^{1,2}_{\mathrm{div},0}(\Omega))^*).$$

Since $W^{1,2}_{\mathrm{div},0}(\Omega)$ is separable and dense in $L^2_{\mathrm{div},0}(\Omega)$ this implies (see Lions [**Lio96**], appendix C)

$$\partial_t \mathbf{u} \in C_{\mathrm{weak}}(I, (L^2_{\mathrm{div},0}(\Omega))^*).$$

Thus

$$\partial_t \mathbf{u} \in C_{\mathrm{weak}}(I, L^2_{\mathrm{div},0}(\Omega)) \tag{6.23}$$

and

$$\|\partial_t \mathbf{u}\|_{C_{\mathrm{weak}}(I, L^2_{\mathrm{div},0}(\Omega))} \le \|\partial_t \mathbf{u}\|_{L^\infty(I, L^2_0(\Omega))}. \tag{6.24}$$

Let $L^2_0(\Omega) := L^2_{\operatorname{div},0}(\Omega) + X$ be the Helmholtz decomposition of $L^2_0(\Omega)$. Since $\operatorname{div} \partial_t \mathbf{u} = 0$ and $\partial_t \mathbf{u} \in L^\infty(I, L^2_0(\Omega))$, there holds

$$\langle \partial_t \mathbf{u}, \mathbf{a} \rangle = 0$$

for all $\mathbf{a} \in X$. Together with (6.23) and (6.24) this implies

$$\partial_t \mathbf{u} \in C_{\text{weak}}(I, L^2_0(\Omega)) \tag{6.25}$$

and

$$\|\partial_t \mathbf{u}\|_{C_{\text{weak}}(I, L^2_0(\Omega))} \le \|\partial_t \mathbf{u}\|_{L^\infty(I, L^2_0(\Omega))}. \tag{6.26}$$

This immediately implies that $(\partial_t \mathbf{u}(t_m))_{m=0,\dots,M}$ is well defined and

$$\|\partial_t \mathbf{u}(t_m)\|_{l^\infty(I_k, L^2_0(\Omega))} \le \|\partial_t \mathbf{u}\|_{C_{\text{weak}}(I, L^2_0(\Omega))}. \tag{6.27}$$

Furthermore for all $m = 1, \dots, M$

$$\begin{aligned} \|d_t(\mathbf{u}(t_m))\|_{L^2_0(\Omega)} &= \left\| \tfrac{1}{k} \int_{t_{m-1}}^{t_m} (\partial_t \mathbf{u})(\tau)\, d\tau \right\|_{L^2_0(\Omega)} \\ &\le \|\partial_t \mathbf{u}\|_{L^\infty(I, L^2_0(\Omega))}. \end{aligned}$$

Thus

$$\|d_t(\mathbf{u}(t_m))\|_{l^\infty(I_k, L^2_0(\Omega))} \le \|\partial_t \mathbf{u}\|_{L^\infty(I, L^2_0(\Omega))}. \tag{6.28}$$

So (6.27) and (6.28) imply

$$\begin{aligned} \|\mathbf{R}^m\|_{l^\infty(I_k, L^2_0(\Omega))} &= \|d_t(\mathbf{u}(t_m)) - \partial_t \mathbf{u}(t_m)\|_{C_{\text{weak}}(I, L^2_0(\Omega))} \\ &\le 2\, \|\partial_t \mathbf{u}\|_{C_{\text{weak}}(I, L^2_0(\Omega))}. \end{aligned}$$

This proves the lemma. □

We continue with the derivation of the error estimates for weak solutions $\mathbf{v}^m, q^m$. Testing (6.17) with $\mathbf{e}^m$ we get

$$\begin{aligned} &\langle d_t \mathbf{e}^m, \mathbf{e}^m \rangle + \langle \boldsymbol{\sigma}^{A,m}_{\mathbf{u},\mathbf{v}} \mathbf{D}\mathbf{e}^m, \mathbf{D}\mathbf{e}^m \rangle \\ &\qquad \le \quad |\langle \mathbf{R}^m, \mathbf{e}^m \rangle| + |\langle \mathbf{S}^A(\mathbf{D}\mathbf{u}^m) - \mathbf{S}(\mathbf{D}\mathbf{u}^m), \mathbf{D}\mathbf{e}^m \rangle| \\ &\qquad\quad + \big| \big\langle (\mathbf{e}^m \cdot \boldsymbol{\nabla}) \mathbf{u}^m, \mathbf{e}^m \big\rangle \big| \\ &\qquad =: K_1 + K_2 + K_3. \end{aligned} \tag{6.29}$$

The first term on the right-hand side of (6.29) K_1 can be estimated as

$$\begin{aligned} K_1 &\le \|\mathbf{R}^m\|_{(W^{1,2}_{\operatorname{div}}(\Omega))^*} \|\mathbf{e}^m\|_{1,2} \\ &\le C_\delta A^{2-p_\infty} \|\mathbf{R}^m\|^2_{(W^{1,2}_{\operatorname{div}}(\Omega))^*} + \delta A^{p_\infty - 2} \|\boldsymbol{\nabla} \mathbf{e}^m\|_2^2. \end{aligned} \tag{6.30}$$

Due to (3.26) we can estimate K_2 by

$$\begin{aligned} K_2 &\le C \int_\Omega \chi_{\{|\mathbf{D}\mathbf{u}^m| \ge A\}} A^{p-2} |\mathbf{D}\mathbf{u}^m| |\mathbf{D}\mathbf{e}^m|\, dx \\ &\le C A^{-1} \int_{\{|\mathbf{D}\mathbf{u}^m| \ge A\}} |\widetilde{D}\mathbf{u}^m|^p |\mathbf{D}\mathbf{e}^m|\, dx. \end{aligned}$$

Let $\Omega_A^m := \{\widetilde{D}\mathbf{u}^m \geq A\}$, then by (8.9) and (8.11) there holds

$$|\Omega_A^m|^{\frac{2}{rp_\infty}} \leq \frac{\|\widetilde{D}\mathbf{u}^m\|_{\frac{rp_\infty}{2},\infty}}{A} \leq \frac{\|(\widetilde{D}\mathbf{u}^m)^{\frac{p}{2}}\|_{r,\infty}^{\frac{2}{p_\infty}}}{A}.$$

So

$$|\Omega_A^m|^{\frac{1}{r}} \leq A^{-\frac{p_\infty}{2}}\|(\widetilde{D}\mathbf{u}^m)^{\frac{p}{2}}\|_{r,\infty} \leq C\,A^{-\frac{p_\infty}{2}}. \tag{6.31}$$

Hence

$$\begin{aligned}
K_2 &\leq CA^{-1}\big\|\chi_{\Omega_A^m}\big(\widetilde{D}\mathbf{u}^m\big)^{\frac{p}{2}}\big\|_4^2\|\mathbf{De}^m\|_2 \\
&\leq CA^{-1}\big\|\chi_{\Omega_A^m}\big\|^2_{L^{\frac{4r}{r-4},1}(\Omega)}\big\|\big(\widetilde{D}\mathbf{u}^m\big)^{\frac{p}{2}}\big\|^2_{L^{r,\infty}(\Omega)}\|\mathbf{De}^m\|_2 \\
&\leq CA^{-1}\big\|\chi_{\Omega_A^m}\big\|^2_{L^{\frac{4r}{r-4},1}(\Omega)}\|\mathbf{De}^m\|_2 \\
&\leq CA^{-1}\Big(\tfrac{4r}{r-4}\big|\Omega_A^m\big|^{\frac{r-4}{4r}}\Big)^2\|\mathbf{De}^m\|_2 \\
&\overset{(6.31)}{\leq} CA^{-1}\Big(\big(\tfrac{4r}{r-4}\big)^{\frac{r-4}{4r}}\big(C\,A^{-\frac{p_\infty}{2}}\big)^{\frac{r-4}{4}}\Big)^2\|\mathbf{De}^m\|_2.
\end{aligned}$$

Since $r > 4$ we have $\frac{r-4}{4r} \in (0, \frac{1}{4})$. Thus

$$\big(\tfrac{4r}{r-4}\big)^{\frac{r-4}{4r}} \leq \sup_{0<t<\frac{1}{4}} t^{-t} = \big(\tfrac{1}{4}\big)^{-\frac{1}{4}} = \sqrt{2}$$

and therefore

$$\begin{aligned}
K_2 &\leq CA^{-1}A^{\frac{-p_\infty(r-4)}{4}}\|\mathbf{De}^m\|_2 \\
&\leq \delta A^{p_\infty-2}\|\mathbf{De}^m\|_2^2 + C_\delta A^{-p_\infty-\frac{p_\infty(r-4)}{2}} \qquad (6.32)\\
&= \delta A^{p_\infty-2}\|\mathbf{De}^m\|_2^2 + C_\delta A^{-\frac{(r-2)p_\infty}{2}}.
\end{aligned}$$

The terms K_1 and K_2 will be responsible for the order of convergence. The third term on the right-hand side of (6.29), which arises from the convection, will not affect the convergence, but is still important, since it does restrict the range of admissible p's. We estimate

$$\begin{aligned}
K_3 &= \big|\big\langle(\mathbf{e}^m\cdot\boldsymbol{\nabla})\mathbf{u}^m, \mathbf{e}^m\big\rangle\big| \\
&= \Big|\sum_{i,j=1}^3\int_\Omega e_j^m(\partial_j u_k^m)e_k^m\,dx\Big| \\
&= \Big|\sum_{i,j=1}^3\int_\Omega e_j^m e_k^m(D_{jk}\mathbf{u}^m)\,dx\Big| \\
&\leq \|\mathbf{Du}^m\|_{L^{\frac{rp_\infty}{2},\infty}(\Omega)}\|\mathbf{e}^m\|^2_{L^{\frac{2rp_\infty}{rp_\infty-2},1}(\Omega)} \\
&\leq C\,\|\mathbf{e}^m\|^2_{L^{\frac{2rp_\infty}{rp_\infty-2},1}(\Omega)}.
\end{aligned}$$

Since $r > \frac{12}{p_\infty(5p_\infty-6)}$ by assumption of theorem 6.3, there exists q with

$$\frac{2rp_\infty}{rp_\infty-2} < q < \frac{12}{12-5p_\infty} < \frac{3p_\infty}{3-p_\infty}.$$

Hence

$$K_3 \le C(q)\, \|\mathbf{e}^m\|^2_{L^q(\Omega)}.$$

Note that the constant $C(q)$ grows, when $q \to \frac{2rp_\infty}{rp_\infty - 2}$, i.e. if $\frac{2rp_\infty}{rp_\infty-2}$ and $\frac{12}{12-5p_\infty}$ are close, which corresponds to $r \searrow \frac{12}{p_\infty(5p_\infty-6)}$. But since we are not interested in the behavior of the constants for $r \searrow \frac{12}{p_\infty(5p_\infty-6)}$, we will neglect this fact and use C instead of $C(q)$. So

$$K_3 \le C\, \|\mathbf{e}^m\|^2_{L^q(\Omega)}.$$

Let $\theta := \frac{3(q-2)p_\infty}{q(5p_\infty-6)}$, then

$$\begin{aligned} K_3 &\le C\, \|\mathbf{e}^m\|_2^{2(1-\theta)} \|\mathbf{e}\|^{2\theta}_{W^{1,p_\infty}(\Omega)} \\ &\le C\, \|\mathbf{e}^m\|_2^{2(1-\theta)} \|\mathbf{D}\mathbf{e}^m\|^{2\theta}_{p_\infty} \qquad \text{(by Korn)} \\ &\le C_\delta\, \|\mathbf{e}^m\|_2^2 (1 + \|\nabla \mathbf{u}^m\|_{p_\infty} + \|\nabla \mathbf{v}^m\|_{p_\infty})^{\frac{(2-p_\infty)\theta}{1-\theta}} \\ &\quad + \delta \|\mathbf{D}\mathbf{e}^m\|^2_{p_\infty} (1 + \|\nabla \mathbf{u}^m\|_{p_\infty} + \|\nabla \mathbf{v}^m\|_{p_\infty})^{p_\infty - 2}. \end{aligned} \tag{6.33}$$

If $\delta > 0$ is small, the last term can be absorbed by the left-hand side of (6.29). To see this, note that by (3.38) there follows

$$\begin{aligned} \big\langle \boldsymbol{\sigma}^{A,m}_{\mathbf{u},\mathbf{v}} \mathbf{D}\mathbf{e}^m, \mathbf{D}\mathbf{e}^m \big\rangle &\ge C \int (1 + |\mathbf{D}\mathbf{u}^m|^2 + |\mathbf{D}\mathbf{v}^m|^2)^{\frac{p_\infty - 2}{2}} |\mathbf{D}\mathbf{e}^m|^2 \, dx \\ &\ge C\, \frac{\|\mathbf{D}\mathbf{e}^m\|^2_{p_\infty}}{(1 + \|\mathbf{D}\mathbf{u}^m\|_{p_\infty} + \|\mathbf{D}\mathbf{v}^m\|_{p_\infty})^{2-p_\infty}}. \end{aligned} \tag{6.34}$$

On the other hand by (3.38) and Korn we get

$$\big\langle \boldsymbol{\sigma}^{A,m}_{\mathbf{u},\mathbf{v}} \mathbf{D}\mathbf{e}^m, \mathbf{D}\mathbf{e}^m \big\rangle \ge c\, A^{p_\infty - 2} \|\mathbf{D}\mathbf{e}^m\|_2^2 \ge c\, A^{p_\infty-2} \|\nabla \mathbf{e}^m\|_2^2. \tag{6.35}$$

Now (6.29), (6.30), (6.32), (6.33), (6.34), and (6.35) imply

$$\begin{aligned} &\langle d_t \mathbf{e}^m, \mathbf{e}^m \rangle + \big\langle \boldsymbol{\sigma}^{A,m}_{\mathbf{u},\mathbf{v}} \mathbf{D}\mathbf{e}^m, \mathbf{D}\mathbf{e}^m \big\rangle \\ &\qquad \le C A^{2-p_\infty} \|\mathbf{R}^m\|^2_{(W^{1,2}_{\operatorname{div}}(\Omega))^*} + C A^{-\frac{(r-2)p_\infty}{2}} \\ &\qquad\quad + \|\mathbf{e}^m\|_2^2 (1 + \|\nabla \mathbf{u}^m\|_{p_\infty} + \|\nabla \mathbf{v}^m\|_{p_\infty})^{\frac{(2-p_\infty)\theta}{1-\theta}}. \end{aligned} \tag{6.36}$$

Since

$$\tfrac12 d_t \|\mathbf{e}^m\|_2^2 \le \tfrac12 d_t \|\mathbf{e}^m\|_2^2 + \tfrac12 k \|d_t \mathbf{e}^m\|_2^2 = \langle d_t \mathbf{e}^m, \mathbf{e}^m \rangle,$$

there holds

$$\begin{aligned} &\tfrac12 d_t \|\mathbf{e}^m\|_2^2 + \big\langle \boldsymbol{\sigma}^{A,m}_{\mathbf{u},\mathbf{v}} \mathbf{D}\mathbf{e}^m, \mathbf{D}\mathbf{e}^m \big\rangle \\ &\qquad \le C A^{2-p_\infty} \|\mathbf{R}^m\|^2_{(W^{1,2}_{\operatorname{div}}(\Omega))^*} + C A^{-\frac{(r-2)p_\infty}{2}} \\ &\qquad\quad + \|\mathbf{e}^m\|_2^2 (1 + \|\nabla \mathbf{u}^m\|_{p_\infty} + \|\nabla \mathbf{v}^m\|_{p_\infty})^{\frac{(2-p_\infty)\theta}{1-\theta}}. \end{aligned}$$

Since $q < \frac{12}{12-5p_\infty}$ and $\theta := \frac{3(q-2)p_\infty}{q(5p_\infty-6)}$ there holds

$$\frac{(2-p_\infty)\theta}{1-\theta} < p_\infty.$$

Since $\|\mathbf{u}^m\|_{l^{p_\infty}(I_k,L^{p_\infty}(\Omega))}+\|\mathbf{v}^m\|_{l^{p_\infty}(I_k,L^{p_\infty}(\Omega))} \leq C$, we can apply the discrete Gronwall's inequality, such that for sufficiently small k there holds

$$\begin{aligned}\tfrac{1}{2}\|\mathbf{e}^m\|^2_{l^\infty(I_k,L^2(\Omega))} &+ \big\|\langle \boldsymbol{\sigma}^{A,m}_{\mathbf{u},\mathbf{v}}\mathbf{De}^m, \mathbf{De}^m\rangle\big\|_{l^1(I_k)} \\ &\leq CA^{2-p_\infty}\|\mathbf{R}^m\|^2_{l^2(I_k,(W^{1,2}_{\operatorname{div}}(\Omega))^*)} + CA^{-\frac{(r-2)p_\infty}{2}}.\end{aligned}$$

By lemma 6.7 there follows

$$\begin{aligned}\tfrac{1}{2}\|\mathbf{e}^m\|^2_{l^\infty(I_k,L^2(\Omega))} &+ \big\|\langle \boldsymbol{\sigma}^{A,m}_{\mathbf{u},\mathbf{v}}\mathbf{De}^m, \mathbf{De}^m\rangle\big\|_{l^1(I_k)} \\ &\leq CA^{2-p_\infty}k^2 + CA^{-\frac{(r-2)p_\infty}{2}}.\end{aligned}$$

We get the best order of convergence if we choose $A^{2-p_\infty}k^2 = A^{-\frac{(r-2)p_\infty}{2}}$, i.e. fix $A := k^{\frac{-4}{4-4p_\infty+rp_\infty}}$. So

$$\begin{aligned}\tfrac{1}{2}\|\mathbf{e}^m\|^2_{l^\infty(I_k,L^2(\Omega))} &+ \big\|\langle \boldsymbol{\sigma}^{A,m}_{\mathbf{u},\mathbf{v}}\mathbf{De}^m, \mathbf{De}^m\rangle\big\|_{l^1(I_k)} \\ &\leq C\,k^2\,A^{2-p_\infty} = C\,k^{\frac{2(r-2)p_\infty}{4-4p_\infty+rp_\infty}} = C\,k^{2\alpha(p_\infty)}.\end{aligned} \tag{6.37}$$

This proves (6.13). Further

$$\big\|\langle \boldsymbol{\sigma}^{A,m}_{\mathbf{u},\mathbf{v}}\mathbf{De}^m, \mathbf{De}^m\rangle\big\|^2_{l^\infty(I_k)} \leq k^{-1}\big\|\langle \boldsymbol{\sigma}^{A,m}_{\mathbf{u},\mathbf{v}}\mathbf{De}^m, \mathbf{De}^m\rangle\big\|^2_{l^2(I_k)} \leq C\,k^{2\alpha(p_\infty)-1}.$$

So the proof of 6.3 is complete.

5. Strong Solutions

Assume that the requirements of theorem 6.4 are satisfied. From theorem 6.3, (6.37), and (3.38) we deduce that

$$\begin{aligned}\|\mathbf{De}^m\|^2_{l^2(I_k,L^2(\Omega))} &\leq A^{2-p_\infty}\big\|\langle \boldsymbol{\sigma}^{A,m}_{\mathbf{u},\mathbf{v}}\mathbf{De}^m, \mathbf{De}^m\rangle\big\|_{l^1(I_k)} \\ &\leq C\,k^2A^{2(2-p_\infty)} \\ &= C\,k^{\frac{2(-4+rp_\infty)}{4-4p_\infty+rp_\infty}}\end{aligned} \tag{6.38}$$

and therefore

$$\|\mathbf{De}^m\|^2_{l^\infty(I_k,L^2(\Omega))} \leq k^{-1}\|\mathbf{De}^m\|^2_{l^2(I_k,L^2(\Omega))} \leq C\,k^{\frac{-12+4p_\infty+rp_\infty}{4-4p_\infty+rp_\infty}}. \tag{6.39}$$

Since $r > \max\{4, \frac{12}{p_\infty(5p_\infty-6)}\}$, there holds $\frac{-12+4p_\infty+rp_\infty}{4-4p_\infty+rp_\infty} \geq 0$. This implies

$$\|\mathbf{De}^m\|^2_{l^\infty(I_k,L^2(\Omega))} \leq C. \tag{6.40}$$

So Korn's inequality gives

$$\|\boldsymbol{\nabla}\mathbf{e}^m\|^2_{l^\infty(I_k,L^2(\Omega))} \leq C. \tag{6.41}$$

The regularity of $\mathbf{u}^m$ implies $\|\boldsymbol{\nabla}\mathbf{v}^m\|^2_{l^\infty(I_k,L^2(\Omega))} \leq C$. Testing (6.6) with $-\Delta\mathbf{v}^m$ gives

$$\begin{aligned}\tfrac{1}{2}d_t\|\boldsymbol{\nabla}\mathbf{v}^m\|_2^2 &+ \mathcal{I}_\Phi{}^A(\mathbf{v}^m) \\ &\leq \|\boldsymbol{\nabla}\mathbf{f}^m\|_2\|\boldsymbol{\nabla}\mathbf{v}^m\|_2 + \Big|\int_\Omega \big((\mathbf{v}^m\cdot\boldsymbol{\nabla})\mathbf{v}^m\big)\Delta\mathbf{v}^m\,dx\Big| \\ &\leq C + \|\boldsymbol{\nabla}\mathbf{v}^m\|_3^3.\end{aligned} \tag{6.42}$$

Note that by lemma 3.14

$$\|\nabla \mathbf{D}\mathbf{v}^m\|_{\frac{4}{4-p_\infty}} \le C\,\mathcal{I}_\Phi{}^A(\mathbf{v}^m)^{\frac{1}{2}}\|(\widetilde{D}\mathbf{v}^m)^{\frac{2-p}{2}}\|_{\frac{4}{2-p_\infty}}$$
$$\le C\,\mathcal{I}_\Phi{}^A(\mathbf{v}^m)^{\frac{1}{2}}\|\widetilde{D}\mathbf{v}^m\|_2^{\frac{2-p_\infty}{2}}.$$

So from (6.40) and lemma 8.5 we deduce

$$\|\mathbf{v}^m\|^2_{1,\frac{12}{8-3p_\infty}} \le C\,\|\nabla^2\mathbf{v}^m\|^2_{\frac{4}{4-p_\infty}} \le C\,\|\nabla\mathbf{D}\mathbf{v}^m\|^2_{\frac{4}{4-p_\infty}} \le C\,\mathcal{I}_\Phi{}^A(\mathbf{v}^m). \tag{6.43}$$

So (6.42) and (6.43) imply

$$\tfrac{1}{2}d_t\|\nabla\mathbf{v}^m\|_2^2 + \|\nabla\mathbf{v}^m\|^2_{\frac{12}{8-3p_\infty}} + \mathcal{I}_\Phi{}^A(\mathbf{v}^m) \le C + C\,\|\nabla\mathbf{v}^m\|_3^3. \tag{6.44}$$

In order to handle $\|\nabla\mathbf{v}^m\|_3^3$ we calculate

$$\|\nabla\mathbf{v}^m\|_3^3 = \int_\Omega |\nabla\mathbf{v}^m|^3\,dx$$
$$\le C\int_\Omega |\nabla\mathbf{v}^m|^{\frac{3}{2}}|\nabla\mathbf{u}^m|^{\frac{3}{2}}\,dx + \int_\Omega |\nabla\mathbf{v}^m|^{\frac{3}{2}}|\nabla\mathbf{e}^m|^{\frac{3}{2}}\,dx$$
$$=: J_1 + J_2.$$

Now

$$J_2 \le C\,\|\nabla\mathbf{v}^m\|_6^{\frac{3}{2}}\|\nabla\mathbf{e}^m\|_2^{\frac{3}{2}} \le \delta A^{p_\infty-2}\|\nabla\mathbf{v}^m\|_6^2 + C_\delta A^{3(2-p_\infty)}\|\nabla\mathbf{e}^m\|_2^6.$$

Due to (3.59) the term $\delta A^{p_\infty-2}\|\nabla\mathbf{v}^m\|_6^2$ can be absorbed by the left-hand side of (6.44). We will show now that the last term can be handled by the discrete version of Gronwall's lemma. By Korn's inequality

$$\begin{aligned}\|\nabla\mathbf{e}^m\|_{l^4(I_k,L^2(\Omega))} &\le C\,k^{-\frac{1}{4}}\|\mathbf{D}\mathbf{e}^m\|_{l^2(I_k,L^2(\Omega))}\\ &\overset{(6.38)}{\le} C\,k^{-\frac{1}{4}}kA^{2-p_\infty}\\ &= C\,k^{\frac{-20-4p_\infty+3rp_\infty}{4(4-4p_\infty+rp_\infty)}}.\end{aligned} \tag{6.45}$$

So

$$\begin{aligned}k\sum_{m=0}^{M}\Big(A^{3(2-p_\infty)}\,\|\nabla\mathbf{e}^m\|_2^4\Big) &= A^{3(2-p_\infty)}\|\nabla\mathbf{e}^m\|^4_{l^4(I_k,L^2(\Omega))}\\ &\overset{(6.45)}{\le} C\;k^3A^{7(2-p_\infty)}\\ &= C\,k^{\frac{-44+16p_\infty+3rp_\infty}{4-4p_\infty+rp_\infty}}.\end{aligned} \tag{6.46}$$

Since $r > \max\{4, \frac{12}{p_\infty(5p_\infty-6)}\}$, there holds $\frac{-44+16p_\infty+3rp_\infty}{4-4p_\infty+rp_\infty} < 0$, so the left-hand side of (6.46) is bounded in some $l^\beta(I_k)$ with $\beta > 1$. So we can apply the discrete version of Gronwall's lemma to control

$$C_\delta A^{3(2-p_\infty)}\|\nabla\mathbf{e}^m\|_2^6 = C_\delta\,\underbrace{A^{3(2-p_\infty)}\|\nabla\mathbf{e}^m\|_2^4}_{\in l^\beta(I_k)}\|\nabla\mathbf{e}^m\|_2^2$$

for sufficiently small k. J_2 is handled and J_1 remains.

$$
\begin{aligned}
J_1 &\le C \left\| |\boldsymbol{\nabla}\mathbf{u}^m|^{\frac{3}{2}} \right\|_{\frac{rp_\infty}{3},\infty} \left\| |\boldsymbol{\nabla}\mathbf{v}^m|^{\frac{3}{2}} \right\|_{\frac{2rp_\infty}{2rp_\infty-6},1} \\
&\le C \, \|\boldsymbol{\nabla}\mathbf{u}^m\|_{\frac{rp_\infty}{2},\infty}^{\frac{3}{2}} \|\boldsymbol{\nabla}\mathbf{v}^m\|_{\frac{3rp_\infty}{2rp_\infty-6},1}^{\frac{3}{2}} \\
&\le C \, \|\mathbf{D}\mathbf{u}^m\|_{\frac{rp_\infty}{2},\infty}^{\frac{3}{2}} \|\boldsymbol{\nabla}\mathbf{v}^m\|_{\frac{3rp_\infty}{2rp_\infty-6},1}^{\frac{3}{2}} \qquad \text{(by Korn)} \\
&\le C \, \|\boldsymbol{\nabla}\mathbf{v}^m\|_{\frac{3rp_\infty}{2rp_\infty-6},1}^{\frac{3}{2}}.
\end{aligned}
$$

If $r > \frac{12}{p_\infty}$ then $\frac{3rp_\infty}{2rp_\infty-2} < 2$ and J_1 is bounded due to (6.41) and the embedding $L^2(\Omega) \hookrightarrow L^{\frac{3rp_\infty}{2rp_\infty-6},1}(\Omega)$. If on the other hand $\max\{4, \frac{12}{p_\infty(5p_\infty-6)}\} < r < \frac{12}{p_\infty}$, then choose θ such that $\frac{2rp_\infty-6}{3rp_\infty} = \frac{(1-\theta)(8-3p_\infty)}{12} + \frac{\theta}{2}$, i.e. $\theta = \frac{3(rp_\infty^2-8)}{rp_\infty(3p_\infty-2)}$ and $1-\theta = \frac{2(12-rp_\infty)}{rp_\infty(3p_\infty-2)}$. So

$$
\begin{aligned}
J_2 &\le C \, \|\boldsymbol{\nabla}\mathbf{v}^m\|_{\frac{12}{8-3p_\infty}}^{\frac{2(12-rp_\infty)}{rp_\infty(3p_\infty-2)}} \|\boldsymbol{\nabla}\mathbf{v}^m\|_2^{\frac{9(rp_\infty-8)}{2rp_\infty(3p_\infty-2)}} \\
&\le C \, \|\boldsymbol{\nabla}\mathbf{v}^m\|_{\frac{12}{8-3p_\infty}}^{\frac{2(12-rp_\infty)}{rp_\infty(3p_\infty-2)}}.
\end{aligned}
$$

Since $r > \max\{4, \frac{12}{p_\infty(5p_\infty-6)}\}$, there holds $\frac{2(12-rp_\infty)}{rp_\infty(3p_\infty-2)} < 1$, so

$$
J_2 \le \delta \|\boldsymbol{\nabla}\mathbf{v}^m\|_{\frac{12}{3-p_\infty}}^2 + C_\delta.
$$

Note that $\delta\|\boldsymbol{\nabla}\mathbf{v}^m\|_{\frac{12}{8-3p_\infty}}^2$ can be absorbed by the left-hand side of (6.44). Overall we have shown

$$
\begin{aligned}
&\tfrac{1}{2} d_t \|\boldsymbol{\nabla}\mathbf{v}^m\|_2^2 + \|\boldsymbol{\nabla}\mathbf{v}^m\|_{\frac{12}{8-3p_\infty}}^2 + \mathcal{I}_\Phi{}^A(\mathbf{v}^m) \\
&\qquad \le C + \underbrace{A^{3(2-p_\infty)} \|\boldsymbol{\nabla}\mathbf{e}^m\|_2^4}_{\in l^\beta(I_k)} \|\boldsymbol{\nabla}\mathbf{e}^m\|_2^2.
\end{aligned}
$$

So the discrete version of Gronwall's inequality gives

$$
\|\boldsymbol{\nabla}\mathbf{v}^m\|_{l^\infty(I_k,L^2(\Omega))}^2 + \|\boldsymbol{\nabla}\mathbf{v}^m\|_{l^2(I_k,L^{\frac{12}{8-3p_\infty}}(\Omega))}^2 + \|\mathcal{I}_\Phi{}^A(\mathbf{v}^m)\|_{l^1(I_k)} \le C. \tag{6.47}
$$

Thus (6.43) implies

$$
\|\mathbf{v}^m\|_{l^2(I_k,W^{2,\frac{4}{4-p_\infty}}(\Omega))} \le C.
$$

Now we are in the situation to apply more test functions to (6.6). We test with $d_t\mathbf{v}^m$ and get

$$
\begin{aligned}
&\|d_t\mathbf{v}^m\|_2^2 + \tfrac{1}{k}\big\langle \mathbf{S}^A(\mathbf{D}\mathbf{v}^m), \mathbf{D}\mathbf{v}^m - \mathbf{D}\mathbf{v}^{m-1} \big\rangle \\
&\qquad \le \big\langle (\mathbf{v}^m \cdot \boldsymbol{\nabla})\mathbf{v}^m, d_t\mathbf{v}^m \big\rangle + C_\delta \|\mathbf{f}^m\|_2^2 + \delta \|d_t\mathbf{v}^m\|_2^2.
\end{aligned}
$$

Since Φ^A is convex, there holds

$$
\begin{aligned}
\big\langle \mathbf{S}^A(\mathbf{D}\mathbf{v}^m), \mathbf{D}\mathbf{v}^m - \mathbf{D}\mathbf{v}^{m-1} \big\rangle &= \big\langle (\boldsymbol{\nabla}\Phi^A)(\mathbf{D}\mathbf{v}^m), \mathbf{D}\mathbf{v}^m - \mathbf{D}\mathbf{v}^{m-1} \big\rangle \\
&\ge c\left(\Phi^A(\mathbf{D}\mathbf{v}^m) - \Phi^A(\mathbf{D}\mathbf{v}^{m-1})\right),
\end{aligned}
$$

so after summing up over $m = 1, \dots, M$ we get

$$\begin{aligned}
&\|d_t \mathbf{v}^m\|_{l^2(I_k,L^2(\Omega))} + \Phi^A(\mathbf{D}\mathbf{v}^M) \\
&\qquad \le \sum_{k=1}^{M} \big(C\,\|(\mathbf{v}^m\cdot\boldsymbol{\nabla})\mathbf{v}^m\|_2\|d_t\mathbf{v}^m\|_2\big) + \Phi^A(\mathbf{D}\mathbf{v}^0) \\
&\qquad \le \sum_{k=1}^{M} \big(C_\delta\,\|\boldsymbol{\nabla}\mathbf{v}^m\|_{\frac{12}{5}}^4 + \delta\|d_t\mathbf{v}^m\|_2^2\big) + \Phi^A(\mathbf{D}\mathbf{u}_0).
\end{aligned}$$

We absorb the term $\delta\|d_t\mathbf{v}^m\|_2^2$ on the left-hand side. Since $p_\infty \ge \frac{4}{3}$ there holds

$$[l^\infty(I_k, L^2(\Omega)), l^2(I_k, L^{\frac{12}{8-3p_\infty}}(\Omega))]_{\frac12} = l^4(I_k, L^{\frac{24}{14-3p_\infty}}(\Omega)) \hookrightarrow l^4(I_k, L^{\frac{12}{5}}(\Omega)).$$

So from (6.47) we deduce for $p_\infty \ge \frac43$ that $\|\boldsymbol{\nabla}\mathbf{v}^m\|_{l^4(I_k,L^{\frac{12}{5}}(\Omega))} \le C$. Thus

$$\|d_t\mathbf{v}^m\|_{l^2(I_k,L^2(\Omega))} + \Phi^A(\mathbf{D}\mathbf{v}^M) \le C. \tag{6.48}$$

If we apply d_t to (6.6) and use $d_t\mathbf{v}^m$ as a test function, then we get

$$\begin{aligned}
&\tfrac12 d_t\|d_t\mathbf{v}^m\|_2^2 - \big\langle d_t \operatorname{div}(\mathbf{S}^A(\mathbf{D}\mathbf{v}^m)), d_t\mathbf{v}^m\big\rangle \\
&\qquad \le \big|\big\langle d_t\big((\mathbf{v}^m\cdot\boldsymbol{\nabla})\mathbf{v}^m\big), d_t\mathbf{v}^m\big\rangle\big| + \underbrace{\|d_t\mathbf{f}^m\|_2^2 + \|d_t\mathbf{v}^m\|_2^2}_{\in l^1(I_k)},
\end{aligned} \tag{6.49}$$

where we have used that $\partial_t\mathbf{f} \in C^\infty(I, L^2(\Omega))$. Note that

$$\begin{aligned}
&\big|\big\langle d_t((\mathbf{v}^m\cdot\boldsymbol{\nabla}\mathbf{v}^m), d_t\mathbf{v}^m\big\rangle\big| \\
&\qquad = \big|\big\langle (d_t\mathbf{v}^m\cdot\boldsymbol{\nabla})\mathbf{v}^m, d_t\mathbf{v}^m\big\rangle + \big\langle (\mathbf{v}^{m-1}\cdot\boldsymbol{\nabla})d_t\mathbf{v}^m, d_t\mathbf{v}^m\big\rangle\big| \\
&\qquad = \big|\big\langle (d_t\mathbf{v}^m\cdot\boldsymbol{\nabla})\mathbf{v}^m, d_t\mathbf{v}^m\big\rangle\big|.
\end{aligned}$$

On the other hand by (3.38)

$$\begin{aligned}
&-\big\langle d_t \operatorname{div}(\mathbf{S}^A(\mathbf{D}\mathbf{v}^m)), d_t\mathbf{v}^m\big\rangle \\
&\qquad = c\,\tfrac{1}{k^2}\big\langle \mathbf{S}^A(\mathbf{D}\mathbf{v}^m) - \mathbf{S}^A(\mathbf{D}\mathbf{v}^{m-1}), \mathbf{D}\mathbf{v}^m - \mathbf{D}\mathbf{v}^{m-1})\big\rangle \\
&\qquad \ge c\,\tfrac{1}{k^2}\int_\Omega (1 + |\mathbf{D}\mathbf{v}^m|^2 + |\mathbf{D}\mathbf{v}^{m-1}|^2)^{\frac{p-2}{2}}|\mathbf{D}\mathbf{v}^m - \mathbf{D}\mathbf{v}^{m-1}|^2\,dx \\
&\qquad = c\int_\Omega (1 + |\mathbf{D}\mathbf{v}^m|^2 + |\mathbf{D}\mathbf{v}^{m-1}|^2)^{\frac{p-2}{2}}|d_t\mathbf{D}\mathbf{v}^m|^2\,dx.
\end{aligned} \tag{6.50}$$

Analogously to (6.43) we can show that this implies

$$-\big\langle d_t \operatorname{div}(\mathbf{S}^A(\mathbf{D}\mathbf{v}^m)), d_t\mathbf{v}^m\big\rangle \ge C\,\|d_t\boldsymbol{\nabla}\mathbf{v}^m\|^2_{\frac{4}{4-p_\infty}} \ge C\,\|d_t\mathbf{v}^m\|^2_{\frac{12}{8-3p_\infty}}. \tag{6.51}$$

From (3.38) and the second line of (6.50) we also deduce that

$$\begin{aligned}
-\big\langle d_t \operatorname{div}(\mathbf{S}^A(\mathbf{D}\mathbf{v}^m)), d_t\mathbf{v}^m\big\rangle &\ge C\,A^{2-p_\infty}\|d_t\boldsymbol{\nabla}\mathbf{v}^m\|_2^2 \\
&\ge C\,A^{2-p_\infty}\|d_t\mathbf{v}^m\|_6^2.
\end{aligned} \tag{6.52}$$

Now we will estimate $|\langle(d_t\mathbf{v}^m\cdot\boldsymbol{\nabla})\mathbf{v}^m, d_t\mathbf{v}^m\rangle|$ as we did with $\|\boldsymbol{\nabla}\mathbf{v}^m\|_3^3$.

$$\big|\big\langle(d_t\mathbf{v}^m\cdot\boldsymbol{\nabla})\mathbf{v}^m, d_t\mathbf{v}^m\big\rangle\big| \le \int\limits_\Omega |d_t\mathbf{v}^m|^2|\boldsymbol{\nabla}\mathbf{u}^m|\,dx + \int\limits_\Omega |d_t\mathbf{v}^m|^2|\boldsymbol{\nabla}\mathbf{e}^m|\,dx$$
$$=: R_1 + R_2.$$

So

$$R_1 \le C\,\|d_t\mathbf{v}^m\|^2_{\frac{2rp_\infty}{rp_\infty-2},1}\|\boldsymbol{\nabla}\mathbf{u}^m\|_{\frac{rp_\infty}{2},\infty} \le C\,\|d_t\mathbf{v}^m\|^2_{\frac{2rp_\infty}{rp_\infty-2},1}.$$

For $r > \frac{12}{p_\infty(5p_\infty-6)}$ there holds $2 < \frac{2rp_\infty}{rp_\infty-2} < \frac{12}{8-3p_\infty}$. Therefore $2 < \frac{2rp_\infty}{rp_\infty-2} + 2 < \frac{12}{8-3p_\infty}$ for some $\varepsilon > 0$. Hence we can use the embedding $L^{\frac{2rp_\infty}{rp_\infty-2}+\varepsilon}(\Omega) \hookrightarrow L^{\frac{2rp_\infty}{rp_\infty-2},1}(\Omega)$ and interpolate between $L^2(\Omega)$ and $L^{\frac{12}{8-3p_\infty}}(\Omega)$ and use Young's inequality to deduce

$$R_1 \le \delta\|d_t\mathbf{v}^m\|^2_{\frac{12}{8-3p_\infty}} + C_\delta.$$

Moreover

$$\begin{aligned}R_2 &\le C\,\|\boldsymbol{\nabla}\mathbf{e}^m\|_2\|d_t\mathbf{v}^m\|_4^2\\ &\le C\,\|\boldsymbol{\nabla}\mathbf{e}^m\|_2\|d_t\mathbf{v}^m\|_2^{\frac12}\|d_t\mathbf{v}^m\|_6^{\frac32}\\ &\le C_\delta A^{3(2-p_\infty)}\|\boldsymbol{\nabla}\mathbf{e}^m\|_2^4\|d_t\mathbf{v}^m\|_2^2 + \delta A^{p_\infty-2}\|d_t\mathbf{v}^m\|_6^2.\end{aligned}$$

Due to (6.52) we can absorb the last term later on the left-hand side. For the other term note that from the calculations for (6.46) we see that for $r > \max\{4, \frac{12}{p_\infty(5p_\infty-6)}\}$ there holds

$$k\sum_{m=0}^M \Big(A^{3(2-p_\infty)}\,\|\boldsymbol{\nabla}\mathbf{e}^m\|_2^4\Big) \le C\,k^\mu$$

for some $\mu > 0$, especially the left-hand side is in $L^q(I_k)$ for some $q > 1$. This enables us apply the discrete version of Gronwall's inequality to handle (for sufficiently small k) the rest of R_2. Taking the sum of (6.49) over k we get

$$\tfrac12\|d_t\mathbf{v}^m\|^2_{l^\infty(I_k,L^2(\Omega))} + \|d_t\boldsymbol{\nabla}\mathbf{v}^m\|^2_{l^2(I_k,L^{\frac{4}{4-p_\infty}}(\Omega))} + \|d_t\mathbf{v}^m\|^2_{l^2(I_k,L^{\frac{12}{8-3p_\infty}}(\Omega))} \le C.$$

Overall we have shown that

$$\begin{aligned}\|\mathbf{v}^m\|_{l^\infty(I_k,W^{1,2}(\Omega))} + \|\mathbf{v}^m\|_{l^2(I_k,W^{1,\frac{12}{8-3p_\infty}}(\Omega))} + \|\mathbf{v}^m\|_{l^2(I_k,W^{2,\frac{4}{4-p_\infty}}(\Omega))} &\le C,\\ \|d_t\mathbf{v}^m\|_{l^\infty(I_k,L^2(\Omega))} + \|d_t\mathbf{v}^m\|_{l^2(I_k,L^{\frac{12}{8-3p_\infty}}(\Omega))} + \|d_t\mathbf{v}^m\|_{l^\infty(I_k,W^{1,\frac{4}{4-p_\infty}}(\Omega))} &\le C.\end{aligned}$$

This proves theorem 6.4.

6. Improved $l^\infty(I_k)$–Error Estimate

Next, we will derive better pointwise in time estimates of the error in case of strong solutions $\mathbf{v}^m, q^m$. Assume that the conditions of theorem 6.5 are satisfied. Then from the derivation of (6.36) we deduce

$$\begin{aligned}\langle d_t\mathbf{e}^m, \mathbf{e}^m\rangle &+ \big\langle \boldsymbol{\sigma}^{A,m}_{\mathbf{u},\mathbf{v}}\mathbf{D}\mathbf{e}^m, \mathbf{D}\mathbf{e}^m\big\rangle \\ &\le |\langle \mathbf{R}^m, \mathbf{e}^m\rangle| + CA^{-\frac{(r-2)p_\infty}{2}} \\ &\quad + \|\mathbf{e}^m\|_2^2(1+\|\boldsymbol{\nabla}\mathbf{u}^m\|_{p_\infty}+\|\boldsymbol{\nabla}\mathbf{v}^m\|_{p_\infty})^{\frac{(2-p_\infty)\theta}{1-\theta}} \\ &\le |\langle \mathbf{R}^m, \mathbf{e}^m\rangle| + CA^{-\frac{(r-2)p_\infty}{2}} + C\,\|\mathbf{e}^m\|_2^2.\end{aligned}$$

Since $A = k^{\frac{4}{4p_\infty - rp_\infty - 4}}$ and $\alpha(p_\infty) = \frac{(r-2)p_\infty}{4-4p_\infty+rp_\infty}$, this implies

$$\begin{aligned}\langle d_t\mathbf{e}^m, \mathbf{e}^m\rangle &+ \big\langle \boldsymbol{\sigma}^{A,m}_{\mathbf{u},\mathbf{v}}\mathbf{D}\mathbf{e}^m, \mathbf{D}\mathbf{e}^m\big\rangle \\ &\le Ck^{2\alpha(p_\infty)} + |\langle \mathbf{R}^m, \mathbf{e}^m\rangle| + C\,\|\mathbf{e}^m\|_2^2.\end{aligned}$$

Hence

$$\begin{aligned}\big\langle \boldsymbol{\sigma}^{A,m}_{\mathbf{u},\mathbf{v}}&\mathbf{D}\mathbf{e}^m, \mathbf{D}\mathbf{e}^m\big\rangle \\ &\le Ck^{2\alpha(p_\infty)} + |\langle \mathbf{R}^m, \mathbf{e}^m\rangle| + C\,\|\mathbf{e}^m\|_2^2 + |\langle d_t\mathbf{e}^m, \mathbf{e}^m\rangle| \\ &\le Ck^{2\alpha(p_\infty)} + \|\mathbf{e}^m\|_2\Big(\|\mathbf{R}^m\|_2 + \|\mathbf{e}^m\|_2 + \|d_t\mathbf{e}^m\|_2\Big) \\ &\le Ck^{2\alpha(p_\infty)} + \|\mathbf{e}^m\|_2\Big(\|\mathbf{R}^m\|_2 + \|\mathbf{e}^m\|_2 + \|d_t\mathbf{u}^m\|_2 + \|d_t\mathbf{v}^m\|_2\Big).\end{aligned}$$

Now theorem 6.4 and lemma 6.7 imply

$$\begin{aligned}\big\langle \boldsymbol{\sigma}^{A,m}_{\mathbf{u},\mathbf{v}}\mathbf{D}\mathbf{e}^m, \mathbf{D}\mathbf{e}^m\big\rangle &\le Ck^{2\alpha(p_\infty)} + \|\mathbf{e}^m\|_2\Big(\|\mathbf{R}^m\|_2 + \|\mathbf{e}^m\|_2 + C\Big) \\ &\le Ck^{2\alpha(p_\infty)} + C\,k^{\alpha(p_\infty)}\Big(C + C\,k^{\alpha(p_\infty)} + C\Big) \\ &\le C\,k^{\alpha(p_\infty)}.\end{aligned}$$

This proves theorem 6.5.

7. Semi Implicit

For practical purposes it would be better to use the semi implicit scheme

$$\begin{aligned} d_t\mathbf{w}^m - \operatorname{div}(\mathbf{S}^A(\mathbf{D}\mathbf{w}^m)) + (\mathbf{w}^{m-1}\cdot\boldsymbol{\nabla})\mathbf{w}^m + \boldsymbol{\nabla}\psi^m &= \mathbf{f}^m \quad \text{on } I_k\times\Omega, \\ \operatorname{div}\mathbf{w}^m &= 0 \quad \text{on } I_k\times\Omega, \\ \mathbf{w}(0) &= \mathbf{u}_0 \quad \text{on } \Omega \end{aligned} \tag{6.53}$$

with $m = 1, 2, \ldots, M$ instead of the fully implicit scheme (6.6). This way, the problem arising from the convective part at every time step is linear. In this section we will see that this scheme has exactly the same properties as the fully implicit scheme (6.6). Especially the existence of weak and strong solutions holds for the same range of p_∞ and the error satisfies the same estimates in terms of the time step size k. The only additional requirement needed is that the solution $\mathbf{u}$ of (6.1) additionally satisfies $\mathbf{u}\in C(I\times\Omega)$, which implies

$$\mathbf{u}^m \in l^\infty(I_k, L^\infty(\Omega)).$$

Since most of the calculations for the semi implicit scheme agree with the ones for the fully implicit scheme, we will only point out the differences to the previous sections.

Since $\langle(\mathbf{w}^{m-1}\cdot\boldsymbol{\nabla})\mathbf{w}^m,\mathbf{w}^m\rangle=0$ it follows exactly as in section 3 that there exists a unique, weak solution $\mathbf{w}^m,\psi^m$ of (6.53). Further $\mathbf{w}^m\in W^{2,2}_{\operatorname{div}}(\Omega)$ and $\psi^m\in W^{1,2}(\Omega)$ for all m, justifying the later calculations.

Let $\mathbf{E}^m:=\mathbf{u}^m-\mathbf{w}^m$ and $\xi^m:=\pi^m-\psi^m$ denote the error, then (compare (6.17) for the error of the fully implicit scheme)

$$
\begin{aligned}
d_t\mathbf{E}^m-\operatorname{div}(\boldsymbol{\sigma}^{A,m}_{\mathbf{u},\mathbf{w}}\mathbf{D}\mathbf{E}^m)&\\
&=\quad \mathbf{R}^m+\operatorname{div}(\mathbf{S}^A(\mathbf{D}\mathbf{u}^m)-\mathbf{S}(\mathbf{D}\mathbf{u}^m))-\boldsymbol{\nabla}\xi^m\\
&\quad\quad -k(d_t\mathbf{u}^m\cdot\boldsymbol{\nabla})\mathbf{u}^m-(\mathbf{E}^{m-1}\cdot\boldsymbol{\nabla})\mathbf{u}^m-(\mathbf{w}^{m-1}\cdot\boldsymbol{\nabla})\mathbf{E}^m\\
\operatorname{div}\mathbf{E}^m&=0,\\
\mathbf{E}^0&=0.
\end{aligned}
\tag{6.54}
$$

The only term that has to be handled differently to the fully implicit case when estimating the error $\mathbf{E}^m$ is $k(d_t\mathbf{u}^m\cdot\boldsymbol{\nabla})\mathbf{u}^m$. Tested with $\mathbf{E}^m$ this term gives

$$
\begin{aligned}
\big|\big\langle k(d_t\mathbf{u}^m\cdot\boldsymbol{\nabla})\mathbf{u}^m,\mathbf{E}^m\big\rangle\big|&=k\big|\big\langle(d_t\mathbf{u}^m\cdot\boldsymbol{\nabla})\mathbf{E}^m,\mathbf{u}^m\big\rangle\big|\\
&\le k\|d_t\mathbf{u}^m\|_2\|\boldsymbol{\nabla}\mathbf{E}^m\|_2\|\mathbf{u}^m\|_\infty\\
&\le k\,C\|\boldsymbol{\nabla}\mathbf{E}^m\|_2\\
&\le k^2\,C_\delta A^{2-p_\infty}+\delta A^{p_\infty-2}\|\boldsymbol{\nabla}\mathbf{E}^m\|_2^2,
\end{aligned}
$$

where we have used that $\mathbf{u}\in C(I\times\Omega)$ and $\partial_t\mathbf{u}\in C(I,L^2(\Omega))$. So there arises a term of the same order as in the handling of $|\langle\mathbf{R}^m,\mathbf{e}^m\rangle|$ (compare with (6.36)). Consequently the calculations for $\mathbf{E}^m$ agree with the one for $\mathbf{e}^m$. Especially we get exactly the same error estimates. Note that whenever there appears $m-1$ instead of m as time step index, e.g. $\mathbf{e}^{m-1}$ instead of $\mathbf{e}^m$, the corresponding estimates become easier to handle: Indeed the natural version of the discrete Gronwall inequality requires $m-1$ as highest time step index on the right-hand side (which has to be controlled). Terms at time step m can only be controlled for k sufficiently small (see the discrete Gronwall in the appendix).

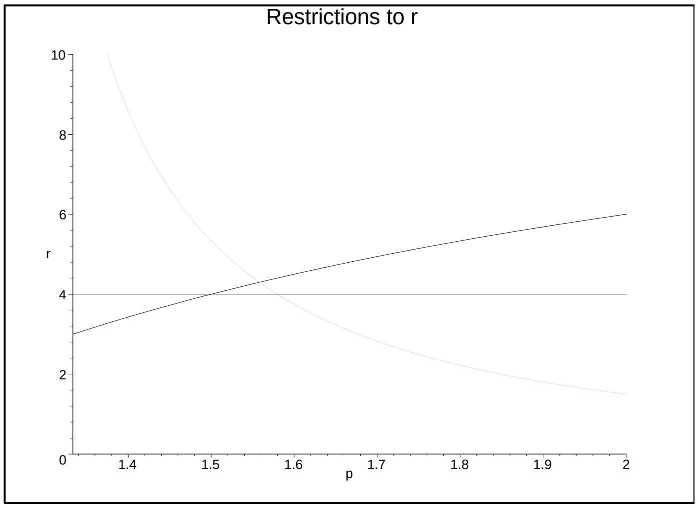
The rest of the calculations for the semi implicit scheme regarding the existence of strong solutions and the improved pointwise in time estimate agree precisely with the ones for the fully implicit scheme. Thus, as stated in theorem 6.6, the theorems 6.1, 6.3, 6.4, and 6.5 also hold true for the semi implicit scheme.

8. Plots

In the end of this chapter we like to present some plots regarding the order of convergence and the requirements used in their proof. In theorem 6.4 (existence of strong solutions) and 6.5 (improved error estimate) we have assumed that the exponent p and the constant r representing the regularity of $\mathbf{u}$ satisfy

$$\tfrac{4}{3}\le p_\infty\le p_0\le 2,\qquad\qquad r>\max\left\{4,\tfrac{12}{p_\infty(5p_\infty-6)}\right\}.$$

Furthermore we know from chapter 5 that for $p_\infty>\frac{7}{5}$ the solution $\mathbf{u}$ has enough regularity to choose $r:=\frac{12(p_\infty-1)}{p_\infty}$ as a minimal value of r. Nevertheless it might be possible to prove better regularity of $\mathbf{u}$ providing a higher value for r. The following plot therefore shows how the requirements for r correspond to the proven minimal value $\frac{12(p_\infty-1)}{p_\infty}$. Although the minimal value is only proven for $p_\infty>\frac{7}{5}$ the range of the plot is $\frac{4}{3} \leq p_\infty \leq 2$, since this is the natural range of theorem 6.4 and 6.5. Although for $p_\infty > 1.57980$ the restriction $r > 4$ is the stronger one, the other



condition $r > \frac{12}{p_\infty(5p_\infty - 6)}$ is the really crucial one. This becomes clear by the fact that the minimal proven r, namely $\frac{12(p_\infty - 1)}{p_\infty}$, is bigger than 4 for all $p_\infty > 1.57980$. Thus $r > 4$ is only a technical requirement. The intersection of $r > \frac{12}{p_\infty(5p_\infty-6)}$ with $r = \frac{12(p_\infty-1)}{p_\infty}$ is what determines the bound $p_\infty > \frac{11+\sqrt{21}}{10} \approx 1.55826$.

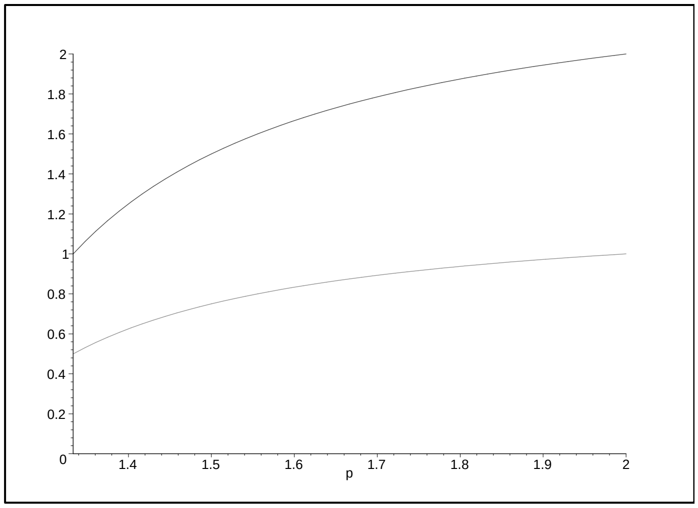
Let us turn to the order of convergence. Recall that by theorem 6.5 and 6.5 we have

$$\frac{1}{2}\|\mathbf{e}^m\|^2_{l^\infty(I_k,L^2(\Omega))} + \left\|\left\langle \boldsymbol{\sigma}^{A,m}_{\mathbf{u},\mathbf{v}} \mathbf{D}\mathbf{e}^m, \mathbf{D}\mathbf{e}^m \right\rangle\right\|_{l^1(I_k)} \leq C\, k^{2\alpha(p_\infty)}$$
$$\left\|\left\langle \boldsymbol{\sigma}^{A,m}_{\mathbf{u},\mathbf{v}} \mathbf{D}\mathbf{e}^m, \mathbf{D}\mathbf{e}^m \right\rangle\right\|_{l^\infty(I_k)} \leq C\, k^{\alpha(p_\infty)}$$

with $\alpha(p_\infty) = \frac{(r-2)p_\infty}{4-4p_\infty+rp_\infty}$. So the order of convergence depends on r (as does the choice of A). Therefore we fix $r := \frac{12(p_\infty-1)}{p_\infty}$ to show the minimal proven order of convergence for the error. With this choice of r the error estimates imply

$$\frac{1}{2}\|\mathbf{e}^m\|^2_{l^\infty(I_k,L^2(\Omega))} + \left\|\left\langle \boldsymbol{\sigma}^{A,m}_{\mathbf{u},\mathbf{v}} \mathbf{D}\mathbf{e}^m, \mathbf{D}\mathbf{e}^m \right\rangle\right\|_{l^1(I_k)} \leq C\, k^{\frac{5p_\infty-6}{2(p_\infty-1)}},$$
$$\left\|\left\langle \boldsymbol{\sigma}^{A,m}_{\mathbf{u},\mathbf{v}} \mathbf{D}\mathbf{e}^m, \mathbf{D}\mathbf{e}^m \right\rangle\right\|_{l^\infty(I_k)} \leq C\, k^{\frac{5p_\infty-6}{4(p_\infty-1)}}.$$

The plot on the next page shows the values of $\frac{5p_\infty-6}{2(p_\infty-1)}$ and $\frac{5p_\infty-6}{4(p_\infty-1)}$ for the range $\frac{4}{3} \leq p_\infty \leq 2$. Note that the optimal order of convergence is 2, resp. 1.


2
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
1.4
1.5
1.6
1.7
1.8
1.9
2
p

CHAPTER 7

Stationary p–Stokes

1. Introduction

In this chapter we examine the stationary p–Stokes problem, i.e.

$$
\begin{aligned}
-\operatorname{div}(\mathbf{S}(\mathbf{Du})) + \boldsymbol{\nabla}\pi &= \mathbf{f}, \quad \text{on } \Omega,\\
\operatorname{div}\mathbf{u} &= 0, \quad \text{on } \Omega,
\end{aligned}
\tag{7.1}
$$

where Ω denotes the d-dimensional torus and $\mathbf{S}$ is induced by a space dependent p–potential with ellipticity constants γ_1, γ_2. Moreover, let $p \in C(\Omega)$ with

$$1 < p_\infty \le p_0 < \infty$$

fulfill one of the equivalent requirements of lemma 2.6, i.e. there exists a constant C_0 such that the module of continuity ω of p satisfies

$$\omega(R) \le \frac{C_0}{-\ln R}$$

for all $0 < R < 1$.

Assume that $\mathbf{u}, \pi$ is a weak solution of (7.1) with $\mathbf{f} \in L^d(\Omega)$, i.e. $\mathbf{u}, \pi$ solve (7.1) in the sense of distributions and there holds

$$|\mathbf{Du}|_{p(\cdot)} \le C. \tag{7.2}$$

This estimate corresponds to the boundedness of the natural energy $\langle \mathbf{S}(\mathbf{Du}), \mathbf{Du}\rangle$. We will show that $\mathbf{u}$ does indeed satisfy a better estimate, namely

$$|\widetilde{D}\mathbf{u}|_{(1+\delta)p(\cdot)} \le C$$

for some $\delta > 0$. This type of inequality is often called Meyer type estimate. We proceed in such a way, that we firstly derive Cacciopoli estimates and then deduce reverse Hölder inequalities. As a consequence we get the desired Meyer type estimate. To be more general we include the pressure stabilized case into our considerations, i.e. we will prove the result above for weak solutions of the system

$$
\begin{aligned}
-\operatorname{div}(\mathbf{S}(\mathbf{Du})) + \boldsymbol{\nabla}\pi &= \mathbf{f}, \qquad \text{on } \Omega,\\
\operatorname{div}\mathbf{u} &= \varepsilon\Delta\pi, \quad \text{on } \Omega,
\end{aligned}
\tag{7.3}
$$

with $\varepsilon \ge 0$.

2. Cacciopoli Estimate

This section is dedicated to the derivation of useful cacciopoli estimates. Let us start with some notation.

NOTATION 7.1. *Let $\mathcal{R}$ be the set of all rigid displacements, that are mappings of the form $\mathbf{x} \mapsto \mathbf{Cx} + \mathbf{b}$, where $\mathbf{C} \in \mathbb{R}^{d\times d}$ is a skew–symmetric matrix and $\mathbf{b} \in \mathbb{R}^d$.*

By $Q_R \Subset \Omega$ we denote an open cube with sides parallel to the axis and side length $2R$. Let x_0 be the center of Q_R, then Q_{2R} denotes the cube that we get by resizing Q_R by the factor 2, where the center stays fixed at x_0. Certainly any other factor is possible.

LEMMA 7.2. *Let $\mathbf{S}$ be a p–potential with ellipticity constants γ_1, γ_2 and $p \in C(\Omega)$, $1 < p_\infty \leq p_0 < \infty$. Further let $\mathbf{u}, \pi$ be a weak solution of (7.3) with $\mathbf{f} \in L^d(\Omega)$. Let $\Lambda := \frac{\gamma_2}{\gamma_1}$. Then there exists $R_0 > 0$, such that for all axis parallel cubes $Q_R \Subset \Omega$, $0 < R < R_0$ there holds*

$$\begin{aligned} \gamma_1 \fint_{Q_{R/2}} (\widetilde{D}\mathbf{u})^p \, dx + \varepsilon \fint_{Q_{R/2}} |\nabla \pi|^2 \, dx \\ \leq C\, \gamma_1 \Lambda^{p_0} \fint_{Q_R} \left|\tfrac{\mathbf{u}-\boldsymbol{\tau}}{R}\right|^{p_{0,Q_R}} dx + C\, \gamma_1 \Lambda^{p_0} \\ + \varepsilon C \fint_{Q_R} \left|\tfrac{\pi-\kappa}{R}\right|^2 dx + C\, \gamma_1^{-p_\infty+1} \|\mathbf{f}\|_d^{p_\infty}, \end{aligned}$$

where $\kappa \in \mathbb{R}$ is arbitrary and $\boldsymbol{\tau}$ is chosen such that

$$\int_{Q_R} (\mathbf{u} - \boldsymbol{\tau}) \cdot \boldsymbol{\varphi} \, dx = 0,$$

for all $\boldsymbol{\varphi} \in \mathcal{R}$. Note that the constants are independent of γ_1, γ_2, ε.

PROOF. Since p is uniformly continuous, there exists $0 < R_0 < 1$, such that

$$\frac{d p_{\infty,Q}}{d + p_{\infty,Q}} < p_{0,Q}$$

for all cubes Q with $\operatorname{diam} Q < R_0$. Fix $x_0 \in \Omega$ and let $Q_R := Q_R(x_0)$ with $0 < R \leq R_0$. Without loss of generality we further assume that R_0 is so small that Q_{2R} is still a strict subset of Ω and does not cover all of Ω.

Let $q_0 := p_{0,Q_R}$ and $q_\infty := p_{\infty,Q_R}$ and let $\boldsymbol{\tau} \in \mathcal{R}$ such that

$$\int_{Q_R} (\mathbf{u} - \boldsymbol{\tau}) \cdot \boldsymbol{\varphi} \, dx = 0$$

for all $\boldsymbol{\varphi} \in \mathcal{R}$. Define $\boldsymbol{\varphi} := (\mathbf{u} - \boldsymbol{\tau})\eta^{q_0} + \mathbf{v}$, where $\eta \in C_0^\infty(Q_{3R/4})$, $0 \leq \eta \leq 1$, $\eta|_{Q_{R/2}} = 1$, $\|\nabla \eta\|_\infty \leq C\, R^{-1}$, and $\mathbf{v}$ will be specified later. The divergence of $\boldsymbol{\varphi}$ is then given by

$$\operatorname{div} \boldsymbol{\varphi} = \eta^{q_0} \operatorname{div} \mathbf{u} + q_0 \eta^{q_0 - 1} (\nabla \eta) \cdot (\mathbf{u} - \boldsymbol{\tau}) + \operatorname{div} \mathbf{v}.$$

Let $g := \eta^{q_0-1}(\nabla \eta) \cdot (\mathbf{u} - \boldsymbol{\tau})$, then $\operatorname{supp}(g) \subset Q_{3R/4}$ and

$$\int_\Omega |g|^{q_0} \, dx \leq C \int_{Q_R} \left|\tfrac{\mathbf{u}-\boldsymbol{\tau}}{R}\right|^{q_0} dx < \infty.$$

Since $q_0 \in [p_\infty, p_0] \subset (1, \infty)$ we can choose by the theorem of Bogovskiĭ (see appendix lemma (8.2)) a vectorial function $\mathbf{v} \in W_0^{1,q_0}(Q_{3R/4})$, such that $\operatorname{div} \mathbf{v} = -g$ and

$$\int\limits_{Q_R} |\boldsymbol{\nabla} \mathbf{v}|^q \, dx \leq C \int\limits_{Q_R} |g|^q \, dx \leq C \int\limits_{Q_R} \left|\tfrac{\mathbf{u}-\boldsymbol{\tau}}{R}\right|^q dx \tag{7.4}$$

for all q with $q_\infty \leq q \leq q_0$. Hence $\operatorname{div} \boldsymbol{\varphi} = \eta^{q_0} \operatorname{div} \mathbf{u} = \eta^{q_0} \varepsilon \Delta \pi$. Further

$$\mathbf{D}\boldsymbol{\varphi} = \eta^{q_0} \mathbf{D}\mathbf{u} + 2\eta^{q_0 - 1}\big((\boldsymbol{\nabla}\eta) \otimes (\mathbf{u} - \boldsymbol{\tau})\big)^{\text{sym}} + \mathbf{D}\mathbf{v}.$$

Testing the equation of motion with $\boldsymbol{\varphi}$ we get

$$\langle \mathbf{S}(\mathbf{D}\mathbf{u}), \mathbf{D}\boldsymbol{\varphi} \rangle - \langle \pi - \kappa, \operatorname{div} \boldsymbol{\varphi} \rangle = \langle \mathbf{f}, \boldsymbol{\varphi} \rangle,$$

where $\kappa \in \mathbb{R}$ can be chosen arbitrarily. Thus

$$\begin{aligned} &\langle \mathbf{S}(\mathbf{D}\mathbf{u}), \eta^{q_0} \mathbf{D}\mathbf{u} \rangle \\ &+ \langle \mathbf{S}(\mathbf{D}\mathbf{u}), 2\eta^{q_0-1}(\boldsymbol{\nabla}\eta \otimes (\mathbf{u} - \boldsymbol{\tau}))^{\text{sym}} \rangle \\ &+ \langle \mathbf{S}(\mathbf{D}\mathbf{u}), \mathbf{D}\mathbf{v} \rangle \\ &- \varepsilon \langle \pi - \kappa, \eta^{q_0} \Delta \pi \rangle \qquad\qquad = \langle \mathbf{f}, \eta^{q_0}(\mathbf{u} - \boldsymbol{\tau}) \rangle + \langle \mathbf{f}, \mathbf{v} \rangle. \end{aligned}$$

We rewrite this equation by $I_1 + I_2 + I_3 + I_4 = I_5 + I_6$ and estimate $I_1, \ldots, I_6$.

$$I_1 \geq \gamma_1 \int\limits_{Q_R} \eta^{q_0} (\widetilde{D}\mathbf{u})^{p-2} |\mathbf{D}\mathbf{u}|^2 \, dx \geq \gamma_1 \int\limits_{Q_{R/2}} (\widetilde{D}\mathbf{u})^p \, dx - \gamma_1 \, C \, R^d$$

and using $ab \leq \delta a^{p'} + \delta^{-p_0+1} b^p$

$$\begin{aligned} |I_2| &\leq \gamma_2 C \int\limits_{Q_R} \eta^{q_0-1} (\widetilde{D}\mathbf{u})^{p-1} \left|\tfrac{\mathbf{u}-\boldsymbol{\tau}}{R}\right| dx \\ &\leq \tfrac{\delta}{2} \gamma_2 \, C \int\limits_{Q_R} \eta^{(q_0-1)p'} (\widetilde{D}\mathbf{u})^p + \gamma_2 \, C \, \delta^{-p_0+1} \int\limits_{Q_R} \left|\tfrac{\mathbf{u}-\boldsymbol{\tau}}{R}\right|^p dx \end{aligned}$$

Since $(q_0 - 1)p' \geq (q_0 - 1)(q_0)' = q_0$ we have

$$\begin{aligned} |I_2| &\leq \tfrac{\delta}{2} \gamma_2 \, C \int\limits_{Q_R} \eta^{q_0} (\widetilde{D}\mathbf{u})^p + \gamma_2 \, C \, \delta^{-p_0+1} \int\limits_{Q_R} \left|\tfrac{\mathbf{u}-\boldsymbol{\tau}}{R}\right|^p dx \\ &\overset{\text{suitable } \delta}{\leq} \tfrac{1}{4} I_1 + C \Lambda^{p_0} \gamma_1 \int\limits_{Q_R} \left|\tfrac{\mathbf{u}-\boldsymbol{\tau}}{R}\right|^p dx \\ &\leq \tfrac{1}{4} I_1 + C \Lambda^{p_0} \gamma_1 \int\limits_{Q_R} \left|\tfrac{\mathbf{u}-\boldsymbol{\tau}}{R}\right|^{q_0} + 1 \, dx \\ &\leq \tfrac{1}{4} I_1 + C \Lambda^{p_0} \gamma_1 \int\limits_{Q_R} \left|\tfrac{\mathbf{u}-\boldsymbol{\tau}}{R}\right|^{q_0} dx + C \Lambda^{p_0} \gamma_1 \, R^d, \end{aligned}$$

and

$$\begin{aligned}|I_3| &\le \gamma_2 \int\limits_{Q_R} (\widetilde{D}\mathbf{u})^{p-1} |\mathbf{D}\mathbf{v}|\,dx \\ &\le \delta\gamma_2 \int\limits_{Q_R} (\widetilde{D}\mathbf{u})^{p}\,dx + \delta^{-p_0+1}\gamma_2 \int\limits_{Q_R} |\nabla\mathbf{v}|^p\,dx \\ &\le \delta\gamma_2 \int\limits_{Q_R} (\widetilde{D}\mathbf{u})^{p}\,dx + \delta^{-p_0+1}\gamma_2 \int\limits_{Q_R} |\nabla\mathbf{v}|^{q_0} + 1\,dx\end{aligned}$$

Using (7.4) we get

$$|I_3| \le \delta_2\gamma_1 \int\limits_{Q_R} (\widetilde{D}\mathbf{u})^{p}\,dx + C\,\delta_2^{-p_0+1}\Lambda^{-p_0}\gamma_1 \Bigg(\int\limits_{Q_R} \big|\tfrac{\mathbf{u}-\boldsymbol{\tau}}{R}\big|^{q_0}\,dx + R^d \Bigg)$$

and

$$\begin{aligned}I_4 &= \varepsilon \int\limits_{Q_R} \nabla(\eta^{q_0}(\pi-\kappa))\nabla\pi\,dx \\ &= \varepsilon \int\limits_{Q_R} \eta^{q_0}|\nabla\pi|^2\,dx + \varepsilon \int\limits_{Q_R} q_0\eta^{q_0-1}(\pi-\kappa)(\nabla\eta)(\nabla\pi)\,dx \\ &=: I_{4,1} + I_{4,2}\end{aligned}$$

and

$$I_{4,1} = \varepsilon \int\limits_{Q_R} \eta^{q_0}|\nabla\pi|^2\,dx \ge \varepsilon \int\limits_{Q_{R/2}} |\nabla\pi|^2\,dx$$

and

$$\begin{aligned}|I_{4,2}| &\le \varepsilon C \int\limits_{Q_R} \eta^{q_0-1}\big|\tfrac{\pi-\kappa}{R}\big||\nabla\pi|\,dx \\ &\le \varepsilon C_{\delta_2} \int\limits_{Q_R} \big|\tfrac{\pi-\kappa}{R}\big|^2\,dx + \delta_2\varepsilon \int\limits_{Q_R} |\nabla\pi|^2\,dx\end{aligned}$$

and

$$
\begin{aligned}
I_5 &= \big\langle \mathbf{f}, \eta^{q_0}(\mathbf{u}-\boldsymbol{\tau})\big\rangle \\
&\le \Big(\int_{Q_R} |\mathbf{f}|^d\,dx\Big)^{\frac{1}{d}} \Big(\int_{Q_R} |\mathbf{u}-\boldsymbol{\tau}|^{d'}\,dx\Big)^{\frac{1}{d'}} \\
&= C\,R^d \Big(\int_{Q_R} |\mathbf{f}|^d\,dx\Big)^{\frac{1}{d}} \Big(\fint_{Q_R} \big|\tfrac{\mathbf{u}-\boldsymbol{\tau}}{R}\big|^{d'}\,dx\Big)^{\frac{1}{d'}} \\
&\le C\,R^d\,\|\mathbf{f}\|_d \Big(\fint_{Q_R} |\boldsymbol{\nabla}(\mathbf{u}-\boldsymbol{\tau})|^{p_\infty}\,dx\Big)^{\frac{1}{p_\infty}} \quad \text{(Poincaré, } d' = 1^* \le p_\infty^*\text{)} \\
&\le C\,R^d\|\mathbf{f}\|_d \Big(\fint_{Q_R} |\mathbf{D}\mathbf{u}|^{p_\infty}\,dx\Big)^{\frac{1}{p_\infty}} \qquad \text{(Korn and } \mathbf{D}\boldsymbol{\tau} = 0\text{)} \\
&\le C\,R^d\|\mathbf{f}\|_d \Bigg(\delta \fint_{Q_R} |\mathbf{D}\mathbf{u}|^{p_\infty}\,dx + \delta^{-p_\infty+1}\Bigg) \qquad \text{(Young)} \\
&\le C\,R^d\|\mathbf{f}\|_d \Bigg(\delta \fint_{Q_R} (\widetilde{D}\mathbf{u})^{p}\,dx + \delta^{-p_\infty+1}\Bigg) \\
&\le \delta\,C\,\|\mathbf{f}\|_d \int_{Q_R} (\widetilde{D}\mathbf{u})^{p}\,dx + R^d\,C\,\|\mathbf{f}\|_d\,\delta^{-p_\infty+1} \\
&\le \delta_2\gamma_1 \int_{Q_R} (\widetilde{D}\mathbf{u})^{p}\,dx + R^d\,C_{\delta_2}\,\gamma_1^{-p_\infty+1}\|\mathbf{f}\|_d^{p_\infty},
\end{aligned}
$$

and

$$
\begin{aligned}
|I_6| &\le \int_{Q_R} |\mathbf{f}||\mathbf{v}|\,dx \le \Big(\int_{Q_R} |\mathbf{f}|^d\,dx\Big)^{\frac{1}{d}} \Big(\int_{Q_R} |\mathbf{v}|^{d'}\,dx\Big)^{\frac{1}{d'}} \\
&= C\,R^d\,\|\mathbf{f}\|_d \Big(\fint_{Q_R} \big|\tfrac{\mathbf{v}}{R}\big|^{d'}\,dx\Big)^{\frac{1}{d'}} \\
&\le C\,R^d\,\|\mathbf{f}\|_d \Big(\fint_{Q_R} |\boldsymbol{\nabla}\mathbf{v}|^{p_\infty}\,dx\Big)^{\frac{1}{p_\infty}} \qquad \text{Poincaré} \\
&\le C\,R^d\|\mathbf{f}\|_d \Big(\fint_{Q_R} \big|\tfrac{\mathbf{u}-\boldsymbol{\tau}}{R}\big|^{p_\infty}\,dx\Big)^{\frac{1}{p_\infty}} \qquad \text{by (7.4)} \\
&\le C\,R^d\|\mathbf{f}\|_d \Big(\fint_{Q_R} |\boldsymbol{\nabla}\mathbf{u}|^{p_\infty}\,dx\Big)^{\frac{1}{p_\infty}} \\
&\le C\,R^d\|\mathbf{f}\|_d \Big(\fint_{Q_R} |\mathbf{D}\mathbf{u}|^{p_\infty}\,dx\Big)^{\frac{1}{p_\infty}} \qquad \text{(Korn)}
\end{aligned}
$$

$$
\begin{aligned}
&\le C\,R^d \|\mathbf{f}\|_d \Big(\delta \fint_{Q_R} |\mathbf{D}\mathbf{u}|^{p_\infty}\,dx + \delta^{-p_\infty+1}\Big) \qquad \text{(Young)}\\
&\le C\,R^d \|\mathbf{f}\|_d \Big(\delta \fint_{Q_R} |\widetilde{D}\mathbf{u}|^{p}\,dx + \delta^{-p_\infty+1}\Big)\\
&\le \delta\, C\,\|\mathbf{f}\|_d \int_{Q_R} |\widetilde{D}\mathbf{u}|^{p}\,dx + R^d\, C\,\|\mathbf{f}\|_d\,\delta^{-p_\infty+1}\\
&\le \delta_2\gamma_1 \int_{Q_R} |\widetilde{D}\mathbf{u}|^{p}\,dx + R^d\, C_{\delta_2}\,\gamma_1^{-p_\infty+1}\|\mathbf{f}\|_d^{p_\infty}.
\end{aligned}
$$

Overall we get

$$
\begin{aligned}
\gamma_1 \int_{Q_{R/2}} (\widetilde{D}\mathbf{u})^p\,dx + \varepsilon \int_{Q_{R/2}} |\nabla\pi|^2\,dx&\\
\le C\,\gamma_1\Lambda^{p_0}\int_{Q_R} \Big|\tfrac{\mathbf{u}-\boldsymbol{\tau}}{R}\Big|^{q_0}\,dx &+ C\,\gamma_1\Lambda^{p_0}R^d\\
&+ \delta_2\varepsilon \int_{Q_R} |\nabla\pi|^2\,dx + \varepsilon C_{\delta_2}\int_{Q_R} \Big|\tfrac{\pi-\kappa}{R}\Big|^2\,dx\\
&+ \delta_2\gamma_1 \int_{Q_R} (\widetilde{D}\mathbf{u})^p\,dx + R^d\,C_{\delta_2}\,\gamma_1^{-p_\infty+1}\|\mathbf{f}\|_d^{p_\infty}.
\end{aligned}
$$

Due to a result of M. Giaquinta and G. Modica (see [**GM82**], lemma 0.5), we can get rid of the third and the fifth term on the right-hand side by increasing the multiplicative constants by a fixed factor, i.e.

$$
\begin{aligned}
\gamma_1 \int_{Q_{R/2}} (\widetilde{D}\mathbf{u})^p\,dx + \varepsilon \int_{Q_{R/2}} |\nabla\pi|^2\,dx&\\
\le C\,\gamma_1\Lambda^{p_0}\int_{Q_R} \Big|\tfrac{\mathbf{u}-\boldsymbol{\tau}}{R}\Big|^{q_0}\,dx &+ C\,\gamma_1\Lambda^{p_0}R^d\\
&+ \varepsilon C\int_{Q_R} \Big|\tfrac{\pi-\kappa}{R}\Big|^2\,dx + R^d\,C\,\gamma_1^{-p_\infty+1}\|\mathbf{f}\|_d^{p_\infty}.
\end{aligned}
$$

Dividing by R^d proves the lemma. □

3. Reverse Hölder Estimates

LEMMA 7.3. *Let Φ be a p–potential with ellipticity constants γ_1, γ_2 and $p \in C(\Omega)$, $1 < p_\infty \le p_0 < \infty$. Let p also satisfy the requirements of lemma 2.6, i.e. the module of continuity ω of p satisfies*

$$
\omega(R) \le \frac{C}{-\ln R}
$$

for all $0 < R < 1$. *Further let* $\mathbf{u}, \pi$ *be a weak solution of* (7.3) *with* $\mathbf{f} \in L^d(\Omega)$. *Let* $\Lambda := \frac{\gamma_2}{\gamma_1}$. *Then there exists* $R_0 > 0$ *and* $0 < \theta < 1$, *such that for all axis parallel cubes* $Q_R \Subset \Omega$ *with* $\operatorname{diam}(Q_R) < R_0$ *there holds*

$$\begin{aligned}\gamma_1 \fint_{Q_{R/2}} (\widetilde{D}\mathbf{u})^p \,dx + \varepsilon \fint_{Q_{R/2}} |\boldsymbol{\nabla}\pi|^2 \,dx \\ \leq C\,\gamma_1 \Lambda^{p_0} C \bigg(\fint_{Q_R} \big(\widetilde{D}\mathbf{u}\big)^{\theta p} \,dx \bigg)^{\frac{1}{\theta}} + C\,\gamma_1 \Lambda^{p_0} \\ + \varepsilon C \bigg(\fint_{Q_{R/2}} |\boldsymbol{\nabla}\pi|^{2\theta} \,dx \bigg)^{\frac{1}{\theta}} + C\,\gamma_1^{-p_\infty+1} \|\mathbf{f}\|_d^{p_\infty}.\end{aligned}$$

Note that the constants are independent of γ_1, γ_2, *and* ε.

Proof. Fix $\theta \in (\max\{\frac{1}{p_\infty}, \frac{d}{d+1}\}, 1)$. Since p is uniformly continuous, there exists $0 < R_0 < 1$, such that

$$\frac{d p_{\infty,Q}}{d + p_{\infty,Q}} < \theta\, p_{0,Q}$$

for all cubes Q with $\operatorname{diam} Q < R_0$. Fix $x_0 \in \Omega$ and let $Q_R := Q_R(x_0)$ with $0 < R \leq R_0$. Without loss of generality we further assume that R_0 is so small that Q_{2R} is still a strict subset of Ω and does not cover all of Ω. By definition of R_0 we have

$$\frac{d q_\infty}{d + q_\infty} < \theta\, q_0 \qquad \text{and} \qquad 1 < \theta\, q_\infty. \tag{7.5}$$

Thus using the Sobolev embedding $W^{1,\theta q_\infty}(Q_R) \hookrightarrow L^{q_0}(Q_R)$ in combination with Korn's inequality we deduce

$$\begin{aligned}\fint_{Q_R} \big|\tfrac{\mathbf{u}-\boldsymbol{\tau}}{R}\big|^{q_0} \,dx &\leq \bigg(\fint_{Q_R} \big|\mathbf{D}\mathbf{u}\big|^{\theta q_\infty} \,dx \bigg)^{\frac{q_0}{\theta q_\infty}} \\ &\leq \bigg(\fint_{Q_R} \big(\widetilde{D}\mathbf{u}\big)^{\theta p} \,dx \bigg)^{\frac{q_0}{\theta q_\infty}}.\end{aligned}$$

Since $\mathbf{u}$ is a weak solution, we have

$$\int_\Omega (\widetilde{D}\mathbf{u})^p \,dx \leq C.$$

This implies

$$\begin{aligned}
\fint_{Q_R} \left|\tfrac{\mathbf{u}-\boldsymbol{\tau}}{R}\right|^{q_0} dx &\le \left(\fint_{Q_R} \big(\widetilde{D}\mathbf{u}\big)^{\theta p} dx\right)^{\frac{q_0}{\theta q_\infty}} \\
&\le \left(\fint_{Q_R} \big(\widetilde{D}\mathbf{u}\big)^{\theta p} dx\right)^{\frac{1}{\theta}} \left(\fint_{Q_R} \big(\widetilde{D}\mathbf{u}\big)^{\theta p} dx\right)^{\frac{q_0-q_\infty}{\theta q_\infty}} \\
&\le \left(\fint_{Q_R} \big(\widetilde{D}\mathbf{u}\big)^{\theta p} dx\right)^{\frac{1}{\theta}} |Q_R|^{-\frac{|q_0-q_\infty|}{\theta q_\infty}}.
\end{aligned}$$

Now lemma 2.6 implies $|Q_R|^{-\frac{|q_0-q_\infty|}{\theta q_\infty}} \le C$, so

$$\fint_{Q_R} \left|\tfrac{\mathbf{u}-\boldsymbol{\tau}}{R}\right|^{q_0} dx \le C \left(\fint_{Q_R} \big(\widetilde{D}\mathbf{u}\big)^{\theta p} dx\right)^{\frac{1}{\theta}}.$$

Thus lemma 7.2 gives

$$\begin{aligned}
&\gamma_1 \fint_{Q_{R/2}} (\widetilde{D}\mathbf{u})^p \, dx + \varepsilon \fint_{Q_{R/2}} |\nabla \pi|^2 \, dx \\
&\quad \le C\,\gamma_1 \Lambda^{p_0} C \left(\fint_{Q_R} \big(\widetilde{D}\mathbf{u}\big)^{\theta p} dx\right)^{\frac{1}{\theta}} + C\,\gamma_1 \Lambda^{p_0} \\
&\qquad + \varepsilon C \fint_{Q_R} \left|\tfrac{\pi-\kappa}{R}\right|^2 dx + C\,\gamma_1^{-p_\infty+1} \|\mathbf{f}\|_d^{p_\infty}.
\end{aligned}$$

On the other hand from $\frac{d}{d+2} < \theta < 1$ and the Sobolev embedding $W^{1,\frac{2d}{d+2}}(Q_R) \hookrightarrow L^2(Q_R)$ there follows

$$\begin{aligned}
\fint_{Q_R} \left|\tfrac{\pi-\kappa}{R}\right|^2 dx &\le \left(\fint_{Q_{R/2}} |\nabla \pi|^{\frac{2d}{d+2}} \, dx\right)^{\frac{d+2}{d}} \\
&\le \left(\fint_{Q_{R/2}} |\nabla \pi|^{2\theta} \, dx\right)^{\frac{1}{\theta}}.
\end{aligned}$$

Thus

$$\gamma_1 \fint_{Q_{R/2}} (\widetilde{D}\mathbf{u})^p \,dx + \varepsilon \fint_{Q_{R/2}} |\boldsymbol{\nabla}\pi|^2 \,dx$$

$$\leq C\,\gamma_1 \Lambda^{p_0} C \left(\fint_{Q_R} \left(\widetilde{D}\mathbf{u}\right)^{\theta p} dx \right)^{\frac{1}{\theta}} + C\,\gamma_1 \Lambda^{p_0}$$

$$+ \varepsilon C \left(\fint_{Q_{R/2}} |\boldsymbol{\nabla}\pi|^{2\theta} \,dx \right)^{\frac{1}{\theta}} + C\,\gamma_1^{-p_\infty+1} \|\mathbf{f}\|_d^{p_\infty}.$$

This proves the lemma. □

4. Meyer Type Estimates

From the reverse Hölder estimate in lemma 7.3 we deduce:

THEOREM 7.4. *Let Φ be a p–potential with ellipticity constants γ_1, γ_2 and $p \in C(\Omega)$, $1 < p_\infty \leq p_0 < \infty$. Let p also satisfy the requirements of lemma 2.6, i.e. the module of continuity ω of p satisfies*

$$\omega(R) \leq \frac{C}{-\ln R}$$

for all $0 < R < 1$. Further let $\mathbf{u}, \pi$ be a weak solution of (7.3) with $\mathbf{f} \in L^d(\Omega)$. Let $\Lambda := \frac{\gamma_2}{\gamma_1}$. Then there exists $\delta > 0$ and $K > 0$, which only depend on p, γ_1, and γ_2, such that $\mathbf{u} \in W^{1,p(\cdot)(1+\delta)}(\Omega)$. Further for all $s \in [1, 1+\delta]$ there holds

$$\left(\fint_{Q_{R/2}} (\widetilde{D}\mathbf{u})^{sp} \,dx \right)^{\frac{1}{s}} + \varepsilon \left(\fint_{Q_{R/2}} |\boldsymbol{\nabla}\pi|^{2s} \,dx \right)^{\frac{1}{s}}$$

$$\leq K \left(\fint_{Q_{R/2}} (\widetilde{D}\mathbf{u})^p \,dx + \varepsilon \fint_{Q_{R/2}} |\boldsymbol{\nabla}\pi|^2 \,dx + 1 + \|\mathbf{f}\|_d^{p_\infty} \right).$$

PROOF. This theorem is a direct consequence of lemma 7.3 and theorem 1.2 in [**Gia82**] (M. Giaquinta). □

CHAPTER 8

Appendix

1. Miscellaneous

LEMMA 8.1 (Aubin–Lions). *Let $1 < \alpha, \beta < \infty$. Let X be a Banach space and let X_0, X_1 be separable and reflexive Banach spaces. Provided that $X_0 \hookrightarrow\hookrightarrow X \hookrightarrow X_1$, we have*

$$\left\{v \in L^\alpha(I, X_0)\,;\, \partial_t v \in L^\beta(I, X_1)\right\} \hookrightarrow\hookrightarrow L^\alpha(I, X).$$

PROOF. See Lions [**Lio69**] (section 1.5) or the survey paper of Simon [**Sim87**]. □

LEMMA 8.2 (Bogovskiĭ). *Let Ω be a bounded domain with Lipschitz boundary in $\mathbb{R}^d$ with $d \geq 2$ and let $1 < q_\infty \leq q_0 < \infty$. Given $f \in L^{q_0}(\Omega)$ with*

$$\int_\Omega f\, dx = 0,$$

there exists a vector field $\mathbf{v} : \Omega \to \mathbb{R}^d$ such that

$$\begin{aligned} \operatorname{div} \mathbf{v} &= f, \\ \mathbf{v} &\in W_0^{1,q_0}(\Omega), \\ \|\nabla \mathbf{v}\|_q &\leq C\, \|f\|_q \quad \text{for all } q \text{ with } q_\infty \leq q \leq q_0, \end{aligned}$$

where $C = C(f, q_\infty, q_0, \Omega)$.

PROOF. See Bogovskiĭ [**Bog80**] and G. P. Galdi [**Gal94**]. □

LEMMA 8.3 (Korn). *Let $1 < p < \infty$. Further let Ω denote the d–dimensional torus, resp. an open bounded domain of $\mathbb{R}^d$ with Lipschitz boundary. Then there exists a constant $C = C(p, \Omega)$, such that for all $\mathbf{v} \in W^{1,p}(\Omega)$, resp. $\mathbf{v} \in W_0^{1,p}(\Omega)$, there holds*

$$\|\nabla \mathbf{v}\|_p \leq C \|\mathbf{D}\mathbf{v}\|_p.$$

PROOF. See for example J. Gobert [**Gob62**] or J. Málek, J. Nečas, M. Rokyta, and M. Růžička [**MNRR96**]. □

LEMMA 8.4. *Let Ω be a bounded domain in $\mathbb{R}^d$ with smooth boundary. Assume that for any $\beta > 0$ and $r > 1$*

$$\mathbf{u} \in L^\infty(I, C^{1,\beta}(\Omega)), \qquad \partial_t \mathbf{u} \in L^r(I, W^{1,r}(\Omega)).$$

Then

$$\mathbf{u} \in C^{1,\alpha}(I \times \Omega),$$

with $\alpha = \min\left\{\beta, \frac{\beta(r-1)}{\beta r + d}\right\}$.

PROOF. See John, Stará [**JS98**] lemma 2.2. □

LEMMA 8.5. *For all (smooth)* $\mathbf{u} : \mathbb{R}^d \to \mathbb{R}^d$ *there holds pointwise*

$$\frac{2}{\sqrt{3}}|\boldsymbol{\nabla}\mathbf{D}^-\mathbf{u}| \le |\boldsymbol{\nabla}^2\mathbf{u}| \le 2\,|\boldsymbol{\nabla}\mathbf{D}\mathbf{u}|, \tag{8.1}$$

where $\mathbf{D}^-\mathbf{u} = \frac{1}{2}(\boldsymbol{\nabla}\mathbf{u} - (\boldsymbol{\nabla}\mathbf{u})^T)$. *If* $d \ge 3$ *these estimates are optimal, even if we restrict ourselves to divergence free functions.*

PROOF. Define $X := \mathbb{R}^{d\times d\times d}$ by

$$X := \{\mathbf{a} \in \mathbb{R}^{d\times d\times d} \,:\, a_{ijk} = a_{jik} \text{ for all } i,j,k = 1,\dots,d\},$$

then $\boldsymbol{\nabla}^2\mathbf{u} \in X$. Further define $\boldsymbol{\tau}, \boldsymbol{\rho}, \boldsymbol{\sigma} : \mathbb{R}^{d\times d\times d} \to \mathbb{R}^{d\times d\times d}$ by

$$\begin{aligned}
(\boldsymbol{\tau}(\mathbf{a}))_{ijk} &:= a_{ikj},\\
(\boldsymbol{\sigma}^+(\mathbf{a}))_{ijk} &:= \tfrac{1}{2}\big(a_{ikj} + a_{ijk}\big),\\
(\boldsymbol{\sigma}^-(\mathbf{a}))_{ijk} &:= \tfrac{1}{2}\big(a_{ikj} - a_{ijk}\big),\\
(\boldsymbol{\rho}(\mathbf{a}))_{ijk} &:= \tfrac{1}{6}\big(a_{ijk} + a_{ikj} + a_{jik} + a_{jki} + a_{kij} + a_{kji}\big)
\end{aligned}$$

for all $i,j,k = 1,\dots,d$, then

$$\begin{aligned}
\boldsymbol{\sigma}^+(\boldsymbol{\nabla}^2\mathbf{u}) &= \boldsymbol{\nabla}\mathbf{D}\mathbf{u},\\
\boldsymbol{\sigma}^-(\boldsymbol{\nabla}^2\mathbf{u}) &= \boldsymbol{\nabla}\mathbf{D}^-\mathbf{u}.
\end{aligned}$$

For all $\mathbf{a} \in X$ there holds

$$\begin{aligned}
\boldsymbol{\rho}(\mathbf{a})\cdot\boldsymbol{\rho}(\mathbf{a}) &= \mathbf{a}\cdot\boldsymbol{\rho}(\mathbf{a})\\
&= \tfrac{1}{6}\sum_{ik} a_{ijk}\big(a_{ijk} + a_{ikj} + a_{jik} + a_{jki} + a_{kij} + a_{kji}\big)\\
&= \tfrac{1}{3}\sum_{ijk} a_{ijk}\big(a_{ijk} + a_{ikj} + a_{jki}\big)\\
&= \tfrac{1}{3}\sum_{ijk} a_{ijk}a_{ijk} + \tfrac{1}{3}\sum_{ijk} a_{ijk}a_{ikj} + \tfrac{1}{3}\sum_{ijk} a_{jik}a_{jki}\\
&= \tfrac{1}{3}\,\mathbf{a}\cdot\mathbf{a} + \tfrac{2}{3}\,\mathbf{a}\cdot\boldsymbol{\tau}(\mathbf{a})\\
&= -\tfrac{1}{3}\,\mathbf{a}\cdot\mathbf{a} + \tfrac{4}{3}\,\mathbf{a}\cdot\boldsymbol{\sigma}^+(\mathbf{a})\\
&= -\tfrac{1}{3}\,\mathbf{a}\cdot\mathbf{a} + \tfrac{4}{3}\,\boldsymbol{\sigma}^+(\mathbf{a})\cdot\boldsymbol{\sigma}^+(\mathbf{a}).
\end{aligned}$$

Thus

$$\mathbf{a}\cdot\mathbf{a} = 4\,\boldsymbol{\sigma}^+(\mathbf{a})\cdot\boldsymbol{\sigma}^+(\mathbf{a}) - 3\,\boldsymbol{\rho}(\mathbf{a})\cdot\boldsymbol{\rho}(\mathbf{a}) \le 4\,\boldsymbol{\sigma}^+(\mathbf{a})\cdot\boldsymbol{\sigma}^+(\mathbf{a}).$$

This proves

$$|\boldsymbol{\nabla}^2\mathbf{u}|^2 \le 4\,|\boldsymbol{\nabla}\mathbf{D}\mathbf{u}|^2.$$

Since

$$\begin{aligned}
\mathbf{a}\cdot\mathbf{a} &= \big(\boldsymbol{\sigma}^+(\mathbf{a}) + \boldsymbol{\sigma}^-(\mathbf{a})\big)\cdot\big(\boldsymbol{\sigma}^+(\mathbf{a}) + \boldsymbol{\sigma}^-(\mathbf{a})\big)\\
&= \boldsymbol{\sigma}^+(\mathbf{a})\cdot\boldsymbol{\sigma}^+(\mathbf{a}) + 2\,\boldsymbol{\sigma}^+(\mathbf{a})\cdot\boldsymbol{\sigma}^-(\mathbf{a}) + \boldsymbol{\sigma}^-(\mathbf{a})\cdot\boldsymbol{\sigma}^-(\mathbf{a})\\
&= \boldsymbol{\sigma}^+(\mathbf{a})\cdot\boldsymbol{\sigma}^+(\mathbf{a}) + \boldsymbol{\sigma}^-(\mathbf{a})\cdot\boldsymbol{\sigma}^-(\mathbf{a}),
\end{aligned}$$

there follows

$$\boldsymbol{\sigma}^-(\mathbf{a})\cdot\boldsymbol{\sigma}^-(\mathbf{a}) = \tfrac{3}{4}\,\mathbf{a}\cdot\mathbf{a} - \tfrac{3}{4}\,\boldsymbol{\rho}(\mathbf{a})\cdot\boldsymbol{\rho}(\mathbf{a}) \le \tfrac{3}{4}\,\mathbf{a}\cdot\mathbf{a}.$$

So $|\boldsymbol{\nabla}^2\mathbf{u}|^2 = |\boldsymbol{\nabla}\mathbf{D}\mathbf{u}|^2 + |\boldsymbol{\nabla}\mathbf{D}^-\mathbf{u}|^2$ and $|\boldsymbol{\nabla}\mathbf{D}^-\mathbf{u}|^2 \leq \frac{3}{4}|\boldsymbol{\nabla}^2\mathbf{u}|$. This proves (8.1).

It remains to prove that for $d \geq 3$ the inequality (8.1) is optimal even for divergence free functions. Obviously it is enough to prove this for $d = 3$. Define $\mathbf{u}(x, y, z) := (yz, xz, 0)^T$, then $\operatorname{div}\mathbf{u} = 0$, $|\boldsymbol{\nabla}^2\mathbf{u}|^2 = 4$, $|\boldsymbol{\nabla}\mathbf{D}\mathbf{u}|^2 = 1$, and $|\boldsymbol{\nabla}\mathbf{D}^-\mathbf{u}|^2 = 3$. This proves the lemma. □

REMARK 8.6. *Note that*

$$\partial_j\partial_k u_m = \partial_j D_{km}\mathbf{u} + \partial_k D_{mj}\mathbf{u} - \partial_m D_{jk}\mathbf{u}. \tag{8.2}$$

This immediately implies $|\boldsymbol{\nabla}^2\mathbf{u}| \leq 3|\boldsymbol{\nabla}\mathbf{D}\mathbf{u}|$, *which is only slightly weaker than lemma 8.5.*

2. Gronwall's Inequality

LEMMA 8.7. **(a local version of Gronwall's lemma)**
Let $T, \alpha > 0$, $h, m \in L^1([0, T])$, $f \in C^1([0, T])$, $f, g, h, m \geq 0$, *and*

$$f'(t) \leq h(t)\, f(t) + m(t)\, f(t)^{1+\alpha},$$

then

$$f(t) \leq \exp(H(t))f(0)\Big(1 - \alpha f(0)^\alpha \int_0^t \exp(\alpha H(s))m(s)\, ds\Big)^{-\frac{1}{\alpha}}$$

for all $0 < t$ *such that* $\alpha f(0)^\alpha \int_0^t \exp(\alpha H(s))m(s)\, ds < 1$, *where*

$$H(t) := \int_0^t h(s)\, ds.$$

PROOF. Define $a : [0, T] \to \mathbb{R}^{\geq 0}$ by

$$a(t) := \exp(H(t))f(0)\Big(1 - \alpha f(0)^\alpha \int_0^t \exp(\alpha H(s))m(s)\, ds\Big)^{-\frac{1}{\alpha}}.$$

Then a solves

$$a'(t) = h(t)\, a(t) + m(t)\, a(t)^{1+\alpha}.$$

Since $a(0) = f(0)$ and $f'(t) \leq a'(t)$ this implies $f(t) \leq a(t)$. This proves the lemma. □

LEMMA 8.8. **(a second local version of Gronwall's lemma)**
Let $T, \alpha > 0$, $g, h \in L^1([0, T])$, $g, h, m \geq 0$. *Let* $f \in C^1([0, T])$, *and* $f \geq 0$. *Define* ε_0 *by*

$$\varepsilon_0 := \big(1 - 2^{-\alpha}\big)\Big(\alpha(1 + f(0))^\alpha \int_0^T \exp(\alpha(G + H)(s))\, ds\Big)^{-1},$$

where

$$G(t) := \int_0^t g(s)\, ds, \qquad H(t) := \int_0^t h(s)\, ds.$$

Then if

$$f'(t) \leq g(t) + h(t)\, f(t) + \varepsilon\, f(t)^{1+\alpha}$$

for some ε *with* $0 \leq \varepsilon < \varepsilon_0$*, there holds*

$$f(t) \leq 2\,(1 + f(0)) \exp\Big(\int_0^t g(s) + h(s)\, ds \Big) - 1$$

for all $0 \leq t \leq T$.

Proof. Let $a(t) := f(t) + 1$, then a satisfies

$$a'(t) \leq g(t) + h(t)\, f(t) + \varepsilon\, f(t)^{1+\alpha} \leq (g(t) + h(t))\, a(t) + \varepsilon\, a(t)^{1+\alpha}.$$

Thus by lemma 8.7

$$\begin{aligned} a(t) &\leq a(0) \exp((G+H)(t)) \Big(1 - \varepsilon \alpha a(0)^{\alpha} \int_0^t \exp(\alpha (G+H)(s))\, ds \Big)^{-\frac{1}{\alpha}}, \\ &\leq a(0) \exp((G+H)(t)) \Big(1 - \varepsilon_0 \alpha a(0)^{\alpha} \int_0^T \exp(\alpha (G+H)(s))\, ds \Big)^{-\frac{1}{\alpha}}, \\ &\leq a(0) \exp((G+H)(t)) \Big(1 - \big(1 - 2^{-\alpha}\big) \Big)^{-\frac{1}{\alpha}} \\ &= 2\, a(0) \exp((G+H)(t)). \end{aligned}$$

This proves the lemma. □

Lemma 8.9. **(discrete explicit version)** *Let* a_m, b_m, c_m *be sequences of non–negative numbers, satisfying*

$$d_t a_m + b_m \leq a_{m-1} c_m, \tag{8.3}$$

for all $m \geq 1$*, where* $d_t a_m = \frac{1}{k}(a_m - a_{m-1})$*. Further let* $c_m \in l^1(I_k)$*. Then there holds*

$$\|a_m\|_{l^\infty(I_k)} + \|b_m\|_{l^1(I_k)} \leq a_0 \exp\big(\|c_m\|_{l^1(I_k)} \big).$$

Lemma 8.10. **(discrete implicit version)** *Let* a_m, b_m, c_m *be sequences of non–negative numbers, satisfying*

$$d_t a_m + b_m \leq a_m c_m, \tag{8.4}$$

for all $m \geq 1$*, where* $d_t a_m = \frac{1}{k}(a_m - a_{m-1})$*. Further let* $c_m \in l^q(I_k)$ *for some* $q > 1$*. Then there exists* $k_0 > 0$ *with* $k_0 = k_0(q)$ *and* $C = C(q)$*, such that if* (8.4) *is fullfilled for some* k *with* $0 < k < k_0$*, then*

$$\|a_m\|_{l^\infty(I_k)} + \|b_m\|_{l^1(I_k)} \leq C\, a_0 \exp\big(\|c_m\|_{l^q(I_k)} \big).$$

3. Lower Semicontinuity

LEMMA 8.11. *Let Ω be a domain in $\mathbb{R}^d$ and $I := [0,T] \subset \mathbb{R}$, $T > 0$. Further let $F : I \times \Omega \times \mathbb{R}^m \times \mathbb{R}^n$ with*

(1) $F \geq 0$,
(2) F measurable in t, x for a.a. y, z,
(3) $F(t, x, \cdot, \cdot)$ continuous for a.a. t, x,
(4) $F(t, x, y, \cdot)$ be convex for all y and a.a. t, x.

If $\mathbf{w}^N \to \mathbf{w}$ in $L^1_{\text{loc}}(I \times \Omega)$ and $\nabla \mathbf{w}^N \rightharpoonup \nabla \mathbf{w}$ in $L^1_{\text{loc}}(I \times \Omega)$, then for all r, s with $1 \leq r < \infty$, $1 < s < \infty$ or $r = s = 1$ there holds

$$\left\| F(t, x, \mathbf{w}, \nabla \mathbf{w}) \right\|_{L^r(I, L^s(\Omega))} \leq \liminf_N \left\| F(t, x, \mathbf{w}^N, \nabla \mathbf{w}^N) \right\|_{L^r(I, L^s(\Omega))}.$$

PROOF. If $r = s = 1$ the result follows immediately from theorem 1 (De Giorgi) from [**GMS98**], pg. 132. So let us assume $1 \leq r < \infty$, $1 < s < \infty$. Since $L^s(\Omega)$ is reflexive, the dual of $L^r(I, L^s(\Omega))$ is $L^{r'}(I, L^{s'}(\Omega))$ (see [**DjU77**]) and for all $f \in L^r(I, L^s(\Omega))$

$$\|f\|_{L^r(I, L^s(\Omega))} = \sup_{\|\varphi\|_{L^{r'}(I, L^{s'}(\Omega))} \leq 1} |\langle f, \varphi \rangle|.$$

For $\varphi \in L^{r'}(I, L^{s'}(\Omega))$ with $\varphi \geq 0$ and $\|\varphi\|_{L^{r'}(I, L^{s'}(\Omega)} \leq 1$ let $F_\varphi : I \times \Omega \times \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}$ be defined by

$$F_\varphi(t, x, y, z) := F(t, x, y, z) \varphi(t, x).$$

Then F_φ fulfills

(1) $F_\varphi \geq 0$,
(2) F_φ measurable in t, x for a.a. y, z,
(3) $F_\varphi(t, x, \cdot, \cdot)$ continuous for a.a. t, x,
(4) $F_\varphi(t, x, y, \cdot)$ be convex for all y and a.a. t, x.

Hence by theorem 1 (De Giorgi) from [**GMS98**], pg. 132

$$\begin{aligned} &\int_I \int_\Omega F(t, x, \mathbf{w}, \nabla \mathbf{w}) \varphi(t, x) \, dx \, dt \\ &\qquad \leq \liminf_N \int_I \int_\Omega F(t, x, \mathbf{w}^N, \nabla \mathbf{w}^N) \varphi(t, x) \, dx \, dt \\ &\qquad \leq \liminf_N \left\| F(t, x, \mathbf{w}^N, \nabla \mathbf{w}^N) \right\|_{L^r(I, L^s(\Omega))}. \end{aligned} \tag{8.5}$$

Since $F \geq 0$, the norm formula for F reduces to

$$\begin{aligned} &\left\| F(t, x, \mathbf{w}, \nabla \mathbf{w}) \right\|_{L^r(I, L^s(\Omega))} \\ &\qquad = \sup_{\substack{\|\varphi\|_{L^{r'}(I, L^{s'}(\Omega))} \leq 1 \\ \varphi \geq 0}} \int_I \int_\Omega F(t, x, \mathbf{w}, \nabla \mathbf{w}) \varphi(t, x) \, dx \, dt. \end{aligned}$$

This and (8.5) proves the lemma. □

Corollary 8.12. *Let Ω be a domain in $\mathbb{R}^n$ and $I := [0,T] \subset \mathbb{R}$, $T > 0$. Let $F : I \times \Omega \times \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}$ and $\mathbf{G} : I \times \Omega \times \mathbb{R}^m \to \mathbb{R}^{n\times n}$ with*

$$F(t,x,y,z) = \sum_{\alpha,\beta=1}^{n} G_{\alpha\beta}(t,x,y) z_\alpha z_\beta.$$

Moreover assume G satisfies

(1) $\mathbf{G} \geq 0$, *i.e. is positive semidefinite,*
(2) $\mathbf{G}$ *measurable in t, x for a.a. y,*
(3) $\mathbf{G}(t,x,\cdot)$ *continuous for a.a. t, x.*

If $\mathbf{w}^N \to \mathbf{w}$ in $L^1_{\text{loc}}(I \times \Omega)$ and $\boldsymbol{\nabla}\mathbf{w}^N \rightharpoonup \boldsymbol{\nabla}\mathbf{w}$ in $L^1_{\text{loc}}(I\times\Omega)$, then for all r, s with $1 \leq r < \infty$, $1 \leq s < \infty$ there holds

$$\Big\|F(t,x,\mathbf{w},\boldsymbol{\nabla}\mathbf{w})\Big\|_{L^r(I,L^s(\Omega))} \leq \liminf_N \Big\|F(t,x,\mathbf{w}^N,\boldsymbol{\nabla}\mathbf{w}^N)\Big\|_{L^r(I,L^s(\Omega))}.$$

Proof. For $\frac{1}{2} < q < 1$ define F_q by

$$F_q(t,x,y,z) := (F(t,x,y,z))^q$$

then F_q fulfills the requirements of lemma 8.11. Thus

$$\begin{aligned} \Big\|F(t,x,\mathbf{w},\boldsymbol{\nabla}\mathbf{w})\Big\|_{L^r(I,L^s(\Omega))} &= \Big\|F_q(t,x,\mathbf{w},\boldsymbol{\nabla}\mathbf{w})\Big\|^{\frac{1}{q}}_{L^{\frac{r}{q}}(I,L^{\frac{s}{q}}(\Omega))} \\ &\leq \liminf_N \Big\|F_q(t,x,\mathbf{w}^N,\boldsymbol{\nabla}\mathbf{w}^N)\Big\|^{\frac{1}{q}}_{L^{\frac{r}{q}}(I,L^{\frac{s}{q}}(\Omega))}. \\ &= \liminf_N \Big\|F(t,x,\mathbf{w}^N,\boldsymbol{\nabla}\mathbf{w}^N)\Big\|_{L^r(I,L^s(\Omega))}. \end{aligned}$$

□

4. Interpolation (Espaces de traces)

In this section we will prove an interpolation theorem, which is very useful in deriving estimates for solutions of parabolic problems pointwise in time. Suppose that u is a solution to a parabolic problem, such that u, resp. $\partial_t u$, are in the Bochner spaces $L^{p_0}(I, A_0)$, resp. $L^{p_1}(I, A_1)$, where A_0 and A_1 are Banach spaces. Then we will see that u is with respect to the time a continuous function with values in the real interpolation space $\overline{A}_{\theta,q} := [A_0, A_1]_{\theta,q}$, where $\theta = \theta(p_0, p_1)$ and $q = q(p_0, p_1)$. The proof will be quite standard and we will mainly compile results which can be found in [**LP64**], [**BL76**] and [**Ada75**]. Nevertheless to the knowledge of the author there exists no exact statement of this result in literature. Since this interpolation result plays a fundamental role in this thesis it is indispensable to prove it in some detail.

Definition 8.13. *Let $A_0 \subset A_1$ be two Banach spaces. Let $1 \leq p_j < \infty$, $\alpha_j \in \mathbb{R}$ and $\eta_j = \alpha_j + \frac{1}{p_j}$ for $j = 0, 1$. Let X denote the space*

$$X(A_0, A_1) = \{u \in L^1_{\text{loc}}(\mathbb{R}^{\geq 0}; A_0), u' \in L^1_{\text{loc}}(\mathbb{R}^{\geq 0}; A_1)\}.$$

We shall work with the Banach spaces $V = V(p_0, \alpha_0, A_0; p_1, \alpha_1, A_1)$ with

$$V = \{u \in X(A_0, A_1); \|u\|_V < \infty\},$$
$$\|u\|_V(u) = \max\left\{\|t^{\alpha_0} u\|_{L^{p_0}(\mathbb{R}^{\geq 0}; A_0)}, \|t^{\alpha_1} u'\|_{L^{p_1}(\mathbb{R}^{\geq 0}; A_1)}\right\}.$$

These spaces have been introduced by J. L. Lions and J. Peetre [**LP64**], but J. Bergh and J. Löfström have defined similar Banach spaces $\widetilde{V}(\overline{A}, \overline{p}, \overline{\eta})$ (see [**BL76**], corollary 3.12.3). The interconnection of these spaces is that both $\widetilde{V}(\overline{A}, \overline{p}, \overline{\eta})$ and $V(p_0, \alpha_0, A_0; p_1\alpha_1, A_1)$ are norm–equivalent, if we choose $\eta_j = \alpha_j + \frac{1}{p_j}$ with $j = 0, 1$ (see [**BL76**], remark 3.14.12).

Lemma 8.14. *Let $u \in X(A_0, A_1)$ then there exists $b \in A_0 + A_1 = A_1$ such that*

$$u(t) = b + \int_1^t u'(\tau) d\tau, \quad a.e.\ in\ \mathbb{R}^{>0}.$$

Especially u is continuous on $(0, \infty)$ with values in A_1.

Proof. The proof of this lemma is standard. We refer to Adams [**Ada75**], lemma 7.12, for a similar result. □

Note that due to this lemma every function $u \in X$ has a limit $u(0^+)$ in $A_0 + A_1 = A_1$. The lemma still holds true if $\mathbb{R}^{>0}$ is replaced by an intervall $I = [0, T]$.

Definition 8.15. *By $T = T(p_0, \alpha_0, A_0; p_1, \alpha_1, A_1)$ we denote the space of traces $u(0^+)$ of functions $u \in V = V(p_0, \alpha_0, A_0; p_1, \alpha_1, A_1)$ equipped with the quotient norm*

$$\|u\|_T = \inf_{v(0^+) = u(0^+)} \|v\|_V.$$

$T(p_0, \alpha_0, A_0; p_1, \alpha_1, A_1)$ is a Banach space.

Theorem 8.16. *Let $1 \le p_j < \infty$, $\alpha_j \in \mathbb{R}$, and $\eta_j = \alpha_j + \frac{1}{p_j}$ with $j = 0, 1$. Further let θ and p be given by*

$$\theta = \frac{\eta_0}{\eta_0 + 1 - \eta_1}, \qquad \frac{1}{p} = \frac{1 - \theta}{p_0} + \frac{\theta}{p_1}.$$

If $\eta_0 > 0$ and $\eta_1 < 1$, then

$$T(p_0, \alpha_0, A_0; p_1, \alpha_1, A_1) = \overline{A}_{\theta, p}.$$

Proof. J. Bergh and J. Löfström have shown that their trace space $T(\overline{A}, \overline{p}, \theta)$ is under the stated conditions $\eta_0 > 0$ and $\eta_1 < 1$ equivalent to the trace space $T(p_0, \alpha_0, A_0; p_1, \alpha_1, A_1)$ of J. L. Lions and J. Peetre (see [**BL76**], remark 3.14.12). Further J. Bergh and J. Löfström showed that $T(\overline{A}, \overline{p}, \theta) = \overline{A}_{\theta, p}$ (see [**BL76**], pg. 72-75). □

Remark 8.17. *Let $1 \le p_0 < \infty$, $1 < p_1 < \infty$ and $\alpha_0 = \alpha_1 = 0$, then $\eta_0 = \frac{1}{p_0} > 0$ and $\eta_1 = \frac{1}{p_1} < 1$. Hence the requirements of theorem 8.16 are automatically fulfilled for $T(p_0, 0, A_0; p_1, 0, A_1)$. This case will later be of great importance. Hence we define:*

Definition 8.18. *For $1 \le p_j < \infty$ with $j = 0, 1$, let*

$$V(p_0, A_0; p_1, A_1) := V(p_0, 0, A_0; p_1, 0, A_1),$$
$$T(p_0, A_0; p_1, A_1) := T(p_0, 0, A_0; p_1, 0, A_1).$$

THEOREM 8.19. *Let $1 \le p_j < \infty$ with $j = 0, 1$ and*

$$\theta = \frac{p_1}{p_1 + p_1 p_0 - p_0}, \tag{8.6}$$

then

$$T(p_0, A_0; p_1, A_1) = \overline{A}_{\theta, \frac{1}{\theta}}$$

and

$$V(p_0, A_0; p_1, A_1) \hookrightarrow L^\infty(\mathbb{R}^{>0}; \overline{A}_{\theta,\frac{1}{\theta}}) \cap C(\mathbb{R}^{>0}; \overline{A}_{\theta,\frac{1}{\theta}}).$$

Furthermore

$$\|u\|_{L^\infty(I;\overline{A}_{\theta,\frac{1}{\theta}})} \le \|u\|_V, \tag{8.7}$$

$$\|u\|_{L^\infty(\mathbb{R}^{\ge 0};\overline{A}_{\theta,\frac{1}{\theta}})} \le C\, \|u\|^{1-\theta}_{L^{p_0}(\mathbb{R}^{\ge 0};A_0)} \|u'\|^{\theta}_{L^{p_1}(\mathbb{R}^{\ge 0};A_1)}. \tag{8.8}$$

PROOF. In the notation of theorem 8.16 we have $\eta_0 = \frac{1}{p_0} > 0$ and $\eta_1 = \frac{1}{p_1} < 1$ and

$$\theta = \frac{\eta_0}{\eta_0 + 1 - \eta_1} = \frac{\frac{1}{p_0}}{\frac{1}{p_0} + 1 - \frac{1}{p_1}} = \frac{p_1}{p_1 + p_1 p_0 - p_0}.$$

Furthermore

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} = \frac{(p_1 - 1) + 1}{p_1 + p_1 p_0 - p_0} = \theta.$$

This proves (8.6).

Due to lemma 8.14 we can assume $u \in C(\mathbb{R}^{\ge 0}; A_0 + A_1 = A_1)$. Let $u \in V = V(p_0, 0, A_0; p_1, 0, A_1)$. For $h > 0$ we define $\tau_h u$ by $(\tau_h u)(t) = u(t+h)$, then $\tau_h u \in V \cap C(\mathbb{R}^{\ge 0}; A_0 + A_1)$. By definition of T and theorem 8.16 there holds

$$\|u(h)\|_{\overline{A}_{\theta,\frac{1}{\theta}}} = \|(\tau_h u)(0)\|_{\overline{A}_{\theta,\frac{1}{\theta}}} \le \|\tau_h u\|_V \le \|u\|_V.$$

This proves the embedding $V \hookrightarrow L^\infty(\mathbb{R}^{>0}; \overline{A}_{\theta,\frac{1}{\theta}})$ and estimate (8.7). In order to show $V \hookrightarrow C(\mathbb{R}^{>0}; \overline{A}_{\theta,\frac{1}{\theta}})$ let $t \ge 0$ and $h \to 0$ (for $t = 0$ let $h \downarrow 0$), then

$$\begin{aligned}
\|u(t+h) - u(t)\|_{\overline{A}_{\theta,\frac{1}{\theta}}} &= \|(\tau_{t+h}u - \tau_t u)(0)\|_{\overline{A}_{\theta,\frac{1}{\theta}}} \\
&\le \|(\tau_{t+h}u - \tau_t u)\|_V \\
&\le \max\Big\{\underbrace{\|\tau_{t+h}u - \tau_t u\|_{L^{p_0}(\mathbb{R}^{>0};A_0)}}_{\xrightarrow{h} 0}, \underbrace{\|\tau_{t+h}u' - \tau_t u'\|_{L^{p_1}(\mathbb{R}^{>0};A_1)}}_{\xrightarrow{h} 0}\Big\} \\
&\xrightarrow{h} 0.
\end{aligned}$$

Still we have to prove the logarithmic convex inequality for the norms of u. This will be done by a scaling argument. For $\lambda > 0$ let $\pi_\lambda u$ be defined by $(\pi_\lambda u)(t) = u(\lambda t)$.

$$\begin{aligned}
\|u(0)\|_{\overline{A}_{\theta,\frac{1}{\theta}}} &= \|(\pi_\lambda u)(0)\|_{\overline{A}_{\theta,\frac{1}{\theta}}} \\
&\le C\, \|\pi_\lambda u\|_V \\
&\le C\, \max\big\{\|\pi_\lambda u\|_{L^{p_0}(\mathbb{R}^{>0};A_0)}, \|(\pi_\lambda)u'\|_{L^{p_1}(\mathbb{R}^{>0};A_1)}\big\} \\
&= C\, \max\big\{\lambda^{-\frac{1}{p_0}}\|u\|_{L^{p_0}(\mathbb{R}^{>0};A_0)}, \lambda^{1-\frac{1}{p_1}}\|u'\|_{L^{p_1}(\mathbb{R}^{>0};A_1)}\big\}.
\end{aligned}$$

The minimum of the right-hand side over all $\lambda > 0$ is attained at λ_0 with

$$\lambda_0^{-\frac{1}{p_0}} \|u\|_{L^{p_0}(\mathbb{R}^{>0};A_0)} = \lambda_0^{1-\frac{1}{p_1}} \|u'\|_{L^{p_1}(\mathbb{R}^{>0};A_1)}.$$

Hence

$$\begin{aligned}\|u(0)\|_{\overline{A}_{\theta,\frac{1}{\theta}}} &\leq C\, \|u\|_{L^{p_0}(\mathbb{R}^{>0};A_0)}^{\frac{1-\frac{1}{p_1}}{\frac{1}{p_0}+1-\frac{1}{p_1}}} \|u'\|_{L^{p_1}(\mathbb{R}^{>0};A_1)}^{\frac{\frac{1}{p_0}}{\frac{1}{p_0}+1-\frac{1}{p_1}}} \\ &= C\, \|u\|^{1-\theta}_{L^{p_0}(\mathbb{R}^{>0};A_0)} \|u'\|^{\theta}_{L^{p_1}(\mathbb{R}^{>0};A_1)}.\end{aligned}$$

This implies the desired inequality for $u(0)$. For $h > 0$ consider

$$\begin{aligned}\|u(h)\|_{\overline{A}_{\theta,\frac{1}{\theta}}} &= \|(\tau_h)u(0)\|_{\overline{A}_{\theta,\frac{1}{\theta}}} \\ &\leq C\, \|\tau_h u\|^{1-\theta}_{L^{p_0}(\mathbb{R}^{>0};A_0)} \|\tau_h'\|^{\theta}_{L^{p_1}(\mathbb{R}^{>0};A_1)} \\ &\leq C\, \|u\|^{1-\theta}_{L^{p_0}(\mathbb{R}^{>0};A_0)} \|u'\|^{\theta}_{L^{p_1}(\mathbb{R}^{>0};A_1)}.\end{aligned}$$

□

Later, when u will be the solution of a parabolic differential equation, we cannot always ensure that u is defined for all $t > 0$. Instead there will be a constant T with $0 < T \leq \infty$ such that u is defined on $(0, T)$.

DEFINITION 8.20. *Let $A_0 \subset A_1$ be two Banach spaces and $I = [0, T]$ with $0 < T \leq \infty$. Let $1 \leq p_j < \infty$ with $j = 0, 1$. Let X_I denote the space*

$$X_I(A_0, A_1) = \{u \in L^1_{\mathrm{loc}}(I; A_0), u' \in L^1_{\mathrm{loc}}(I; A_1)\}.$$

We shall work with the Banach spaces $V_I = V(I; p_0, \alpha_0, A_0; p_1, \alpha_1, A_1)$ with

$$\begin{aligned}V_I &= \{u \in X_I(A_0, A_1); \|u\|_{V_I} < \infty\}, \\ \|u\|_{V_I}(u) &= \max\left\{\|t^{\alpha_0} u\|_{L^{p_0}(I;A_0)}, \|t^{\alpha_1} u'\|_{L^{p_1}(I;A_1)}\right\}.\end{aligned}$$

Note that $V(p_0, \alpha_0, A_0; p_1, \alpha_1, A_1) = V_{\mathbb{R}^{>0}}(p_0, \alpha_0, A_0; p_1, \alpha_1, A_1)$.

THEOREM 8.21. *Let $0 < T \leq \infty$, $I = [0, T]$, $1 \leq p_j < \infty$ with $j = 0, 1$ and*

$$\theta = \frac{p_1}{p_1 + p_1 p_0 - p_0},$$

then

$$V_I(p_0, A_0; p_1, A_1) \hookrightarrow C(I; \overline{A}_{\theta,\frac{1}{\theta}}).$$

Furthermore

$$\|u\|_{L^\infty(I;\overline{A}_{\theta,\frac{1}{\theta}})} \leq C \max\left\{\|u\|_{L^{p_0}(I;A_0)},\ \|u\|^{1-\theta}_{L^{p_0}(I;A_0)} \|u'\|^{\theta}_{L^{p_1}(I;A_1)}\right\}.$$

PROOF. Let $u \in V_I := V_I(p_0, A_0; p_1, A_1)$. Let $\varphi \in C_0^\infty(\mathbb{R})$ with $0 \leq \varphi \leq 1$, $\varphi|_{[0,\frac{2}{3}T]} = 1$, and $\varphi|_{[\frac{4}{5}T,\infty)} = 0$. Let $w = u \cdot \varphi$, then $w \in V := V(p_0, A_0; p_1, A_1)$. By theorem 8.19 we conclude $w \in C(\mathbb{R}^{\geq 0}; \overline{A}_{\theta,\frac{1}{\theta}})$, hence $u \in C([0, \frac{2}{3}T]; \overline{A}_{\theta,\frac{1}{\theta}})$. If we substitute **u** by $\overline{u}$ with $\overline{u}(t) := u(T - t)$ and w by $\overline{w}$ with $\overline{w} = \overline{u} \cdot \varphi$, we get $u \in$

$C([\frac{1}{3}T,T];\overline{A}_{\theta,\frac{1}{\theta}})$. Altogether we have proven $u \in C(I;\overline{A}_{\theta,\frac{1}{\theta}})$, which implies $V_I \to C(I;\overline{A}_{\theta,\frac{1}{\theta}})$ (as a linear mapping). Furthermore we know that

$$\begin{aligned}\|u\|_{L^\infty(I;\overline{A}_{\theta,\frac{1}{\theta}})} &\le \|w\|_{L^\infty([0;\frac{T}{2}];\overline{A}_{\theta,\frac{1}{\theta}})} + \|\overline{w}\|_{L^\infty([\frac{T}{2};T];\overline{A}_{\theta,\frac{1}{\theta}})}\\ &\le \|w\|_V + \|\overline{w}\|_V.\end{aligned}$$

We will now show that the right-hand side is bounded independently of u. For this we will only examine $\|w\|_V$, for $\|\overline{w}\|_V$ can be handled by exchanging w by $\overline{w}$ in the following calculations. Recall that $\|u\|_{V_I} \le 1$, so by lemma 8.14 there holds $\|u'\|_{L^\infty(I;A_1)} \le C\|u\|_{V_I}$.

$$\|w\|_{L^{p_0}(\mathbb{R}^{>0},A_0)} = \|u\cdot\varphi\|_{L^{p_0}(\mathbb{R}^{>0},A_0)} \le \|\varphi\|_\infty \|u\|_{L^{p_0}(I,A_0)} \le C_\varphi \|u\|_{L^{p_0}(I,A_0)}.$$

$$\begin{aligned}\|w'\|_{L^{p_1}(\mathbb{R}^{>0},A_1)} &= \|(u\cdot\varphi)'\|_{L^{p_1}(\mathbb{R}^{>0},A_1)}\\ &\le \|u'\cdot\varphi\|_{L^{p_1}(\mathbb{R}^{>0},A_1)} + \|\,u\cdot\varphi'\|_{L^{p_1}(\mathbb{R}^{>0},A_1)}\\ &= \|u'\cdot\varphi\|_{L^{p_1}(\mathbb{R}^{>0},A_1)} + \|\,u\cdot\varphi'\|_{L^{p_1}(I,A_1)}\\ &\le \|\varphi\|_\infty \underbrace{\|u'\|_{L^{p_1}(I,A_1)}}_{\le \|u\|_{V_I}} + \|\varphi\|_{1,\infty} \underbrace{\|u\|_{L^{p_1}(I,A_1)}}_{\le C\,\|u\|_{V_I}}\\ &\le \big(1 + 2\,C\,\|\varphi\|_{1,\infty}\big)\,\|u\|_{V_I}\\ &\le C_\varphi\,\|u\|_{V_I}.\end{aligned}$$

Hence

$$\begin{aligned}\|w\|_V &\le \|w\|^{1-\theta}_{L^{p_0}(\mathbb{R}^{>0},A_0)} \|w'\|^{\theta}_{L^{p_1}(\mathbb{R}^{>0},A_1)}\\ &\le C_\varphi \|u\|^{1-\theta}_{L^{p_0}(I,A_0)} \|u\|^{\theta}_{V_I}\\ &\le C_\varphi \|u\|_{V_I}.\end{aligned}$$

This shows on the one hand that $V_I \to C(I;\overline{A}_{\theta,\frac{1}{\theta}})$ and on the other hand that

$$\begin{aligned}\|u\|_{L^\infty(I;\overline{A}_{\theta,\frac{1}{\theta}})} &\le \|w\|_{L^\infty([0;\frac{T}{2}];\overline{A}_{\theta,\frac{1}{\theta}})} + \|\overline{w}\|_{L^\infty([\frac{T}{2};T];\overline{A}_{\theta,\frac{1}{\theta}})}\\ &\le C_\varphi \|u\|^{1-\theta}_{L^{p_0}(I,A_0)} \|u\|^{\theta}_{V_I}\\ &= C_\varphi \max\big\{\|u\|_{L^{p_0}(I,A_0)}, \|u\|^{1-\theta}_{L^{p_0}(I,A_0)} \|u'\|^{\theta}_{L^{p_1}(I,A_1)}\big\}.\end{aligned}$$

This proves the theorem. □

5. Lorentz Spaces

In chapter 4, 5, and 6 we make use of the Lorentz spaces $L^{q,r}(\Omega)$ with $r, q \in [1,\infty]$. Therefore we give a short overview of the properties of these spaces (see [**BL76**] for reference).

Let Ω be an arbitrary domain ($\Omega \subset \mathbb{R}^d$ or the d-dimensional torus) and let $f : \Omega \to \mathbb{R}$ be measurable. For $\lambda > 0$ let

$$m(\lambda, f) := |\{y \in \Omega \,:\, |f(y)| > \lambda\}|.$$

Then

$$f^*(t) := \inf\big\{\lambda > 0 \,:\, m(\lambda, f) \le t\big\}, \qquad t \in [0,\infty)$$

is the decreasing rearrangement of f. This is a non–negative and non–increasing function on $(0,\infty)$, which is continuous on the right and has the property

$$m(\lambda, f^*) = m(\lambda, f), \qquad \lambda \geq 0.$$

Now the Lorentz space $L^{q,r} := L^{q,r}(\Omega)$ is defined as follows. We have $f \in L^{q,r}$, $1 \leq q \leq \infty$, if and only if

$$\|f\|_{L^{q,r}} := \Big(\int_0^\infty \big(t^{\frac{1}{q}} f^*(t)\big)^r \frac{dt}{t}\Big)^{\frac{1}{r}} < \infty \qquad \text{when } 1 \leq r < \infty,$$

$$\|f\|_{L^{q,\infty}} := \sup_{t>0} t^{\frac{1}{q}} f^*(t) < \infty \qquad \text{when } r = \infty.$$

For $1 \leq q \leq \infty$ we further introduce the weak L^q spaces denoted by ${L_{\text{weak}}}^q$. The space ${L_{\text{weak}}}^q := {L_{\text{weak}}}^q(\Omega)$, $1 \leq q < \infty$, consists of all measurable f such that

$$\|f\|_{{L_{\text{weak}}}^q} := \sup_{\lambda>0} m(\lambda, f)^{\frac{1}{q}} < \infty. \tag{8.9}$$

In the limiting case $q = \infty$ we put ${L_{\text{weak}}}^q := L^\infty$. Note that $\|\cdot\|_{{L_{\text{weak}}}^q}$ is not a norm if $1 \leq q < \infty$ but a quasi norm thus defining a topology. For all $1 \leq q \leq \infty$ we have

$$L^{q,q}(\Omega) = L^q(\Omega), \tag{8.10}$$

$$L^{q,\infty}(\Omega) = {L_{\text{weak}}}^q(\Omega) \tag{8.11}$$

with equality of (quasi–)norms. Especially a set $M \subset L^{q,\infty}(\Omega)$ is bounded with respect to $\|\cdot\|_{L^{q,\infty}}$ if and only if it is bounded with respect to $\|\cdot\|_{{L_{\text{weak}}}^q}$. For more details regarding Lorentz spaces see Bergh, Löfström [**BL76**] and Triebel [**Tri78**].

Notation

C	generic constant
Ω	d–dimensional torus
$\langle\cdot,\cdot\rangle$	scalar product in space, i.e. $\langle f,g\rangle = \int\limits_\Omega fg\,dx$
C_0^∞	smooth functions with compact support
L^q	Lebesgue space
$L^{q,r}$	Lorentz space
$W^{k,q}$	Sobolev space
$W_0^{k,q}(\Omega)$	Sobolev space on torus with mean value zero
$B^s_{p,q}(\Omega)$	Besov space
$L^q(I,X)$	Bochner space (with $I=[0,T]$)
I_k	discretized time interval, i.e. $I_k=[0,k,2k,\dots,T]$
	with time step size $k=\frac{T}{m}$
$l^q(I_k)$	time discretized version of $L^q(I)$
	with norm $\|f\|_{l^q(I_k)} = (k\sum_{m=0}^M \lvert f_m\rvert^q)^{\frac{1}{q}}$ for $1\le q<\infty$
	with norm $\|f\|_{l^\infty(I_k)} = \sup_{m=0,\dots,M}\lvert f_m\rvert$ for $q=\infty$
$\mathbf{A}^{\mathrm{sym}}$	symmetric part of the matrix $\mathbf{A}$
$\mathbf{A}^{\mathrm{anti}}$	scew symmetric (anti symmetric) part of the matrix $\mathbf{A}$
$\mathbf{Du}$	symmetric part of the gradient $\boldsymbol{\nabla}\mathbf{u}$
$\fint\limits_B \dots\,dx$	mean value integral over the set B

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