# The Radiation Field Tensor in Classical Electrodynamics

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# Der Strahlungstensor in der Klassischen Elektrodynamik

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**Abstract**

This master thesis is devoted to the study of the electromagnetic field associated with radiation. The primary objective is to present a rigorous proof of the validity of the well known radiation term. Specifically, we demonstrate that for a sequence of points $(x_n)_{n\in\mathbb{N}}$ in space-time converging to a point $q_{(\tau^*)}$ on the world-line of a charged particle, the tensor components of the radiation field converge to:

$$\lim_{n\to\infty} F^{\mu\nu}_{Rad(x_n)} = \frac{4e}{3}\left(\dddot{q}^{\mu}_{(\tau^*)}\dot{q}^{\nu}_{(\tau^*)} - \dddot{q}^{\nu}_{(\tau^*)}\dot{q}^{\mu}_{(\tau^*)}\right), \tag{1}$$

where $F^{\mu\nu}_{Rad} = F^{\mu\nu-} - F^{\mu\nu+}$ is the difference between the retarded, $F^{\mu\nu-}$, and advanced, $F^{\mu\nu+}$, Lienard-Wiechert fields of the trajectory $q_{(\tau)}$. These fields are known to be divergent at the particle's position, see equations (4.1) and (4.2). Therefore, the limit can only be achieved by a delicate cancellation between the divergent terms in both fields. To achieve this result, the thesis is structured into four chapters:

1. **Chapter One** provides a motivation for the topic, revisiting key concepts from electrodynamics. It offers a refresher on general principles of the theory, defines the radiation field, and addresses aspects of Dirac's theory of radiation in classical electrodynamics which are relevant to this work.

2. **Chapter Two** marks the beginning of the main body of this work which is presented in a mathematical rigorous manner. Here, the foundational concepts of space-time, world-lines, and retarded/advanced positions are introduced and defined. Specifically, this chapter establishes the existence and uniqueness of the advanced and retarded positions (Lemma 2.9), demonstrates the continuity of the advanced and retarded times across all of space-time (Lemma 2.12), and proves their differentiability outside the world-line (Lemma 2.13).

3. **Chapter Three** develops a physical intuition for the radiation fields. In this chapter, the mechanism of the divergence cancellation between $F^{\mu\nu-}$ and $F^{\mu\nu+}$ are analyzed by shifting the spatial position by an infinitesimal amount in an arbitrary direction. In this chapter, we work rather with the electromagnetic fields instead of the tensors to also study the role of Coulomb fields in radiation. For this, we investigate the electromagnetic Lienard-Wiechert fields (equations (3.3) and (3.4)) in different scenarios.

   It is shown, through explicit computation, that the radiation fields vanish for a particle moving with constant velocity (Results 3.1 and 3.2). Expansions of the advanced and retarded times near the particle are derived (Results 3.4 and 3.5), as well as for the normal vectors (Results 3.6 and 3.7). These results came as an interesting surprise, as these quantities are not differentiable at the trajectory, which means that a conventional Taylor expansion is not available. A particularly interesting observation presented at the end of this chapter is that for the case of motion with constant acceleration, it is shown that the Coulomb fields diverge in such a way that they cancel a divergence of the far fields, making the entire expression convergent in the end (Result 3.11 and equations (3.32) and (3.33)). Hence, any study of equation (1) for general trajectories must include both the near and the far fields.

4. **Chapter Four** is the final chapter of this thesis and is again presented in a mathematical rigorous manner. In this chapter, the announced proof of the convergence of Dirac's radiation term (1) is provided, where the main difficulty is to control the cancellation mechanism between $F^{\mu\nu-}$ and $F^{\mu\nu+}$. In order to achieve this, the existence of a special point on the world-line (referred to as "Dirac's choice") is shown. This point is evaluated at parameter $\tau_n$ at which $\dot{q}^{\mu}_{(\tau_n)}(x_\mu - q_{\mu(\tau_n)}) = 0$ holds (Lemma 4.3). Due to this property, it serves as a convenient expansion point because it simplifies the computations of the field tensors. It is then demonstrated that the parameter $\tau_n$ at this point must converge, along with the advanced and retarded parameters $\tau_n^{\pm}$, to the same parameter $\tau^*$ on the world-line (Corollary 4.4).

   From this point, an expansion of the retarded and advanced field tensors is performed around $\tau_n$, and their divergence at the trajectory is studied (Lemma 4.7). In the final section, the limit of the radiation field and the error term are computed explicitly, which constitutes the main result of the chapter (Theorem 4.2).

5. **Chapter Five** concludes with short outlook in this topic.

# Contents

# Chapter 1

# Radiation in Classical Electrodynamics

The goal of this chapter is to provide a self-contained introduction to the topic of radiation reaction. In the first section, we present a brief overview of Maxwell's equations, the Lorentz force and the interplay between them. In the second section, we discuss the radiation phenomena, where we briefly introduce the historical methods used to tackle this problem. Here, we will focus on the developments in the model of a point particle. We also present a discussion about the validity of P.A.M. Dirac's assumption introduced in his famous paper *"Classical theory of radiating electrons"* [Dir38], where he motivated that the radiation field tensor is the difference of the retarded and the advanced Lienard-Wiechert fields.

## 1.1 Electrodynamics

Today, the electromagnetic theory remains as one of the most successful frameworks of physics. From Maxwell´s original work [Max10] in 1873, we have seen the theory to evolve into a well established and robust field with the help of which a wide spectrum of phenomena can be grouped.

The classical theory of electrodynamics, as understood today, is described in terms of the electromagnetic fields and charge distributions. A charged body moving in space-time is modeled with the charge density $\rho : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}$, which tell us about the spatial distribution of the charge in space, and the current density $\overrightarrow{J} : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3$, which provides information about the movement of the charge. Given these two quantities, it is possible to solve the **Maxwell´s equations**[1] [Gri11, p. 417]

$$
\begin{aligned}
\nabla \cdot \overrightarrow{E}_{(t,\overrightarrow{x})} &= \frac{\rho_{(t,\overrightarrow{x})}}{\epsilon_0}, \\
\nabla \cdot \overrightarrow{B}_{(t,\overrightarrow{x})} &= 0, \\
\nabla \times \overrightarrow{E}_{(t,\overrightarrow{x})} &= -\frac{\partial \overrightarrow{B}_{(t,\overrightarrow{x})}}{\partial t}, \\
\nabla \times \overrightarrow{B}_{(t,\overrightarrow{x})} &= \mu_0 \overrightarrow{J}_{(t,\overrightarrow{x})} + \mu_0 \epsilon_0 \frac{\partial \overrightarrow{E}_{(t,\overrightarrow{x})}}{\partial t},
\end{aligned}
$$

for the electric $\overrightarrow{E} : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3$ and magnetic fields $\overrightarrow{B} : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3$. If, on the other hand, we are given the electromagnetic fields, then we can compute the **Lorentz Force** that acts on a charged body. For point particles with charge $q$ at position $\overrightarrow{r}_{(t)}$ it reads [Gri11, p. 272, 446]

$$
\overrightarrow{F}_{(t,\overrightarrow{r}_{(t)})} = q \left( \overrightarrow{E}_{(t,\overrightarrow{r}_{(t)})} + \overrightarrow{v}_{(t)} \times \overrightarrow{B}_{(t,\overrightarrow{r}_{(t)})} \right),
$$

and for extended charge models one would integrate over the charge density, which is normally assumed to be supported on a compact subset $V$ of $\mathbb{R}^3$. For our case of interest, we would like to mix the Lorentz force with the Maxwell's equations. This is so because we would like to model a charged particle which is moving in space-time and therefore is producing time varying

[1] Here we use SI units, such that $\epsilon_0 \approx 8.8 * 10^{-12} \frac{F}{m}$ is the vacuum permittivity and $\mu_0 \approx 1.2 * 10^{-6} \frac{N}{A^2}$ is the vacuum permeability.

electromagnetic fields, which, at the same time, produce a force that acts on the particle, changing in this way its trajectory.

To understand better why this procedure is problematic, we discuss now how to solve the Maxwell's equations and provide as an example the fields of a point particle. It is of common practice to rewrite the Maxwell's equations in terms of the **electromagnetic potentials**, which are defined as [Gri11, p. 526]

$$\vec{E}_{(t,\vec{x})} = -\nabla\phi_{(t,\vec{x})} - \frac{\partial \vec{A}_{(t,\vec{x})}}{\partial t},$$

and

$$\vec{B}_{(t,\vec{x})} = \nabla \times \vec{A}_{(t,\vec{x})},$$

where $\phi$ is called the **scalar potential** and $\vec{A}$ is the **vector potential**. These four new functions are not uniquely defined, one can perform what is called a gauge transformation and obtain equally valid expressions[2].

It can be shown [Gri11, p. 531], that under the Lorentz gauge ($\nabla \cdot \vec{A}_{(t,\vec{x})} = -\mu_0\epsilon_0 \frac{\partial \phi_{(t,\vec{x})}}{\partial t}$), the Maxwell's equations can be given in the form

$$\Box\phi_{(t,\vec{x})} = -\frac{\rho_{(t,\vec{x})}}{\epsilon_0},$$
$$\Box\vec{A}_{(t,\vec{x})} = -\mu_0 \vec{J}_{(t,\vec{x})}.$$

The theory of special relativity provides us with a comfortable framework in which the last equations are melted into one, using the so called **four potential** [Gri11, p. 671]

$$A^\mu_{(t,\vec{x})} := \begin{pmatrix} \phi_{(t,\vec{x})}/c \\ \vec{A}_{(t,\vec{x})} \end{pmatrix},$$

where we introduce the index $\mu \in \{0, ..., 3\}$ and $A^\mu$ gives us the vector components. The same can be done for the quantities associated with the charge distribution. We define the **four current** as [Gri11, p. 668]

$$J^\mu_{(t,\vec{x})} := \begin{pmatrix} c\rho_{(t,\vec{x})} \\ \vec{J}_{(t,\vec{x})} \end{pmatrix}.$$

In this way, one can rewrite the original Maxwell's equations as

$$\Box A^\mu_{(t,\vec{x})} = -\mu_0 J^\mu_{(t,\vec{x})}. \tag{1.1}$$

One of the main focus in the theory of electrodynamics is to study how to solve equation (1.1). In this section we follow [Jac14, p. 708]. We begin by calculating the **Green's functions** of the D'Alembertian, i.e. we look for a function $G_{(x^\mu, x'^\mu)}$ that solves

$$\Box G_{(x^\mu, x'^\mu)} = \delta^4_{(x^\mu - x'^\mu)}, \tag{1.2}$$

where $\delta^4_{(x^\mu - x'^\mu)}$ is the four dimensional Dirac delta function. Because the D'Alembert operator is invariant under translations, we look functions of the form $G_{(x^\mu - x'^\mu)}$. Performing a four dimensional Fourier transformation on equation (1.2) we obtain

$$-k_\mu k^\mu \tilde{G} = 1 \Rightarrow \tilde{G} = -\frac{1}{k_\mu k^\mu},$$

and get (using the shorthand notation $z^\mu = x^\mu - x'^\mu$)

$$G_{(z^\mu)} = -\frac{1}{(2\pi)^4} \int \frac{e^{-ik_\alpha z^\alpha}}{k_\mu k^\mu} d^4k.$$

[2]The Gauge transformations are given by $\vec{A}'_{(t,\vec{x})} = \vec{A}_{(t,\vec{x})} + \nabla\lambda_{(t,\vec{x})}$ and $\phi'_{(t,\vec{x})} = \phi_{(t,\vec{x})} - \frac{\partial\lambda_{(t,\vec{x})}}{\partial t}$, with a scalar, differentiable function $\lambda_{(t,\vec{x})}$ [Gri11, p. 530]. Inserting these expressions in the equations for the electromagnetic fields show that the choice of $\lambda$ does not change the fields $\vec{E}$ and $\vec{B}$.

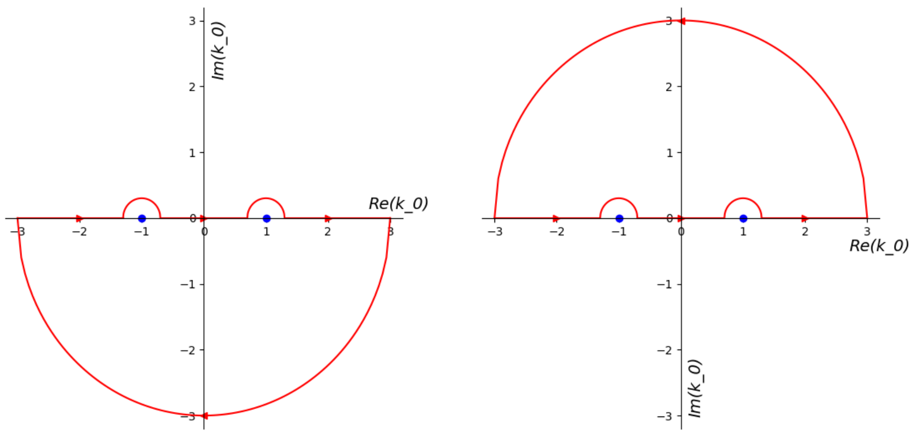


Figure 1.1: Possible paths of integration

Since the integral diverges for $k_\mu k^\mu = 0$, we use complex analysis to evaluate it at the poles. We write then

$$G_{(z^\mu)} = -\frac{1}{(2\pi)^4}\int d^3k e^{i\overrightarrow{k}\cdot\overrightarrow{z}}\int dk^0 \frac{e^{-ik^0z^0}}{(k^0)^2-\overrightarrow{k}^2},$$

where the last integral over $k^0$ can be evaluated using one of two possible paths, as shown in figure 1.1. Using the residue theorem, we obtain

$$G_{(x^\mu - x'^\mu)} = \frac{\Theta_{(x^0-x'^0)}}{(2\pi)^3}\int d^3k \frac{\sin\left(\|\overrightarrow{k}\|_{\mathbb{R}^3}(x^0-x'^0)\right)e^{i\overrightarrow{k}\cdot(\overrightarrow{x}-\overrightarrow{x}')}}{\|\overrightarrow{k}\|_{\mathbb{R}^3}},$$

where $\Theta_{(x^0-x'^0)}$ is the Heaviside step function and $\|\cdot\|_{\mathbb{R}^3}$ denotes the euclidean norm in $\mathbb{R}^3$. The last integral can be solved using spherical coordinates. We use the relation between the Dirac's delta function and the integral of the exponential function and finally arrive at

$$G^- := G^-_{(x^\mu-x'^\mu)} = \frac{\Theta_{(x^0-x'^0)}}{4\pi}\frac{1}{\|\overrightarrow{x}-\overrightarrow{x}'\|_{\mathbb{R}^3}}\delta(x^0-x'^0-\|\overrightarrow{x}-\overrightarrow{x}'\|_{\mathbb{R}^3}).$$

This result is know as the **retarded or causal Green's function of the D'Alembert operator**. If, on the contrary, one selects the second path to perform the integration (figure 1.1 right), the result obtained is the so called **advanced Green's function of the D'Alembert operator**

$$G^+ := G^+_{(x^\mu-x'^\mu)} = \frac{\Theta_{(x'^0-x^0)}}{4\pi}\frac{1}{\|\overrightarrow{x}-\overrightarrow{x}'\|_{\mathbb{R}^3}}\delta(x^0-x'^0+\|\overrightarrow{x}-\overrightarrow{x}'\|_{\mathbb{R}^3}).$$

Notice that there is no mathematical motivation to choose one result over the other. But the physical implications of the last result is that *one must know the particle's position in the future in order to calculate the actual electromagnetic fields*, as shown below.

The Green's functions are of great use when solving equation (1.1) because it holds that

$$A^{\mu\pm}_{(t,\overrightarrow{x})} = \int d^4x' G^\pm J^\mu_{(t',\overrightarrow{x}')}. \tag{1.3}$$

## Example 1.1: The Lienard-Wiechert Potentials

In order to study the motion of a point particle, we set

$$J^\mu_{(t,\overrightarrow{x})} = e\delta^3(\overrightarrow{x}-\overrightarrow{q}_{(t)})\begin{pmatrix} c \\ \overrightarrow{v}_{(t)}, \end{pmatrix}$$

where $e$ is the electric charge and $\overrightarrow{q}_{(t)}$ denoted the position of the particle at a given time $t$. If we put this expression in the last integral, we obtain two different outcomes. For $G^-$ the result is the *retarded four potential*

$$A^{\mu -}_{(t,\vec{x})} = \frac{\mu_0 e}{4\pi} \frac{1}{\|\vec{x} - \vec{q}^{\,-}\|_{\mathbb{R}^3} \left(1 - \frac{\vec{v}^{\,-}\cdot\vec{n}^{\,-}}{c}\right)} \begin{pmatrix} c \\ \vec{v}^{\,-} \end{pmatrix},$$

where all quantities with a subscript "minus" must be evaluated at the **retarded time** $t^-$, which shall fulfill

$$t^-_{(t,\vec{x})} = t - \frac{1}{c}\|\vec{x} - \vec{q}^{\,-}\|_{\mathbb{R}^3}.$$

Using $G^+$ we obtain the *advanced four potential* instead

$$A^{\mu +}_{(t,\vec{x})} = \frac{\mu_0 e}{4\pi} \frac{1}{\|\vec{x} - \vec{q}^{\,+}\|_{\mathbb{R}^3} \left(1 + \frac{\vec{v}^{\,+}\cdot\vec{n}^{\,+}}{c}\right)} \begin{pmatrix} c \\ \vec{v}^{\,+} \end{pmatrix},$$

where now the quantities with the subscript "plus" must be evaluated at the **advanced time** $t^+$ which shall fulfill

$$t^+_{(t,\vec{x})} = t + \frac{1}{c}\|\vec{x} - \vec{q}^{\,+}\|_{\mathbb{R}^3}.$$

From this last example we can see the appearance of a factor

$$\frac{1}{\|\vec{x} - \vec{q}^{\,\pm}\|_{\mathbb{R}^3}} \tag{1.4}$$

in both the advanced and retarded four-potentials. It will be shown later (Corollary 2.10 in chapter two) that if $\vec{x} = \vec{q}_{(t)}$, then $\vec{q}_{(t^\pm_{(x)})} = \vec{q}_{(t)}$ and therefore the equation (1.4) is divergent at the particle's position. This problem is also transported to the electromagnetic fields. This is better seen from the electromagnetic field tensor, which is defined as [Gri11, p. 666]

$$F^{\mu\nu}_{(t,\vec{x})} := \partial^\mu A^\nu_{(t,\vec{x})} - \partial^\nu A^\mu_{(t,\vec{x})},$$

where $\mu, \nu \in \{0, ..., 3\}$. Comparing this equation with the definition of the electromagnetic potentials allows us to check that the components of $F^{\mu\nu}$ are the components of the electromagnetic fields. From $A^{\mu\pm}_{(t,\vec{x})}$ we can compute the advanced and retarded field tensors. The calculation is given in Appendix E and here we will only show the final result. Lets assume that we are given the trajectory of a particle in space-time, denoted as $q_{(\tau)}$, where $\tau$ may be the proper time of the particle. Then, at a point $x = (t, \vec{x})$ in space-time, the Lienard-Wiechert fields are given by

$$F^{\mu\nu\pm}_{(x)} = \frac{\mp e}{\dot{q}_{(\tau^\pm)} \cdot (x - q_{(\tau^\pm)})} \left[ \frac{\mathrm{d}}{\mathrm{d}\tau} \frac{\dot{q}^\nu_{(\tau)}(x^\mu - q^\mu_{(\tau)}) - \dot{q}^\mu_{(\tau)}(x^\nu - q^\nu_{(\tau)})}{\dot{q}_{(\tau)} \cdot (x - q_{(\tau)})} \right] \Bigg|_{\tau = \tau^\pm}, \tag{1.5}$$

where in this equation $\tau^\pm$ denotes the proper times at which the equations

$$t^\pm_{(t,\vec{x})} = t \pm \frac{1}{c}\|\vec{x} - \vec{q}^{\,+}\|_{\mathbb{R}^3}$$

hold. $\tau^\pm$ and their properties will be discussed in detail in the next chapter. For now, we only need these formulas to state the problem we are facing. As the four-potentials, the field tensors also diverge at the particle's position. So there is no clear way to compute the Lorentz force using just the retarded or the advanced fields.

In the next section we provide an overview of what the people have done in order to model radiation and its influence in the movement of the particle.

## 1.2 Models of the Radiation Phenomena

One of the main results of the theory of electrodynamics is the fact that the Maxwell's equations obey also the wave equation. The speed at which the wave propagates in vacuum is (in SI units) the constant $c = \frac{1}{\sqrt{\epsilon_0 \mu_0}} = 299792458\ m/s$, i.e. the speed of light. This let Maxwell propose that light was in fact "made of" electromagnetic fields. The process in which charged objects emit light is called *radiation*. In other words, it refers to the way electromagnetic fields carry energy away from their source, [Zan13, p. 730]. For this reason, the emission and absorption of radiation is a subject of high interest in many fields of study. One of the core formulas in this topic was derived by Larmor in the paper *"LXIII. On the theory of the magnetic influence on spectra; and on the radiation from moving ions ", 1897*, [Lar97] and reads

$$P_{(t^-)} = \frac{1}{4\pi\epsilon_0}\frac{2e^2\|\overrightarrow{a}_{(t^-)}\|^2_{\mathbb{R}^3}}{3c^3}. \tag{1.6}$$

A derivation of this formula can be found in Appendix F.

Unfortunately, this formula provides us only with information about the power radiated at spatial infinity. Other people attempted to derive models in which one gets more information about the movement of the particle pursuing the following line of reasoning. If we think about our understanding of how nature operates, we assume the principle of energy conservation to be valid, then the electromagnetic fields that are carrying energy away from a particle must also posses information about its change of movement. In other words, the particle must lose energy through some mechanism which one wishes to model as a force acting back on the particle, in that way changing its movement until there is no more energy to be radiated and therefore achieving a state of constant motion. As mentioned before, it is not possible to work directly with the Lienard-Wiechert fields as these are divergent at the particle's trajectory.

One approach to avoiding this problem is discussed in [Spo04], where, instead of studying a point particle, an extended charge model is considered. However, two unavoidable issues arise: if one works with a rigid body, first introduced by Max Abraham in 1902 [Abr02], then one will violate the theory of special relativity and will end up with an effective mass which depends on minus the inverse of the radius of the charged body. So, for example, in the case of an electron if the radius is smaller than the classical radius, the bare mass is be negative [Spo04, p. 77]. On the other hand, if the model adheres to the theory of special relativity, as in the Lorentz model introduced by Hendrik Lorentz in 1903 [Lor03], one encounters another problem of "mass renormalization." Here, the effective mass of the electron have an extra term due to the angular momentum and to control this term, the bare mass of the electron must tend to zero. In this case it also happens that the equator of the electron rotates with the speed of light (see [Spo04, p. 51]).

Another path was proposed by Paul Dirac in 1938 in [Dir38]. In this paper he split the retarded field tensor as

$$F^{\mu\nu-}_{(x)} = \frac{1}{2}(F^{\mu\nu-}_{(x)} + F^{\mu\nu+}_{(x)}) + \frac{1}{2}(F^{\mu\nu-}_{(x)} - F^{\mu\nu+}_{(x)}), \tag{1.7}$$

and use it to calculate new equations of motion. We see the appearance of the difference

$$F^{\mu\nu}_{Rad(t,\overrightarrow{x})} = F^{\mu\nu-}_{(t,\overrightarrow{x})} - F^{\mu\nu+}_{Rad(t,\overrightarrow{x})}, \tag{1.8}$$

which was identified by Dirac as the radiation field produced by the particle [Dir38, p. 151]. In order to illustrate his idea, we study a simple experiment, see Figure 1.2. Here we assume that a charged particle moves with constant velocity before entering a zone where it interacts with an external force. In this zone (colored cyan), the particle may absorb and emit light and in turn accelerate. At the end of the interaction, the particle comes again to a state of constant motion where no radiation is emitted or absorbed. Let us explain in detail what we are supposing:

1. The blue line represents the particle's trajectory $q(\mathbb{R})$. (A full definition of world-line is given in the next chapter, see Definition 2.6).

2. As asymptotic behavior, the particle shall move with some constant velocities $v_1$ before and $v_2$ after the interaction. In space-time, the asymptotics are straight-lines and here, the particle fulfills the second Newton's law with external force equal zero.

3. In the cyan colored zone, the particle accelerates due to the interaction with $f_{ext}$. Here, radiation may occur.

The advantage of this scattering picture is that, before and after the interaction, we can separate in a physical meaningful manner the field that belongs to the particle and external fields. So, at some time $T^-$ before the interaction occurs, the total field in space-time can be written as

$$F_{total,T^-} = F_{in,T^-} + F^{v_1}_{Coulomb,T^-},$$

where we write the sum of $F_{in,T^-}$ called the incoming field (radiation that may interact later with the particle, which is a homogeneous solution of the Maxwell's equations) and the Coulomb field of the particle moving at constant speed $v_1$. The same can be done after the interaction occurs, such that we write at some other time $T^+$ the total field as

$$F_{total,T^+} = F_{out,T^+} + F^{v_2}_{Coulomb,T^+},$$

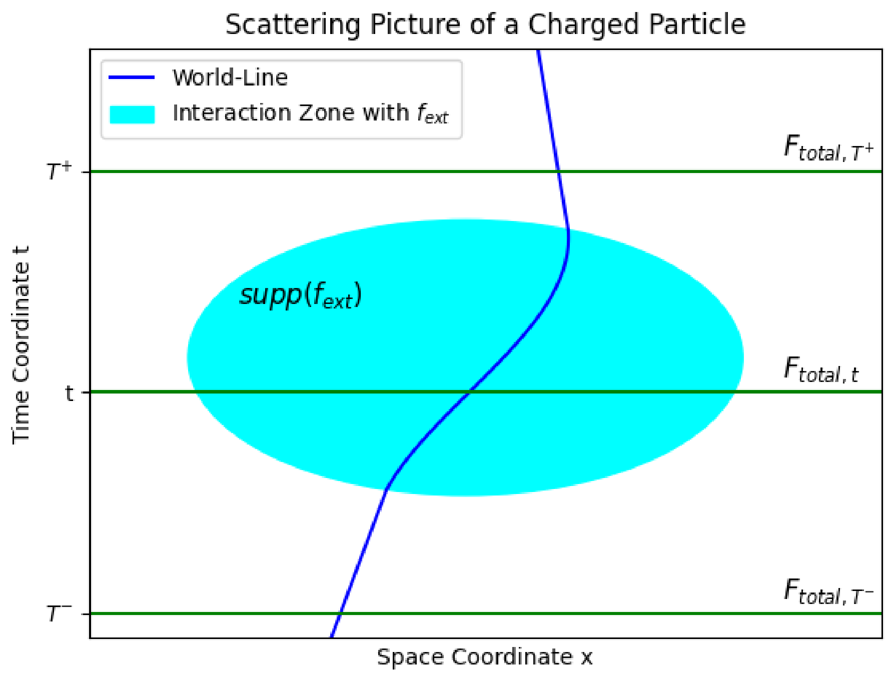


Figure 1.2: Motivation of the Radiation Field

where in this case we do not have an incoming field but rather another homogeneous solution of the Maxwell's equations, called the outgoing field $F_{out,T^+}$ (we think of it as radiation moving away from the particle).

In this scattering picture, we can intuitively define the radiation field. Before the interaction occurred, the particle had not radiated, such that the incoming field corresponds to radiation external to the particle. But after the interaction, because the particle radiated and then it moves again with constant velocity, the outgoing field must contain all external radiation plus the time-evolved radiated field by the particle during the interaction. In other words, advancing the incoming field forwards in time until $T^+$, we define

$$F_{Rad,T^+} := F_{out,T^+} - F^{evolved}_{in,T^+}.$$

In this last equation, $F^{evolved}_{in,T^+}$ represents the solution of the homogeneous Maxwell's equations evaluated at time $T^+$[3]. In the region where the interaction happens, at some time $t$, we can do the same procedure and evolve $F_{total,T^-}$ forward and $F_{total,T^+}$ backward in time. Here both fields must result in the same field, obtaining in this way

$$F_t = F^{evolved}_{in,t} + F^-_t = F^{evolved}_{out,t} + F^+_t.$$

We define again the radiation field as the difference between the evolved outgoing and incoming fields, namely

$$F_{Rad,t} := F^{evolved}_{out,t} - F^{evolved}_{in,t},$$

or, if we use the equation for $F_t$ to substitute them, we obtain

$$F_{Rad,t} = F^-_t - F^+_t. \tag{1.9}$$

This physical motivation is used here to define the radiation field. With the help of this term, Dirac derived the following formula for the radiation field

$$F^{\mu\nu}_{Rad(q_{(\tau)})} = \frac{4e}{3}\left(\dddot{q}^{\mu}_{(\tau)}\dot{q}^{\nu}_{(\tau)} - \dddot{q}^{\nu}_{(\tau)}\dot{q}^{\mu}_{(\tau)}\right) \tag{1.10}$$

and he also derived equations of motion, which allowed him to study some solutions. This term is interesting because of three aspects:

[3]See for example [Dec10, p. 62], Corollary 4.13.

1. It is an homogeneous solution of the Maxwell's equations.

2. As the difference of two divergent terms, it has a chance of being divergence free.

3. Dirac's result is compatible with Larmor's formula, see [Bar80, p. 189].

The mathematical rigorous study of the term $F^{\mu\nu -}_{(t,\vec{x})} - F^{\mu\nu +}_{(t,\vec{x})}$ and how it leads to equation (1.10) is the main topic of this thesis. In chapter four, we demonstrate that this term can be expanded, providing an explicit calculation of the remainder. Dirac's approach also presents two significant issues, which remain unsolved: First, because of the splitting in equation (1.7), one also has to deal with the divergent expression

$$\frac{1}{2}(F^{\mu\nu -}_{(x)} + F^{\mu\nu +}_{(x)})$$

which, to be controlled, one needs to let the "bare mass" of the particle to diverge. Second, the solutions of the differential equation derived by Dirac are not of physical nature, where for example, a particle radiating would approach the speed of light exponentially fast, phenomena which is known as the "runaway solutions". In Appendix G we provide some examples of the solutions obtained by this approach.

It is important to mention that there were some works done in this topic after Dirac's paper was published. For example, in 1945, R. Feynman and J. Wheeler published a paper in which they introduced the "absorber theory" [WF45]. In this theory, it is not the accelerated motion of a particle that causes radiation, but rather the interaction with other particles, referred to as the absorber medium. There is no self-force and no divergences, as the force acting on a charged particle always stems from other particles. The motivation for this paper was Wheeler and Feynman's criticism ([WF45, p. 159]) of the works of Lorentz and Dirac: Lorentz "provided an incomplete expression of the self-force," while Dirac "offered no explanation for the origin of radiative damping." The two authors derived an expression for the radiation reaction which is in accordance with Dirac's result (1.10).

Since equation (1.10) appears quite often in the literature of radiation reaction and its mathematical status is not clear, we provide a theorem studying under which conditions equation (1.10) holds and give the convergence rate in chapter Four, Theorem 4.2. We will make use of the Dirac's ideas which simplify the calculations.

# Chapter 2

# Mathematical Structure of Minkowski Space-Time

In contrast to chapter one, in this second chapter we aim to construct and define the basic objects needed for the rest of the work in a mathematical rigorous way. We want to show the existence of the advanced and retarded times and study some of their basic properties. Explicitly, we will show that the advanced and retarded times are continuous everywhere in the Minkowski space-time.

**Summary of this Chapter**

In this chapter, we present a collection of essential results required for the convergence of the radiation field. Specifically, we demonstrate

1. the existence and uniqueness of the retarded and advanced positions (Lemma 2.9).
2. the continuity of the retarded and advanced times (Lemma 2.12) and their differentiability outside the world-line (Lemma 2.13).

## 2.1 Definition of Minkowski Space-Time

Here we present a rigorous definition of all elements needed in classical electrodynamics. We establish a solid mathematical foundation and demonstrate that a sequence of points in Minkowski spacetime, which converges to the world-line of a particle, gives rise to two distinct sequences of points along the world-line, namely the advanced and retarded positions.

**Definition 2.1. (Pseudo-scalar Product)** Given two vectors $x, y \in \mathbb{R}^4$, which in Cartesian coordinates are denoted by $x = (x^0, x^1, x^2, x^3)^T$ and $y = (y^0, y^1, y^2, y^3)^T$, we call the map

$$\begin{cases} \mathbb{R}^4 \times \mathbb{R}^4 \to \mathbb{R} \\ (x, y) \mapsto x \cdot y := x^0 y^0 - x^1 y^1 - x^2 y^2 - x^3 y^3 \end{cases}$$

the pseudo-scalar product of $x$ and $y$. For the special case $x \cdot x$ we also may write $x^2$.

**Definition 2.2. (Metric Tensor)** Given two vectors $x, y \in \mathbb{R}^4$, which in Cartesian coordinates are denoted by $x = (x^0, x^1, x^2, x^3)^T$ and $y = (y^0, y^1, y^2, y^3)^T$, we call the matrix $\overleftrightarrow{\eta}$ whose components satisfy the equation

$$x \cdot y = x^T \overleftrightarrow{\eta} y$$

the metric tensor. Its components can be read off from the defining equation of the pseudo-scalar product, i.e.

$$\overleftrightarrow{\eta} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

**Definition 2.3. (Einstein's Summation Convention)** When working in the framework of special relativity we may also denote the vectors $x \in \mathbb{R}^4$ by their components $x^\mu$, where $\mu$ is an index going from zero to three. Then, it is possible to write the pseudo-scalar product of two vectors as

$$x \cdot y = x^\mu y_\mu = x^\mu \eta_{\mu\nu} y^\nu, \tag{2.1}$$

where the repeated indices above and below are summed over. In this context, $(\eta_{\mu\nu})_{(\mu,\nu)\in\{0,\ldots,3\}^2}$ denote the components of metric tensor. We also note the implicit definition in the way of writing, namely $y_\mu = \eta_{\mu\nu} y^\nu$. We may say that the metric tensor can be used to "lower indices".

**Definition 2.4. (Minkowski-Spacetime)** We define the set of vectors in $\mathbb{R}^4$ doted with the pseudo-scalar product $x \cdot y$ for all $x$,$y \in \mathbb{R}^4$ as the Minkowski-Spacetime $\mathbb{M}$.

*Remark.* The name of $n_{\mu\nu}$ as a tensor is well supported by its behavior under coordinate transformations. That is, when choosing two different coordinate systems $x^\mu$ and $x^{\mu'}$, we find the transformation relation

$$\eta_{\mu'\nu'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\nu}{\partial x^{\nu'}} \eta_{\mu\nu},$$

whenever these coordinates are either rotations or Lorentz-boosts in some direction.

The last set of definitions give us the vocabulary needed to talk about moving particles and fields in space-time. As always in the case of special relativity, we embed the zeroth component of a vector $x$ with the meaning of the time component. We will also refer to the vectors in $\mathbb{M}$ as *four-vectors*.

Additionally, we need a meaning for convergence and we will use this word in the normal mathematical sense using the euclidean norm rather than the Minkowski-metric. Whenever we write $\|x\|_{\mathbb{R}^n}$ for $x \in \mathbb{R}^n$, we actually mean

$$\|x\|_{\mathbb{R}^n} = \left[\sum_{i=1}^n x_i^2\right]^{1/2}$$

and as ee will always refer explicitly to the dimension in which the norm has to be taken, there should be no confusion between the pseudo-scalar product of $\mathbb{M}$ and the conventional scalar product of $\mathbb{R}$.

**Definition 2.5. (Converge of Sequences in $\mathbb{M}$)** A sequence of four-vectors $(x_n)_{n\in\mathbb{N}}$ with $x_n \in \mathbb{M}$ for all $n \in \mathbb{N}$ is called convergent with limiting vector $q \in \mathbb{M}$ if it holds true that

$$\forall \epsilon > 0 \quad \exists N \in \mathbb{N} \quad \forall n \geq N \quad : \quad x_n \in \mathcal{B}_\epsilon(q) := \{y \in \mathbb{R}^4 : \|y - q\|_{\mathbb{R}^4} < \epsilon\}.$$

## 2.2 Physical Objects in Space-Time

We now aim to define precisely what is meant by a particle moving through spacetime and introduce the concept of the light-cone.

**Definition 2.6. (World-Line of a Particle)** We call the maps

$$q : \begin{cases} \mathbb{R} \longrightarrow \mathbb{M} \\ \tau \longmapsto q_{(\tau)} := \begin{pmatrix} \tilde{t}_{(\tau)} \\ \vec{q}_{(\tilde{t}_{(\tau)})} \end{pmatrix} \end{cases}, \tag{2.2}$$

world-lines, with $\tilde{t}_{(\tau)}$ the time coordinate at world-line parameter $\tau \in \mathbb{R}$, $\vec{q}_{(\tilde{t}_{(\tau)})}$ the position at time $\tilde{t}_{(\tau)} \in \mathbb{R}$ and $\vec{v}_{(\tilde{t})} := \dfrac{\mathrm{d}\vec{q}}{\mathrm{d}\tilde{t}}$ the velocity of the particle, if the following properties are satisfied:

1. $q \in \mathcal{C}^\infty_{(\mathbb{R},\mathbb{M})}$
2. $\tilde{t} : \mathbb{R} \to \mathbb{R}$ is bijective
3. The four-velocity is time-like and positive oriented, i.e[1]

$$\forall \tau \in \mathbb{R} : \dot{\tilde{t}}_{(\tau)} > \left\| \dot{\vec{q}}_{(\tilde{t}_{(\tau)})} \right\|_{\mathbb{R}^3} \geq 0. \tag{2.3}$$

[1]Here the dot is the derivative with respect to $\tau$.

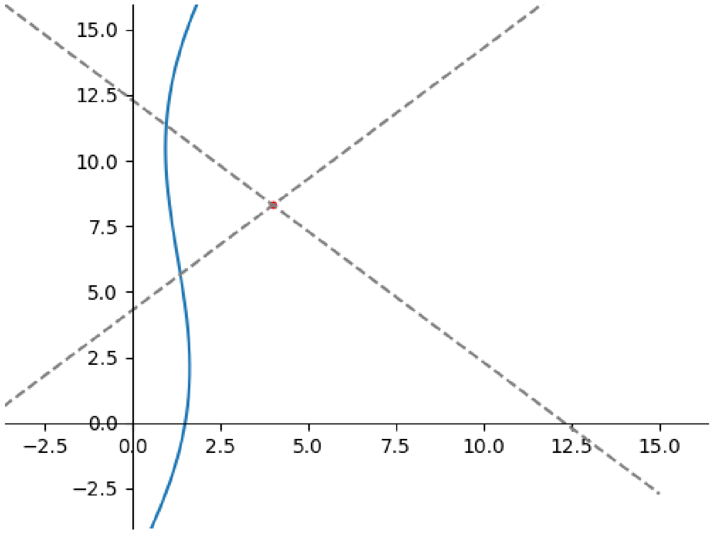


Figure 2.1: Illustration of a light cone centered at a general point x.

4. There exist a maximal velocity smaller than the speed of light, i.e

$$\forall \tilde{t} \in \mathbb{R} : \left\| \overrightarrow{v}_{(\tilde{t})} \right\|_{\mathbb{R}^3} = \left\| \frac{\mathrm{d}\overrightarrow{q}_{(\tilde{t})}}{\mathrm{d}\tilde{t}} \right\|_{\mathbb{R}^3} \leq v_{max} < 1. \tag{2.4}$$

for some $v_{max} \in [0, 1)$.

Sometimes we also refer to the set $q(\mathbb{R})$ as the world-line of the particle. From the context should be clear when we talk about the set and when we talk about the map as a function of the parameter $\tau$.

*Remark.* Normally one would use $\tau$ as the proper time of the particle. But as this definition is of general character, we leave open the choice of the meaning of the parameter $\tau$.

**Definition 2.7. (Light Cone)** For any point $x \in \mathbb{M}$, we call the set

$$L_x := \{y \in \mathbb{M} : (y - x)^2 = 0\} \tag{2.5}$$

the light cone centered at $x$.

From a physical point of view, this set contains all points that can "communicate" with $x$ using a beam of light. This definition allows us to introduce the important concept of advanced and retarded positions.

**Definition 2.8. (Advanced/Retarded Positions)** Given a world-line $q(\mathbb{R})$ as in Definition 2.6 and a point $x \in \mathbb{M}$ we construct the set

$$L_x \cap q(\mathbb{R}) = \{y \in \mathbb{M} : (y - x)^2 = 0 \quad \wedge \quad \exists \tau \in \mathbb{R} : q_{(\tau)} = y\}. \tag{2.6}$$

Its elements are called the **retarded positions** with respect to $x$ if for $y = (t_y, \overrightarrow{y})$ and for $x = (t_x, \overrightarrow{x})$ it holds that $t_y < t_x$. We call the elements of this set the **advanced positions** with respect to $x$ if, on the contrary, it holds that $t_y > t_x$.

Using a physical perspective, this positions denote the points of the world-line that can interact with the point $x$ by radiating or absorbing light (see figure 2.1.) It might seems clear that there is only one retarded and one advanced position, but instead of leaving a vague argument, we will proof the correctness of this statement.

**Lemma 2.9. (Existence and Uniqueness of the Retarded and Advanced Positions)** Let $q$ be a world-line as given in Definition 2.6 and let $x$ be any point in $\mathbb{M} \setminus q(\mathbb{R})$. Then, the set $L_x \cap q(\mathbb{R})$ has exactly two elements, one retarded and one advanced position with respect to $x$.

*Proof.* 1. Existence:

Let $x \in \mathbb{M} \setminus q(\mathbb{R})$ have the components $(t, \overrightarrow{x})$ and since $\tilde{t}$ is bijective, there exists one $\tau_x$ such that $q_{(\tau_x)} = (t, \overrightarrow{q}_{(t)})$. We define the set

$$C_v(q_{(\tau_x)}) := \{y \in \mathbb{M} \quad \Big| \quad \forall s \in \mathbb{R}, v \in [-v_{max}, v_{max}], \overrightarrow{e} \in \mathbb{R}^3 \text{ with } \|\overrightarrow{e}\|_{\mathbb{R}^3} = 1 : y = (t+s, \overrightarrow{q}_{(t)} + sv\overrightarrow{e})^T\}$$

with the following properties

(a) $q(\mathbb{R}) \subset C_v(q_{(\tau_x)})$.
We study the non trivial case $s \neq 0$ since for $s = 0$ we get $q_{(\tau_x)} \in C_v(q_{(\tau_x)})$ by definition. Let $\tau \in \mathbb{R}$ be a real number and $\overrightarrow{e}$ be any unit vector in $\mathbb{R}^3$, then we look for solutions $(s, v')$ of the following set of equations

$$q_{(\tau)} = \begin{pmatrix} \tilde{t}_{(\tau)} \\ \overrightarrow{q}_{(\tilde{t}_{(\tau)})} \end{pmatrix} = \begin{pmatrix} t+s \\ \overrightarrow{q}_{(t)} + sv'\overrightarrow{e} \end{pmatrix}.$$

The first equation has a unique solution $s = \tilde{t}_{(\tau)} - t \neq 0$ for all $\tau$ and $t$ For the second equation we have

$$v' = \left( \frac{\overrightarrow{q}_{(\tilde{t}_{(\tau)})} - \overrightarrow{q}_{(t)}}{\tilde{t}_{(\tau)} - t} \right) \cdot \overrightarrow{e},$$

for all $\tau$ and $t$. This expression is also well defined since the denominator is not equal to zero. From the mean value theorem we see, that there exist at least one time $w \in (min\{\tilde{t}_{(\tau)}, t\}, max\{\tilde{t}_{(\tau)}, t\})$ such that

$$v' = \overrightarrow{v}_{(w)} \cdot \overrightarrow{e}$$

(because $q$ is differentiable) and therefore

$$|v'| \leq \|\overrightarrow{v}_{(w)}\|_{\mathbb{R}^3} \leq v_{max}.$$

So for all $\tau \in \mathbb{R}$ it is possible to choose the pair $(s, v')$ such that for all unit vectors $\overrightarrow{e} \in \mathbb{R}^3$ holds $q_{(\tau)} \in C_v(q_{(\tau_x)})$.

(b) $C_v(q_{(\tau_x)}) \cap L_x \neq \emptyset$.
Here we follow a similar strategy. Let $y = (t+s, \overrightarrow{q}_{(t)} + sv\overrightarrow{e})^T$ be a four-vector in $C_v(q_{(\tau_x)})$. Then

$$y - x = \begin{pmatrix} s \\ \overrightarrow{q}_{(t)} - \overrightarrow{x} + sv\overrightarrow{e} \end{pmatrix},$$

and when we look for solutions of the equation $(y-x)^2 = 0$ we obtain

$$\begin{aligned} (y-x)^2 &= s^2 - (\overrightarrow{q}_{(t)} - \overrightarrow{x} + sv\overrightarrow{e})^2 \\ &= (1-v^2)s^2 - 2v\overrightarrow{e} \cdot (\overrightarrow{q}_{(t)} - \overrightarrow{x})s - (\overrightarrow{q}_{(t)} - \overrightarrow{x})^2 = 0 \end{aligned}$$

$$\Rightarrow s = \frac{2v\overrightarrow{e} \cdot (\overrightarrow{q}_{(t)} - \overrightarrow{x}) \pm \sqrt{v^2[\overrightarrow{e} \cdot (\overrightarrow{q}_{(t)} - \overrightarrow{x})]^2 + (1-v^2)(\overrightarrow{q}_{(t)} - \overrightarrow{x})^2}}{1-v^2},$$

which is always defined for all $x \in \mathbb{M}$. That is, for a given four-vector $x$, we can freely choose $\overrightarrow{e}$ and $|v| \leq v_{max}$ such that $s \in \mathbb{R}$ can always be calculated from the last equation and we obtain $y \in C_v(q_{(\tau_x)}) \cap L_x$.

(c) $q(\mathbb{R}) \setminus (C_v(q_{(\tau_x)}) \cap L_x)$ is split into three disjoint sets.
Let $y \in C_v(q_{(\tau_x)}) \setminus (C_v(q_{(\tau_x)}) \cap L_x)$ and let $(s, v)$ be chosen such that $y = q_{(\tau)}$ for some $\tau \in \mathbb{R}$. Then we know that $(q_{(\tau)} - x)^2 \neq 0$ and therefore

$$|\tilde{t}_{(\tau)} - t| \neq \pm\|\overrightarrow{q}_{(\tilde{t}_{(\tau)})} - \overrightarrow{x}\|_{\mathbb{R}^3}.$$

Here we have four mutually excluding possibilities:
1) $\tilde{t}_{(\tau)} > t$ with $\tilde{t}_{(\tau)} > t + \|\overrightarrow{q}_{(\tilde{t}_{(\tau)})} - \overrightarrow{x}\|_{\mathbb{R}^3}$, 2) $\tilde{t}_{(\tau)} > t$ with $\tilde{t}_{(\tau)} < t + \|\overrightarrow{q}_{(\tilde{t}_{(\tau)})} - \overrightarrow{x}\|_{\mathbb{R}^3}$, 3) $\tilde{t}_{(\tau)} < t$ with $\tilde{t}_{(\tau)} > t - \|\overrightarrow{q}_{(\tilde{t}_{(\tau)})} - \overrightarrow{x}\|_{\mathbb{R}^3}$, and 4) $\tilde{t}_{(\tau)} < t$ with $\tilde{t}_{(\tau)} < t - \|\overrightarrow{q}_{(\tilde{t}_{(\tau)})} - \overrightarrow{x}\|_{\mathbb{R}^3}$.

And so we can build the following disjoint sets

$$q(\mathbb{R})^{up} = \{q_{(\tau)} \in q(\mathbb{R}) : \tilde{t}_{(\tau)} > t + \|\overrightarrow{q}_{(\tilde{t}_{(\tau)})} - \overrightarrow{x}\|_{\mathbb{R}^3}\},$$
$$q(\mathbb{R})^{middle} = \{q_{(\tau)} \in q(\mathbb{R}) : t + \|\overrightarrow{q}_{(\tilde{t}_{(\tau)})} - \overrightarrow{x}\|_{\mathbb{R}^3} > \tilde{t}_{(\tau)} > t - \|\overrightarrow{q}_{(\tilde{t}_{(\tau)})} - \overrightarrow{x}\|_{\mathbb{R}^3}\},$$
$$q(\mathbb{R})^{down} = \{q_{(\tau)} \in q(\mathbb{R}) : \tilde{t}_{(\tau)} < t - \|\overrightarrow{q}_{(\tilde{t}_{(\tau)})} - \overrightarrow{x}\|_{\mathbb{R}^3}\},$$

which are not empty since $\tilde{t}$ is bijective.

(d) From the continuity of $q$ and the intermediate value theorem it follows that $q(\mathbb{R})$ must intersect $C_v(q_{(\tau_x)}) \cap L_x$. Just by choosing two parameters $\tau_1$ such that $q_{(\tau_1)} \in q(\mathbb{R})^{down}$ and $\tau_2$ such that $q_{(\tau_2)} \in q(\mathbb{R})^{up}$, we notice that $q^\mu_{(\tau)}$ has to take all values in the interval $[q^\mu_{(\tau_1)}, q^\mu_{(\tau_2)}]$. In other words, there exist a $\tau \in (\tau_1, \tau_2)$ such that $q_{(\tau)} \in C_v(q_{(\tau_x)}) \cap L_x$.

$$\Rightarrow q(\mathbb{R}) \cap (C_v(q_{(\tau_x)}) \cap L_x) = q(\mathbb{R}) \cap L_x \neq \emptyset. \tag{2.7}$$

2. **Uniqueness**

In order to show that $q(\mathbb{R}) \cap L_x = \{q_{(\tau^-)}, q_{(\tau^+)}\}$ for two unique proper times $\tau^-, \tau^+$ with the property $\tilde{t}_{(\tau^-)} \leq t \leq \tilde{t}_{(\tau^+)}$, we use a proof by contradiction. First we show the uniqueness of $\tau^-$.

Suppose that there exist two different proper times $\tau^-$ and $\lambda^-$ and without loss of generality we can assume $\tau^- < \lambda^-$ such that $\tilde{t}_{(\tau^-)} < \tilde{t}_{(\lambda^-)} < t$. Then, because $q_{(\tau)}$ is a differentiable function in $\mathbb{R}$, we can use the mean value theorem and state that there exists another proper time $\omega^- \in (\tau^-, \lambda^-)$ such that

$$\dot{q}_{(\omega^-)} = \frac{q_{(\lambda^-)} - q_{(\tau^-)}}{\lambda^- - \tau^-} = \frac{1}{\lambda^- - \tau^-} \begin{pmatrix} \tilde{t}_{(\lambda^-)} - \tilde{t}_{(\tau^-)} \\ \overrightarrow{q}_{(\tilde{t}_{(\lambda^-)})} - \overrightarrow{q}_{(\tilde{t}_{(\tau^-)})} \end{pmatrix}.$$

Using the third property of the world-line in Definition 2.6, we see that a time like four-velocity must have the time component greater than the spatial component, i.e.

$$\left|\tilde{t}_{(\lambda^-)} - \tilde{t}_{(\tau^-)}\right| > \left\|\overrightarrow{q}_{(\tilde{t}_{(\lambda^-)})} - \overrightarrow{q}_{(\tilde{t}_{(\tau^-)})}\right\|_{\mathbb{R}^3}.$$

If we use the fact that $\tilde{t}_{(\lambda^-)} > \tilde{t}_{(\tau^-)}$ and the condition of the retarded time $\tilde{t} = t - \|\overrightarrow{q}_{(\tilde{t})} - \overrightarrow{x}\|_{\mathbb{R}^3}$ we can rewrite

$$\tilde{t}_{(\lambda^-)} - t + t - \tilde{t}_{(\tau^-)} = \left\|\overrightarrow{x} - \overrightarrow{q}_{(\tilde{t}_{(\tau^-)})}\right\|_{\mathbb{R}^3} - \left\|\overrightarrow{x} - \overrightarrow{q}_{(\tilde{t}_{(\lambda^-)})}\right\|_{\mathbb{R}^3}$$

and get

$$\left\|\overrightarrow{x} - \overrightarrow{q}_{(\tilde{t}_{(\tau^-)})}\right\|_{\mathbb{R}^3} - \left\|\overrightarrow{x} - \overrightarrow{q}_{(\tilde{t}_{(\lambda^-)})}\right\|_{\mathbb{R}^3} > \left\|\overrightarrow{q}_{(\tilde{t}_{(\lambda^-)})} - \overrightarrow{q}_{(\tilde{t}_{(\tau^-)})}\right\|_{\mathbb{R}^3},$$

which leads to the contradiction

$$\left\|\overrightarrow{x} - \overrightarrow{q}_{(\tilde{t}_{(\tau^-)})}\right\|_{\mathbb{R}^3} > \left\|\overrightarrow{x} - \overrightarrow{q}_{(\tilde{t}_{(\tau^-)})}\right\|_{\mathbb{R}^3}.$$

This means that our assumption was incorrect, therefore there can not exist two different parameters $\tau^-$ and $\lambda^-$ under which $q_{(\tau^-)}$ and $q_{(\lambda^-)}$ are both retarded positions with respect to the four-vector $x$.

The proof of the uniqueness of the advanced position follows the same steps as we showed, just using the condition $\tilde{t} = t + \|\overrightarrow{q}_{(\tilde{t})} - \overrightarrow{x}\|_{\mathbb{R}^3}$.

□

**Corollary 2.10.** For the situation given in Lemma 2.9, if $x \in q(\mathbb{R})$, then the set $L_x \cap q(\mathbb{R})$ has only one element.

*Proof.* Let $x \in q(\mathbb{R})$. So there exists a $\tau^* \in \mathbb{R}$ such that $x = q_{(\tau^*)}$. Then we look for elements of the set

$$L_{q_{(\tau^*)}} \cap q(\mathbb{R}) = \{q_{(\tau)} \in q(\mathbb{R}) : (q_{(\tau)} - q_{(\tau^*)})^2 = 0\}.$$

Clearly $q_{(\tau)} = q_{(\tau^*)}$ solves the equation, so we have $q_{(\tau^*)} \in L_{q_{(\tau^*)}} \cap q(\mathbb{R})$. Now suppose that there exists other $\tau^{**}$ with $q_{(\tau^{**})} \in L_{q_{(\tau^*)}} \cap q(\mathbb{R})$. We then have

$$(q_{(\tau^{**})} - q_{(\tau^*)})^2 = 0$$

$$\Rightarrow \left(\frac{q_{(\tau^{**})} - q_{(\tau^{*})}}{\tau^{**} - \tau^{*}}\right)^2 = 0.$$

Now, because of the mean value theorem, there exist a $\omega \in (min\{\tau^*, \tau^{**}\}, max\{\tau^*, \tau^{**}\})$ such that

$$\dot{q}_{(\omega)} = \frac{q_{(\tau^{**})} - q_{(\tau^{*})}}{\tau^{**} - \tau^{*}}$$

$$\Rightarrow \dot{q}^2_{(\omega)} = 0$$

which is a contradiction to the definition of a world-line 2.6 in which the four-velocity must be time-like. Therefore there can not exist another parameter $\tau^{**}$ such that the four-position belongs to $L_x \cap q(\mathbb{R})$ and it follows

$$L_{q_{(\tau^*)}} \cap q(\mathbb{R}) = \{q_{(\tau^*)}\}.$$

□

The existence and uniqueness of the retarded and advanced positions allows us to examine the time components in more detail. As the following definition shows, we interpret these components also as a function of a point in space-time.

**Definition 2.11. (Retarded and Advanced Times)** The zeroth component of the retarded/advanced positions are called the retarded/advanced times.

*Remark.* Because of the Light-cone equation, we can also study the retarded and advanced times as a function of the position, namely

$$t^{\pm} : \begin{cases} \mathbb{M} \to \mathbb{R} \\ x \mapsto t^{\pm}_{(x)} := t \pm \|\vec{x} - \vec{q}_{(t^{\pm}_{(x)})}\|_{\mathbb{R}^3} \end{cases} \tag{2.8}$$

One can notice the difficulty of this expression. The retarded and advanced times are implicitly defined where the expression of the position can be very complicated. We will calculate explicit formulas for $t^{\pm}$ in the next chapter when we study a particle that moves with constant velocity.

**Lemma 2.12. (Continuity of the Advanced and Retarded Times)** Let $(x_n)_{n\in\mathbb{N}}$ be a sequence with $x_n \in \mathbb{M} \setminus q(\mathbb{R})$ for all $n \in \mathbb{N}$ and $q$ be a world-line as in Definition 2.6. Let this sequence converge to a point $y \in \mathbb{M}$. Then, the retarded and advanced times are continuous, i.e.

$$\forall y \in \mathbb{M} : \quad \lim_{n\to\infty} t^{\pm}_{(x_n)} = t^{\pm}_{(\lim_{n\to\infty} x_n)} = t^{\pm}_{(y)}. \tag{2.9}$$

*Proof.* We divide the proof in two steps:

1. $(t^{\pm}_{(x_n)})_{n\in\mathbb{N}}$ is a Cauchy sequence:

   Let $m, n \in \mathbb{N}$ and let $x_n$, $x_m$ be two elements of the sequence $(x_n)_{n\in\mathbb{N}}$ with components $(t_n, \vec{x}_n)$ and $(t_m, \vec{x}_m)$. Let $t^{\pm}_{(x_n)}$ be given as in Definition 2.11. Then

$$\left|t^{\pm}_{(x_n)} - t^{\pm}_{(x_m)}\right| = \left|t_n - t_m \pm \left(\|\vec{x}_n - \vec{q}_{(t^{\pm}_{(x_n)})}\|_{\mathbb{R}^3} - \|\vec{x}_m - \vec{q}_{(t^{\pm}_{(x_m)})}\|_{\mathbb{R}^3}\right)\right|.$$

   Using the triangle inequality we obtain

$$\left|t^{\pm}_{(x_n)} - t^{\pm}_{(x_m)}\right| \leq |t_n - t_m| + \left|\|\vec{x}_n - \vec{q}_{(t^{\pm}_{(x_n)})}\|_{\mathbb{R}^3} - \|\vec{x}_m - \vec{q}_{(t^{\pm}_{(x_m)})}\|_{\mathbb{R}^3}\right|$$

   and if we notice that $\big|\|a\| - \|b\|\big| \leq \|a - b\|$ for all vectors [2] in $\mathbb{R}^3$, we can rewrite the inequality as

$$\left|t^{\pm}_{(x_n)} - t^{\pm}_{(x_m)}\right| \leq |t_n - t_m| + \|\vec{x}_n - \vec{x}_m - \vec{q}_{(t^{\pm}_{(x_n)})} + \vec{q}_{(t^{\pm}_{(x_m)})}\|_{\mathbb{R}^3},$$

   and the triangle inequality yields again

$$\left|t^{\pm}_{(x_n)} - t^{\pm}_{(x_m)}\right| \leq |t_n - t_m| + \|\vec{x}_n - \vec{x}_m\|_{\mathbb{R}^3} + \|\vec{q}_{(t^{\pm}_{(x_n)})} - \vec{q}_{(t^{\pm}_{(x_m)})}\|_{\mathbb{R}^3}.$$

[2] This is valid since we have the following case distinction

(a) $\|a\| \geq \|b\|$: In this case we have $\big|\|a\| - \|b\|\big| = \|a\| - \|b\| \leq \|a - b\|$.

(b) $\|a\| < \|b\|$: In this case we have $\big|\|a\| - \|b\|\big| = \|b\| - \|a\| \leq \|b - a\| = \|a - b\|$.

Now we use the fact that the trajectory is per definition differentiable with respect to $t^{\pm}$. We apply the mean value theorem, rewrite the last term as

$$\|\overrightarrow{q}_{(t^{\pm}_{(x_n)})} - \overrightarrow{q}_{(t^{\pm}_{(x_m)})}\|_{\mathbb{R}^3} = \frac{\left|t^{\pm}_{(x_n)} - t^{\pm}_{(x_m)}\right|}{\left|t^{\pm}_{(x_n)} - t^{\pm}_{(x_m)}\right|}\|\overrightarrow{q}_{(t^{\pm}_{(x_n)})} - \overrightarrow{q}_{(t^{\pm}_{(x_m)})}\|_{\mathbb{R}^3}$$

$$= \left|t^{\pm}_{(x_n)} - t^{\pm}_{(x_m)}\right| \left\|\frac{\overrightarrow{q}_{(t^{\pm}_{(x_n)})} - \overrightarrow{q}_{(t^{\pm}_{(x_m)})}}{t^{\pm}_{(x_n)} - t^{\pm}_{(x_m)}}\right\|_{\mathbb{R}^3} = \left|t^{\pm}_{(x_n)} - t^{\pm}_{(x_m)}\right| \|\overrightarrow{v}_{(t')}\|_{\mathbb{R}^3}$$

for some $t^{'}$ between $t^{\pm}_{(x_n)}$ and $t^{\pm}_{(x_m)}$ and because the velocity is assumed to have an overall maximum, we obtain

$$\|\overrightarrow{q}_{(t^{\pm}_{(x_n)})} - \overrightarrow{q}_{(t^{\pm}_{(x_m)})}\|_{\mathbb{R}^3} \leq v_{max}\left|t^{\pm}_{(x_n)} - t^{\pm}_{(x_m)}\right|. \tag{2.10}$$

Inserting this result in equation (1), we get

$$\left|t^{\pm}_{(x_n)} - t^{\pm}_{(x_m)}\right| \leq \frac{|t_n - t_m| + \|\overrightarrow{x}_n - \overrightarrow{x}_m\|_{\mathbb{R}^3}}{1 - v_{max}} \tag{2.11}$$

and since the sequence $(x_n)_{n\in\mathbb{N}}$ converges, it is always possible to find a $N \in \mathbb{N}$ for all $\epsilon > 0$ such that $|t_n - t_m| < \epsilon$ and $\|\overrightarrow{x}_n - \overrightarrow{x}_m\|_{\mathbb{R}^3} < \epsilon$ for all $n, m > N$. Therefore,

$$\left|t^{\pm}_{(x_n)} - t^{\pm}_{(x_m)}\right| \leq \frac{2}{1 - v_{max}}\epsilon =: \tilde{\epsilon}.$$

In summary,

$$\forall\tilde{\epsilon} > 0 \quad \exists N \in \mathbb{N}: \quad \left|t^{\pm}_{(x_n)} - t^{\pm}_{(x_m)}\right| \leq \tilde{\epsilon} \quad \forall m, n \geq N$$

as a Cauchy sequence $(t^{\pm}_{(x_n)})_{n\in\mathbb{N}}$ always converges to a real limit in $\mathbb{R}$.

2. Now we know that the limit

$$\lim_{n\to\infty} t^{\pm}_{(x_n)}$$

exists and we want to determine its value. For $x_n \to y$ as $n \to \infty$ with $y = (t_y, \overrightarrow{y})$ we calculate the following difference

$$\left|\lim_{n\to\infty} t^{\pm}_{(x_n)} - t^{\pm}_{(\lim_{n\to\infty} x_n)}\right| = \left|\lim_{n\to\infty} t^{\pm}_{(x_n)} - t^{\pm}_{(y)}\right|$$

$$= \left|\lim_{n\to\infty} t_n \pm \lim_{n\to\infty} \|\overrightarrow{x}_n - \overrightarrow{q}_{(t^{\pm}_{(x_n)})}\|_{\mathbb{R}^3} - t_y \mp \|\overrightarrow{y} - \overrightarrow{q}_{(t^{\pm}_{(y)})}\|_{\mathbb{R}^3}\right|,$$

and, when using the continuity of the norm and the trajectory, we obtain

$$\left|\lim_{n\to\infty} t^{\pm}_{(x_n)} - t^{\pm}_{(\lim_{n\to\infty} x_n)}\right| = \left|\|\overrightarrow{y} - \overrightarrow{q}_{(\lim_{n\to\infty} t^{\pm}_{(x_n)})}\|_{\mathbb{R}^3} - \|\overrightarrow{y} - \overrightarrow{q}_{(t^{\pm}_{(y)})}\|_{\mathbb{R}^3}\right|$$

$$\leq \|\overrightarrow{q}_{(\lim_{n\to\infty} t^{\pm}_{(x_n)})} - \overrightarrow{q}_{(t^{\pm}_{(y)})}\|_{\mathbb{R}^3} \leq v_{max}\left|\lim_{n\to\infty} t^{\pm}_{(x_n)} - t^{\pm}_{(\lim_{n\to\infty} x_n)}\right|,$$

where we have implemented again the equation (2.10) and the inequality $\big|\|a\| - \|b\|\big| \leq \|a - b\|$ for vectors in $\mathbb{R}^3$. We can then conclude, since $v_{max} < 1$, that

$$\left|\lim_{n\to\infty} t^{\pm}_{(x_n)} - t^{\pm}_{(\lim_{n\to\infty} x_n)}\right| \leq 0,$$

which, in other words, implies that the retarded and advanced times are continuous.

□

Unfortunately, in our theory, these times are not differentiable at the world-line of a particle. However, the following lemma demonstrates that if we remain outside the world-line, we can compute the partial derivatives of $t^{\pm}$.

**Lemma 2.13. (Differentiability of $t^{\pm}$)** Let $t^{\pm}$ be the retarded and advanced times as given in Definition 2.11 for all four-vectors $x \in \mathbb{M}$ with components $(t, \overrightarrow{x})$. Then, if the point $x$ is chosen outside of the world-line, the retarded and advanced times are differentiable. In fact, their partial derivatives are given by

$$\frac{\partial t^{\pm}_{(x)}}{\partial t} = \frac{1}{1 \pm \overrightarrow{v}_{(t^{\pm}_{(x)})} \cdot \overrightarrow{n}_{(t^{\pm}_{(x)}, \overrightarrow{x})}} \tag{2.12}$$

and

$$\frac{\partial t^{\pm}_{(x)}}{\partial x^i} = \frac{\pm n^i_{(t^{\pm}_{(x)}, \overrightarrow{x})}}{1 \pm \overrightarrow{v}_{(t^{\pm}_{(x)})} \cdot \overrightarrow{n}_{(t^{\pm}_{(x)}, \overrightarrow{x})}}, \tag{2.13}$$

where $\overrightarrow{n}$ denotes the normal vector

$$\overrightarrow{n}_{(t^{\pm}_{(x)}, \overrightarrow{x})} = \frac{\overrightarrow{x} - \overrightarrow{q}_{(t^{\pm}_{(x)})}}{\|\overrightarrow{x} - \overrightarrow{q}_{(t^{\pm}_{(x)})}\|_{\mathbb{R}^3}}. \tag{2.14}$$

*Proof.* 1. First we compute $\partial t^{\pm}/\partial t$ by definition, i.e.

$$\frac{\partial t^{\pm}_{(x)}}{\partial t} = \lim_{\epsilon \to 0} \frac{t^{\pm}_{(t+\epsilon, \overrightarrow{x})} - t^{\pm}_{(t, \overrightarrow{x})}}{\epsilon}.$$

The numerator yields

$$\begin{aligned} t^{\pm}_{(t+\epsilon, \overrightarrow{x})} - t^{\pm}_{(t, \overrightarrow{x})} &= t + \epsilon \pm \|\overrightarrow{x} - \overrightarrow{q}_{(t^{\pm}_{(t+\epsilon, \overrightarrow{x})})}\|_{\mathbb{R}^3} - t \mp \|\overrightarrow{x} - \overrightarrow{q}_{(t^{\pm}_{(t, \overrightarrow{x})})}\|_{\mathbb{R}^3} \\ &= \epsilon \pm \left( \|\overrightarrow{x} - \overrightarrow{q}_{(t^{\pm}_{(t+\epsilon, \overrightarrow{x})})}\|_{\mathbb{R}^3} - \|\overrightarrow{x} - \overrightarrow{q}_{(t^{\pm}_{(t, \overrightarrow{x})})}\|_{\mathbb{R}^3} \right). \end{aligned}$$

Now, because we assume $\overrightarrow{x} \neq \overrightarrow{q}_{(t^{\pm}_{(t, \overrightarrow{x})})}$, we can interpret the norm as a function of $t^{\pm}$ and expand it around $t^{\pm}_{(t, \overrightarrow{x})}$. This results in

$$\begin{aligned} \|\overrightarrow{x} - \overrightarrow{q}_{(t^{\pm}_{(t+\epsilon, \overrightarrow{x})})}\|_{\mathbb{R}^3} =& \|\overrightarrow{x} - \overrightarrow{q}_{(t^{\pm}_{(t, \overrightarrow{x})})}\|_{\mathbb{R}^3} + \\ & \left( t^{\pm}_{(t+\epsilon, \overrightarrow{x})} - t^{\pm}_{(t, \overrightarrow{x})} \right) \left( \frac{d}{ds} \|\overrightarrow{x} - \overrightarrow{q}_{(s)}\|_{\mathbb{R}^3} \right) \Big|_{s = t^{\pm}_{(t, \overrightarrow{x})}} + R, \end{aligned}$$

where $R$ is the remainder term of the Taylor expansion[3], and it is of second order. Performing the derivative we obtain

$$\begin{aligned} \|\overrightarrow{x} - \overrightarrow{q}_{(t^{\pm}_{(t+\epsilon, \overrightarrow{x})})}\|_{\mathbb{R}^3} =& \|\overrightarrow{x} - \overrightarrow{q}_{(t^{\pm}_{(t, \overrightarrow{x})})}\|_{\mathbb{R}^3} - \\ & \left( t^{\pm}_{(t+\epsilon, \overrightarrow{x})} - t^{\pm}_{(t, \overrightarrow{x})} \right) \overrightarrow{v}_{(t^{\pm}_{(t, \overrightarrow{x})})} \cdot \overrightarrow{n}_{(t^{\pm}_{(t, \overrightarrow{x})}, \overrightarrow{x})} + R. \end{aligned}$$

From here it follows

$$t^{\pm}_{(t+\epsilon, \overrightarrow{x})} - t^{\pm}_{(t, \overrightarrow{x})} = \epsilon \pm \left( - \left( t^{\pm}_{(t+\epsilon, \overrightarrow{x})} - t^{\pm}_{(t, \overrightarrow{x})} \right) \overrightarrow{v}_{(t^{\pm}_{(t, \overrightarrow{x})})} \cdot \overrightarrow{n}_{(t^{\pm}_{(t, \overrightarrow{x})}, \overrightarrow{x})} + R \right).$$

or, after a quick rearrange of the terms

$$t^{\pm}_{(t+\epsilon, \overrightarrow{x})} - t^{\pm}_{(t, \overrightarrow{x})} = \frac{\epsilon \pm R}{1 \pm \overrightarrow{v}_{(t^{\pm}_{(t, \overrightarrow{x})})} \cdot \overrightarrow{n}_{(t^{\pm}_{(t, \overrightarrow{x})}, \overrightarrow{x})}},$$

where we can see that, in order to perform the limit of the partial derivative, we must know the behavior of $R/\epsilon$. For this purpose we use the remainder form given by Lagrange and write

$$R = \frac{\left( t^{\pm}_{(t+\epsilon, \overrightarrow{x})} - t^{\pm}_{(t, \overrightarrow{x})} \right)^2}{2} \left( \frac{d^2}{ds^2} \|\overrightarrow{x} - \overrightarrow{q}_{(s)}\|_{\mathbb{R}^3} \right) \Big|_{s=\lambda},$$

[3] As an abuse of notation, here and later we omit the point at which $R$ is evaluated. It should be understood that, e.g. in this case, the remainder is evaluated at some time between $(t, \overrightarrow{x})$ and $(t + \epsilon, \overrightarrow{x})$.

with a $\lambda$ between $t^{\pm}_{(t+\epsilon,\overrightarrow{x})}$ and $t^{\pm}_{(t,\overrightarrow{x})}$. Let us call this last derivative $C_\epsilon$ to spare some writing[4]. Then, we obtain

$$\frac{R}{\epsilon}=\frac{(t^{\pm}_{(t+\epsilon,\overrightarrow{x})}-t^{\pm}_{(t,\overrightarrow{x})})^2}{2\epsilon}C_\epsilon=\frac{C_\epsilon(t^{\pm}_{(t+\epsilon,\overrightarrow{x})}-t^{\pm}_{(t,\overrightarrow{x})})}{2}\cdot\frac{t^{\pm}_{(t+\epsilon,\overrightarrow{x})}-t^{\pm}_{(t,\overrightarrow{x})}}{\epsilon}.$$

We can insert this last equation in the limit we want to calculate and get

$$\begin{aligned}\frac{t^{\pm}_{(t+\epsilon,\overrightarrow{x})}-t^{\pm}_{(t,\overrightarrow{x})}}{\epsilon}=&\frac{1}{1\pm\overrightarrow{v}_{(t^{\pm}_{(t,\overrightarrow{x})})}\cdot\overrightarrow{n}_{(t^{\pm}_{(t,\overrightarrow{x})},\overrightarrow{x})}}\pm\frac{R}{\epsilon}\frac{1}{1\pm\overrightarrow{v}_{(t^{\pm}_{(t,\overrightarrow{x})})}\cdot\overrightarrow{n}_{(t^{\pm}_{(t,\overrightarrow{x})},\overrightarrow{x})}}\\ =&\frac{1}{1\pm\overrightarrow{v}_{(t^{\pm}_{(t,\overrightarrow{x})})}\cdot\overrightarrow{n}_{(t^{\pm}_{(t,\overrightarrow{x})},\overrightarrow{x})}}\times\\ &\left(1\pm\frac{C_\epsilon(t^{\pm}_{(t+\epsilon,\overrightarrow{x})}-t^{\pm}_{(t,\overrightarrow{x})})}{2}\cdot\frac{t^{\pm}_{(t+\epsilon,\overrightarrow{x})}-t^{\pm}_{(t,\overrightarrow{x})}}{\epsilon}\right),\end{aligned}$$

which can again be rearranged into

$$\frac{t^{\pm}_{(t+\epsilon,\overrightarrow{x})}-t^{\pm}_{(t,\overrightarrow{x})}}{\epsilon}=\frac{1}{1\pm\overrightarrow{v}_{(t^{\pm}_{(t,\overrightarrow{x})})}\cdot\overrightarrow{n}_{(t^{\pm}_{(t,\overrightarrow{x})},\overrightarrow{x})}}\cdot\frac{1}{1\mp\frac{C_\epsilon(t^{\pm}_{(t+\epsilon,\overrightarrow{x})}-t^{\pm}_{(t,\overrightarrow{x})})}{2}}.$$

Now we can perform the limit of the last equation and obtain

$$\begin{aligned}\lim_{\epsilon\to0}\frac{t^{\pm}_{(t+\epsilon,\overrightarrow{x})}-t^{\pm}_{(t,\overrightarrow{x})}}{\epsilon}=&\lim_{\epsilon\to0}\left(\frac{1}{1\pm\overrightarrow{v}_{(t^{\pm}_{(t,\overrightarrow{x})})}\cdot\overrightarrow{n}_{(t^{\pm}_{(t,\overrightarrow{x})},\overrightarrow{x})}}\cdot\frac{1}{1\mp\frac{C_\epsilon(t^{\pm}_{(t+\epsilon,\overrightarrow{x})}-t^{\pm}_{(t,\overrightarrow{x})})}{2}}\right)\\ =&\frac{1}{1\pm\overrightarrow{v}_{(t^{\pm}_{(t,\overrightarrow{x})})}\cdot\overrightarrow{n}_{(t^{\pm}_{(t,\overrightarrow{x})},\overrightarrow{x})}}\cdot\lim_{\epsilon\to0}\frac{1}{1\mp\frac{C_\epsilon(t^{\pm}_{(t+\epsilon,\overrightarrow{x})}-t^{\pm}_{(t,\overrightarrow{x})})}{2}}\\ =&\frac{1}{1\pm\overrightarrow{v}_{(t^{\pm}_{(t,\overrightarrow{x})})}\cdot\overrightarrow{n}_{(t^{\pm}_{(t,\overrightarrow{x})},\overrightarrow{x})}},\end{aligned}$$

which is the desired result.

2. For the partial derivatives $\partial t^{\pm}/\partial x^i$ we can do all calculations in a similar manner. We consider, for example, the partial derivative with respect to $x$, i.e.

$$\frac{\partial t^{\pm}}{\partial x}=\lim_{\epsilon\to0}\frac{t^{\pm}_{(t,x+\epsilon,y,z)}-t^{\pm}_{(t,x,y,z)}}{\epsilon}.$$

Here the steps are little bit trickier than in the previous calculation because the numerator is equal to

$$t^{\pm}_{(t,x+\epsilon,y,z)}-t^{\pm}_{(t,x,y,z)}=t\pm\left\|\begin{pmatrix}x+\epsilon\\y\\z\end{pmatrix}+\overrightarrow{q}_{(t^{\pm}_{(t,x+\epsilon,y,z)})}\right\|_{\mathbb{R}^3}-t\mp\left\|\begin{pmatrix}x\\y\\z\end{pmatrix}+\overrightarrow{q}_{(t^{\pm}_{(t,x,y,z)})}\right\|_{\mathbb{R}^3}.$$

In order to perform the Taylor expansion of the first norm in the right way, we have to consider it as a function of the four variables $(t^{\pm}_{(t,x+\epsilon,y,z)},x+\epsilon,y,z)$ and expand it around

---

[4] Explicitly we have $C_\epsilon=-\overrightarrow{a}_{(\lambda)}\cdot\overrightarrow{n}_{(\lambda,\overrightarrow{x})}+\frac{\overrightarrow{v}^2_{(\lambda)}-(\overrightarrow{v}_{(\lambda)}\cdot\overrightarrow{n}_{(\lambda,\overrightarrow{x})})^2}{\|\overrightarrow{x}-\overrightarrow{q}_{(\lambda)}\|_{\mathbb{R}^3}}$ for some $\lambda$ between $t^{\pm}_{(t,\overrightarrow{x})}$ and $t^{\pm}_{(t+\epsilon,\overrightarrow{x})}$. In the limit $\epsilon\to0$, the velocity, acceleration and normal vector are well defined and finite since we are always outside the world-line. Moreover, assuming the existence of a maximal acceleration, we have

$$|C_\epsilon|\le a_{max}+\frac{2v^2_{max}}{\min_{s\in\left(t^{\pm}_{(t,\overrightarrow{x})},t^{\pm}_{(t+\epsilon,\overrightarrow{x})}\right)}\|\overrightarrow{x}-\overrightarrow{q}_{(s)}\|_{\mathbb{R}^3}}.$$

the four-dimensional vector $(t^{\pm}_{(t,x,y,z)}, x, y, z)^T$. This Taylor expansion is given by

$$\left\| \begin{pmatrix} x+\epsilon \\ y \\ z \end{pmatrix} + \overrightarrow{q}_{(t^{\pm}_{(t,x+\epsilon,y,z)})} \right\|_{\mathbb{R}^3} = \| \overrightarrow{x} - \overrightarrow{q}_{(t^{\pm}_{(t,x,y,z)})} \|_{\mathbb{R}^3} + \left( t^{\pm}_{(t,x+\epsilon,y,z)} - t^{\pm}_{(t,x,y,z)} \right) \times$$

$$\left( \frac{\partial}{\partial s} \left\| \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \overrightarrow{q}_{(s)} \right\|_{\mathbb{R}^3} \right) \Bigg|_{s=t^{\pm}_{(t,x,y,z)}} +$$

$$(x+\epsilon-x) \left( \frac{\partial}{\partial s'} \left\| \begin{pmatrix} s' \\ y \\ z \end{pmatrix} + \overrightarrow{q}_{(t^{\pm}_{(t,s',y,z)})} \right\|_{\mathbb{R}^3} \right) \Bigg|_{s'=x} + R,$$

where again we have a remainder term $R$. After doing the partial derivatives we obtain

$$\left\| \begin{pmatrix} x+\epsilon \\ y \\ z \end{pmatrix} + \overrightarrow{q}_{(t^{\pm}_{(t,x+\epsilon,y,z)})} \right\|_{\mathbb{R}^3} = \| \overrightarrow{x} - \overrightarrow{q}_{(t^{\pm}_{(t,x,y,z)})} \|_{\mathbb{R}^3} -$$

$$\left( t^{\pm}_{(t,x+\epsilon,y,z)} - t^{\pm}_{(t,x,y,z)} \right) \overrightarrow{v}_{(t^{\pm}_{(t,\overrightarrow{x})})} \cdot \overrightarrow{n}_{(t^{\pm}_{(t,\overrightarrow{x})}, \overrightarrow{x})} +$$

$$\epsilon n^1_{(t^{\pm}_{(t,\overrightarrow{x})}, \overrightarrow{x})} + R,$$

where we have written $n^1$ for the $x$-component of the normal vector. Inserting this result in the numerator yields

$$t^{\pm}_{(t,x+\epsilon,y,z)} - t^{\pm}_{(t,x,y,z)} = \mp \left( t^{\pm}_{(t,x+\epsilon,y,z)} - t^{\pm}_{(t,x,y,z)} \right) \overrightarrow{v}_{(t^{\pm}_{(t,\overrightarrow{x})})} \cdot \overrightarrow{n}_{(t^{\pm}_{(t,\overrightarrow{x})}, \overrightarrow{x})} \pm$$

$$\epsilon n^1_{(t^{\pm}_{(t,\overrightarrow{x})}, \overrightarrow{x})} \pm R,$$

or, if re rearrange the terms in the last equation, we acquire

$$\frac{t^{\pm}_{(t,x+\epsilon,y,z)} - t^{\pm}_{(t,x,y,z)}}{\epsilon} = \frac{\pm n^1_{(t^{\pm}_{(t,\overrightarrow{x})}, \overrightarrow{x})} \pm R/\epsilon}{1 \pm \overrightarrow{v}_{(t^{\pm}_{(t,\overrightarrow{x})})} \cdot \overrightarrow{n}_{(t^{\pm}_{(t,\overrightarrow{x})}, \overrightarrow{x})}}. \tag{2.15}$$

So we see again, that is necessary to check the behavior of the remainder term with respect to epsilon. Because this is rather a tedious computation, we provide a full calculation in Appendix D, where it is shown that the remainder term vanishes in the limit $\epsilon \to 0$. Therefore, we obtain

$$\frac{\partial t^{\pm}}{\partial x} = \lim_{\epsilon \to 0} \frac{t^{\pm}_{(t,x+\epsilon,y,z)} - t^{\pm}_{(t,x,y,z)}}{\epsilon} = \frac{\pm n^1_{(t^{\pm}_{(t,\overrightarrow{x})}, \overrightarrow{x})}}{1 \pm \overrightarrow{v}_{(t^{\pm}_{(t,\overrightarrow{x})})} \cdot \overrightarrow{n}_{(t^{\pm}_{(t,\overrightarrow{x})}, \overrightarrow{x})}}, \tag{2.16}$$

which is the desired result. We can see, that the partial derivatives with respect to $y$ and $z$ are computed in the same way as it was shown here.

□

**Definition 2.14. (Advanced and Retarded Proper Parameters)** For the situation given in the Definition 2.11, we call the maps

$$\tau^{\pm} : \begin{cases} \mathbb{M} \to \mathbb{R} \\ x \mapsto \tau^{\pm}_{(x)} = \tilde{t}^{-1}_{(t \pm \| \overrightarrow{x} - \overrightarrow{q}_{(\tilde{t}_{(x)})} \|_{\mathbb{R}^3})} \end{cases} \tag{2.17}$$

the advanced (with plus sign) and retarded (with minus sign) proper parameters. Because of the bijectivity of $\tilde{t}$ and Lemma 2.9 these maps are well defined.

**Corollary 2.15.** Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of points outside the world-line and let it converge to a four-position $q_{(\tau^*)} \in q(\mathbb{R})$ with $\tau^* \in \mathbb{R}$. Then, the advanced and retarded parameters $\tau^{\pm}$ as given in definition above, converge to $\tau^*$.

*Proof.* The map

$$\tilde{t} : \begin{cases} \mathbb{R} \to \mathbb{R} \\ \tau \mapsto \tilde{t}_{(\tau)} \end{cases} \tag{2.18}$$

is per definition of the world-line 2.6 bijective and differentiable at all $\tau \in \mathbb{R}$, therefore the map $\tilde{t}^{-1}$ is also bijective and differentiable. From here it follows

$$\lim_{n\to\infty} \tau^{\pm}_{(x_n)} = \lim_{n\to\infty} \tilde{t}^{-1}_{(t^{\pm}_{(x_n)})} = \tilde{t}^{-1}_{(\lim_{n\to\infty} t^{\pm}_{(x_n)})}$$

and using the continuity of $t^{\pm}$ we obtain

$$\lim_{n\to\infty} \tau^{\pm}_{(x_n)} = \tilde{t}^{-1}_{(t^{\pm}_{(\lim_{n\to\infty} x_n)})} = \tilde{t}^{-1}_{(t^{\pm}_{(q_{(\tau^*)})})}.$$

Because of corollary 2.10, we know that at the world-line there is only one element of $L_{q_{(\tau^*)}} \cap q(\mathbb{R})$ and that element is actually $q_{(\tau^*)}$. Therefore

$$t^{\pm}_{(q_{\tau^*})} = \tilde{t}_{(\tau^*)}$$

from which follows

$$\lim_{n\to\infty} \tau^{\pm}_{(x_n)} = \tilde{t}^{-1}_{(\tilde{t}_{(\tau^*)})} = \tau^*.$$

□

Because the times $t^{\pm}_{(x)}$ correspond to the time-coordinates of the advanced and retarded positions, $\tau^{\pm}_{(x)}$ are the parameter at which the four-position is an element of $L_x \cap q(\mathbb{R})$, i.e. $q_{(\tau^{\pm}_{(x)})}$ are the advanced and retarded positions. The following results provide a limit for the induced sequence $(q_{(\tau^{\pm}_{(x_n)})})_{n\in\mathbb{N}}$.

**Corollary 2.16.** In the context of the last corollary, the advanced and retarded positions $q_{(\tau^{\pm})}$ as given in Definition 2.7 and in Lemma 2.9 converge to $q_{(\tau^*)}$.

*Proof.* Using the continuity of $q$ and the last corollary we obtain

$$\lim_{n\to\infty} q_{\left(\tau^{\pm}_{(\tilde{t}_{(x_n)})}\right)} = q_{\left(\lim_{n\to\infty} \tau^{\pm}_{(\tilde{t}_{(x_n)})}\right)} = q_{(\tau^*)}. \tag{2.19}$$

□

# Chapter 3

# Radiation Fields Under a Simple Expansion

In this chapter, we explore the radiation field in a scenario where the spatial component is shifted by a small amount in an arbitrary direction. Our primary aim is to develop a physical intuition for the behavior of the radiation field, so the approach here is more conceptual than mathematical. We focus on understanding the physics behind the motion of a particle in various specific scenarios.

The chapter is structured as follows: the first section introduces the notation used throughout the discussion. In the second section, we revisit the basic case of a particle moving with constant velocity, using this as a foundation to explore more complex scenarios in the subsequent sections. The primary aim is to investigate the role of Coulomb fields in the radiation process.


**Summary of this Chapter**

A series of results is derived, here we

1. show that, with an explicit computation, for a particle moving with constant velocity, it is possible to calculate both the retarded and advanced electromagnetic fields, and that the radiation field vanishes (see Results 3.1, 3.2 and the equation (3.16)).

2. compute a novel expansion of the retarded and advanced times (see Results 3.4 and 3.5).

3. calculate the limit of the normal vectors at the particle's position (see Results 3.6 and 3.7).

4. demonstrate that the difference between the retarded and advanced Coulomb fields is, in fact, divergent. Furthermore, we show that this divergence cancels out with a corresponding divergence in the difference of the far fields (see Results 3.11 and equations (3.32) and (3.33)).


## 3.1 The Lienard-Wiechert Fields

### 3.1.1 Important Remarks

In this chapter, we will engage in extensive calculations where detailing every argument may become cumbersome. To streamline this process, we introduce a concise and efficient notation that will help distinguish between the retarded and advanced expressions.

1. The retarded and advanced times will be denoted as $t^{\pm}$, where the proper definition was given the second chapter and reads

$$t^{\pm} := t^{\pm}_{(t,\vec{x})} = t \pm \left\| \vec{x} - \vec{q}\left(t^{\pm}_{(t,\vec{x})}\right) \right\|_{\mathbb{R}^3}. \tag{3.1}$$

2. The retarded and advanced positions and velocities are expressed as

$$\vec{q}^{\,\pm} := \vec{q}\left(t^{\pm}_{(t,\vec{x})}\right),$$

and
$$\overrightarrow{v}^{\pm} := \overrightarrow{v}_{\left(t^{\pm}_{(t,\overrightarrow{x})}\right)}.$$

3. The retarded and advanced **normal vectors** are given by
$$\overrightarrow{n}^{\pm} = \overrightarrow{n}_{\left(t^{\pm}_{(t,\overrightarrow{x})},\overrightarrow{x}\right)} := \frac{\overrightarrow{x} - \overrightarrow{q}^{\pm}}{\|\overrightarrow{x} - \overrightarrow{q}^{\pm}\|_{\mathbb{R}^3}}. \tag{3.2}$$

4. In this chapter, we will use the **Lienard-Wiechert Fields** of a point particle of chage $e = 1$. For an extensive study of their properties, we refer to the PhD thesis of Priv.-Doz. Dr. Dirk-André Deckert *"Electrodynamic absorber theory"* [Dec10, p. 66]. These fields are written as
$$\overrightarrow{E}^{\pm} = \overrightarrow{E}_{\left(t^{\pm}_{(t,\overrightarrow{x})},\overrightarrow{x}\right)} = \frac{(\overrightarrow{n}^{\pm} \pm \overrightarrow{v}^{\pm})(1 - \overrightarrow{v}^{\pm 2})}{\|\overrightarrow{x} - \overrightarrow{q}^{\pm}\|^2_{\mathbb{R}^3}(1 \pm \overrightarrow{n}^{\pm} \cdot \overrightarrow{v}^{\pm})^3} + \frac{\overrightarrow{n}^{\pm} \times [(\overrightarrow{n}^{\pm} \pm \overrightarrow{v}^{\pm}) \times \overrightarrow{a}^{\pm}]}{\|\overrightarrow{x} - \overrightarrow{q}^{\pm}\|_{\mathbb{R}^3}(1 \pm \overrightarrow{n}^{\pm} \cdot \overrightarrow{v}^{\pm})^3} \tag{3.3}$$
for the electric field and
$$\overrightarrow{B}^{\pm} = \overrightarrow{B}_{\left(t^{\pm}_{(t,\overrightarrow{x})},\overrightarrow{x}\right)} = \mp \overrightarrow{n}^{\pm} \times \overrightarrow{E}^{\pm} \tag{3.4}$$
for the magnetic field. Here we make a split the fields in two different components, the *Coulomb fields* and the *far fields*. The Coulomb electric fields are defined as
$$\overrightarrow{E}^{\pm}_{C} := \frac{(\overrightarrow{n}^{\pm} \pm \overrightarrow{v}^{\pm})(1 - \overrightarrow{v}^{\pm 2})}{\|\overrightarrow{x} - \overrightarrow{q}^{\pm}\|^2_{\mathbb{R}^3}(1 \pm \overrightarrow{n}^{\pm} \cdot \overrightarrow{v}^{\pm})^3}$$
and the far electric fields are given by
$$\overrightarrow{E}^{\pm}_{f} := \frac{\overrightarrow{n}^{\pm} \times [(\overrightarrow{n}^{\pm} \pm \overrightarrow{v}^{\pm}) \times \overrightarrow{a}^{\pm}]}{\|\overrightarrow{x} - \overrightarrow{q}^{\pm}\|_{\mathbb{R}^3}(1 \pm \overrightarrow{n}^{\pm} \cdot \overrightarrow{v}^{\pm})^3}.$$
The Coulomb and far magnetic fields are then computed from equation (3.4).

Because we want to perform an expansion around $\overrightarrow{q_{(t)}}$, we choose $\overrightarrow{x} = \overrightarrow{q_{(t)}} + \epsilon \overrightarrow{e}$, with an unitary vector $\overrightarrow{e}$ and a small parameter $\epsilon$. It will be shown later that this choice allows us to derive explicit formulas for $t^{\pm}$, and that the normal vectors converge to some result that depends on the direction $\overrightarrow{e}$ chosen to approach $\overrightarrow{q_{(t)}}$.

In our notation, the radiation fields are written as
$$\overrightarrow{E}_{rad} := \overrightarrow{E}^{-} - \overrightarrow{E}^{+} \quad \text{and} \quad \overrightarrow{B}_{rad} := \overrightarrow{B}^{-} - \overrightarrow{B}^{+}.$$

Since a Taylor expansion does not exists at the particles position $\overrightarrow{q_{(t)}}$, we aim to derive some formulas to compute these quantities in a feasible manner.

## 3.2 The Case of Constant Velocity

If we want to calculate the radiation reaction, we need to develop some useful techniques that will help us in the general set up. Because of this, we will first study the simple case of a particle moving along the x-direction. In general, we dispose of two methods to get the radiation field: we could use the machinery of special relativity or the brute force. We will see that the latter method yields important results that are applicable in general. Therefore, both approaches will be examined in detail.

### 3.2.1 Special Relativity

We suppose that a charged particle moves along the x-direction with constant velocity, i.e., the velocity vector is of the form $\overrightarrow{v} = (v, 0, 0)^T$, where $v$ is constant. In this case, we can calculate the retarded and advanced fields of the particle in its rest frame and then transform the fields into a coordinate frame in which the particle moves with velocity $\overrightarrow{v}$. It is seen from equations (3.3) and (3.4), setting $v = 0$, that in the rest frame of the particle the retarded and advanced electric fields are equal to the static Coulomb field and that the retarded and advanced magnetic fields are zero. Writing $\overrightarrow{q}^{\pm} = \overrightarrow{q}_0$, we get
$$\overrightarrow{E}^{+} = \overrightarrow{E}^{-} = \frac{\overrightarrow{x} - \overrightarrow{q}_0}{\|\overrightarrow{x} - \overrightarrow{q}_0\|^3_{\mathbb{R}^3}} \tag{3.5}$$

and

$$\overrightarrow{B}^{+} = \overrightarrow{B}^{-} = \overrightarrow{0}.$$

This means that the radiation fields are equal to zero too.

Now, we boost the fields from the rest frame of the particle using the following formulas (with the speed of light c set equal to one) [Jac14, p. 645]

$$\overrightarrow{E'} = \gamma(\overrightarrow{E} - \overrightarrow{v} \times \overrightarrow{B}) - \frac{\gamma^2}{\gamma+1}\overrightarrow{v}(\overrightarrow{v} \cdot \overrightarrow{E}),$$
$$\overrightarrow{B'} = \gamma(\overrightarrow{B} + \overrightarrow{v} \times \overrightarrow{E}) - \frac{\gamma^2}{\gamma+1}\overrightarrow{v}(\overrightarrow{v} \cdot \overrightarrow{B}).$$

Here we denote the Lorentz factor as $\gamma = 1/\sqrt{1-v^2}$. We are able to calculate the radiation field of a particle moving at a constant speed

$$\begin{aligned}\overrightarrow{E'}_{rad} = \overrightarrow{E'}^{-} - \overrightarrow{E'}^{+} &= (\gamma\overrightarrow{E}^{-} - \frac{\gamma^2}{\gamma+1}\overrightarrow{\beta}(\overrightarrow{\beta} \cdot \overrightarrow{E}^{-})) - (\gamma\overrightarrow{E}^{+} - \frac{\gamma^2}{\gamma+1}\overrightarrow{\beta}(\overrightarrow{\beta} \cdot \overrightarrow{E}^{+})) \\ &= \gamma(\overrightarrow{E}^{-} - \overrightarrow{E}^{+}) - \frac{\gamma^2}{\gamma+1}\overrightarrow{\beta}(\overrightarrow{\beta} \cdot (\overrightarrow{E}^{-} - \overrightarrow{E}^{+}) \overset{Eq.(3.5)}{=} \overrightarrow{0}\end{aligned} \tag{3.6}$$

and for the magnetic radiation field we obtain

$$\overrightarrow{B'}_{rad} = \overrightarrow{B'}^{-} - \overrightarrow{B'}^{+} = -\gamma\overrightarrow{\beta} \times \overrightarrow{E}^{-} + \gamma\overrightarrow{\beta} \times \overrightarrow{E}^{+} = -\gamma\overrightarrow{\beta} \times (\overrightarrow{E}^{-} - \overrightarrow{E}^{+}) \overset{Eq.(3.5)}{=} \overrightarrow{0}. \tag{3.7}$$

So we see, in a very simple manner, that the radiation field and therefore the radiation reaction is zero for a particle moving with constant velocity.

### 3.2.2 Explicit Calculation

In this section we want to repeat the results (3.6) and (3.7), but this time we are going to compute the fields near the particle position $\overrightarrow{q_{(t)}}$ and show that they vanish. This explicit computation is rather tedious, but we will be rewarded with some expressions that will hold in more general setups and will serve as guidance when we confront a general movement. First, we notice that $t^{\pm}$ can be computed explicitly in this case. Since we impose a constant velocity, the particle's trajectory is given by $\overrightarrow{q}_{(t)} = \overrightarrow{v}t$ and $\overrightarrow{q}^{\pm} = \overrightarrow{v}t^{\pm}$. Then we use the implicit equation that defines both $t^{-}$ and $t^{+}$ and get

$$\begin{aligned}&t^{\pm}_{(t,\overrightarrow{x})} := t^{\pm} = t \pm \|\overrightarrow{x} - \overrightarrow{q}^{\pm}\|_{\mathbb{R}^3} \\ &\Rightarrow (t^{\pm} - t)^2 = \|\overrightarrow{x} - \overrightarrow{q}^{\pm}\|^2_{\mathbb{R}^3} = \left\| \begin{pmatrix} x - vt^{\pm} \\ y \\ z \end{pmatrix} \right\|^2_{\mathbb{R}^3} \\ &\Rightarrow (t^{\pm} - t)^2 = (x - vt^{\pm})^2 + y^2 + z^2\end{aligned} \tag{3.8}$$

This equation is a polynomial of second order in $t^{\pm}$ and after rearranging all the terms we get

$$(1 - v^2)t^{\pm} + 2(xv - t)t^{\pm} + t^2 - \|\overrightarrow{x}\|_{\mathbb{R}^3} = 0 \tag{3.9}$$

with zeros at the retarded and advanced times

$$t^{\pm}_{(t,\overrightarrow{x})} = \frac{t - xv \pm \sqrt{(x - vt)^2 + (1 - v^2)(y^2 + z^2)}}{1 - v^2}. \tag{3.10}$$

For a charged particle with constant velocity we obtain exact equations. Plotting the retarded time for some velocity as in Figure 3.1 shows in an explicit manner, why these functions are not differentiable at the world-line. The plot reveals that $t^{-}$ is at $y = 0$ and $z = 0$ given by two planes that intersect at the line $\overrightarrow{x} = \overrightarrow{v}t$, i.e. the retarded time is **continuous everywhere** but not **differentiable at the particle's trajectory**, a result showed in the last chapter.

Even so, $t^{\pm}$ behave well enough to perform an expansion around the particle's position. This expansion is dependent on the direction chosen to approach the trajectory. To illustrate this idea we define an arbitrary unit vector

$$\overrightarrow{e} = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}$$

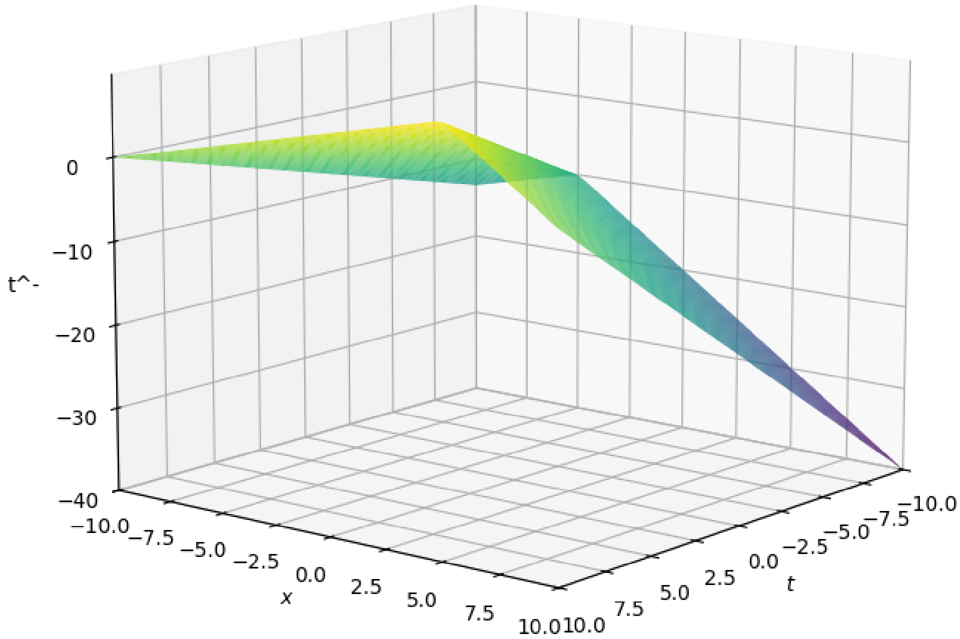


Figure 3.1: Retarded time for constant velocity as a function of t and x for $v = 1/2$

where $\alpha, \beta, \gamma \in \mathbb{R}$ and $\alpha^2 + \beta^2 + \gamma^2 = 1$ and set for any $\epsilon > 0$

$$\vec{x} = \vec{q}_{(t)} + \epsilon \vec{e}. \tag{3.11}$$

We can now investigate how do the different relevant functions look like near the particle's trajectory in the direction of $\vec{e}$ and at the end, we can use our results to compute the radiation fields. We can also get explicit formulas for some quantities. Using (3.11), we obtain

$$\vec{x} = \begin{pmatrix} vt + \epsilon\alpha \\ \epsilon\beta \\ \epsilon\gamma \end{pmatrix}$$

$$\Rightarrow \quad t^-_{(t, \vec{q}_{(t)} + \epsilon \vec{e})} = t - \epsilon \left( \frac{v\alpha + \sqrt{1 - v^2(\beta^2 + \gamma^2)}}{1 - v^2} \right)$$

It is interesting to note here that $\epsilon$ is an arbitrary positive number and at this point, it is not necessary to impose $\epsilon$ to be an infinitesimal quantity. Therefore, there is no remainder associated with this expression. Now we can calculate the retarded normal vector inserting our expression for $\vec{x}$

$$\vec{n}^- = \frac{\epsilon \vec{e} + \vec{q}_{(t)} - \vec{q}^-_{\left(t^-_{(t, \vec{q}_{(t)} + \epsilon \vec{e})}\right)}}{\left\| \epsilon \vec{e} + \vec{q}_{(t)} - \vec{q}^-_{\left(t^-_{(t, \vec{q}_{(t)} + \epsilon \vec{e})}\right)} \right\|_{\mathbb{R}^3}} = \frac{\vec{e} + \frac{\vec{q} - \vec{q}^-}{\epsilon}}{\left\| \vec{e} + \frac{\vec{q} - \vec{q}^-}{\epsilon} \right\|_{\mathbb{R}^3}}$$

where in the last step we have introduced our abuse of notation to make the equations simpler. Inserting everything together and remembering that $\vec{q}^- = \vec{v} t^-$, we are able to calculate the retarded normal vector

$$\vec{n}^- = \frac{1}{\sqrt{1 + 2v\alpha\sqrt{1 - v^2(\beta^2 + \gamma^2)} + v^2(1 - 2(\beta^2 + \gamma^2))}} \begin{pmatrix} \alpha + v\sqrt{1 - v^2(\beta^2 + \gamma^2)} \\ (1 - v^2)\beta \\ (1 - v^2)\gamma \end{pmatrix}. \tag{3.12}$$

This equation shows that the normal vector is independent of the parameter $\epsilon$. In other words, when performing the limit $\epsilon \to 0$, $\vec{n}^-$ is a quantity that depends on the direction chosen to take the limit.

## Example 3.1: $\vec{n}^-$ for some directions $\vec{e}$

We want to give some explicit values and see how do the normal vector looks like. We can split any vector in components parallel and perpendicular to $\vec{e}$ to simplify some scalar products. So, for example, we write $\vec{v} = v_\parallel \vec{e} + \vec{v_\perp}$ where $v_\parallel$ and $v_\perp$ denote the parallel and perpendicular components of $\vec{v}$ with respect to $\vec{e}$. We get then the following results:

$$\text{if} \quad \alpha = 1, \beta = 0, \gamma = 0 \qquad \Rightarrow \quad \vec{n}^- = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \sqrt{1 - v_\perp^2}\,\vec{e} + \vec{v_\perp} \quad \text{with} \quad v_\perp = 0$$

$$\text{if} \quad \alpha = 0, \beta = 1, \gamma = 0 \quad \Rightarrow \quad \vec{n}^- = \begin{pmatrix} v \\ \sqrt{1 - v^2} \\ 0 \end{pmatrix} = \sqrt{1 - v_\perp^2}\,\vec{e} + \vec{v_\perp} \quad \text{with} \quad v_\perp = v$$

$$\text{if} \quad \alpha = 0, \beta = \gamma = \frac{1}{\sqrt{2}} \qquad \Rightarrow \quad \vec{n}^- = \begin{pmatrix} v \\ \frac{\sqrt{1-v^2}}{\sqrt{2}} \\ \frac{\sqrt{1-v^2}}{\sqrt{2}} \end{pmatrix} = \sqrt{1 - v_\perp^2}\,\vec{e} + \vec{v_\perp} \quad \text{with} \quad v_\perp = v$$

We see an interesting feature in these examples, the retarded normal vector seems to take the form $\sqrt{1 - v_\perp^2}\,\vec{e} + \vec{v_\perp}$ in all cases. We will show later in the section 3.3.2, that actually the retarded normal vector can be replaced with this equation when the limit $\epsilon \to 0$ is taken.

We are ready to calculate the retarded electric and magnetic fields. Using the equation (3.3) and setting the acceleration to zero we obtain the following long equation:

## Result 3.1: Retarded Electric Field of a Point Particle with Constant Velocity

$$\begin{aligned}
\vec{E}^- = \frac{(\vec{n}^- - \vec{v})(1 - \vec{v}^2)}{\|\vec{x} - \vec{q}^-\|^2_{\mathbb{R}^3}(1 - \vec{n}^- \cdot \vec{v})^3} = \\
\frac{(1-v^2)^3}{\epsilon^2\left(\sqrt{1 + 2v\alpha\sqrt{1 - v^2(\beta^2+\gamma^2)} + v^2(1 - 2(\beta^2+\gamma^2))} - v\alpha - v^2\sqrt{1 - v^2(\beta^2+\gamma^2)}\right)^3} \times \\
\begin{pmatrix} \alpha + v\sqrt{1 - v^2(\beta^2+\gamma^2)} - v\sqrt{1 + 2v\alpha\sqrt{1 - v^2(\beta^2+\gamma^2)} + v^2(1 - 2(\beta^2+\gamma^2))} \\ (1-v^2)\beta \\ (1-v^2)\gamma \end{pmatrix}
\end{aligned} \tag{3.13}$$

We can repeat these steps but using the advanced time $t^+$. The results are just listed here since the calculations are the same as for the retarded time

$$t^+_{(t, \vec{q}_t + \epsilon\vec{e})} = t^+ = t + \epsilon\left(\frac{-\alpha v + \sqrt{1 - v^2(\beta^2+\gamma^2)}}{1 - v^2}\right)$$

and

$$\vec{n}^+ = \frac{1}{\sqrt{1 - 2v\alpha\sqrt{1 - v^2(\beta^2+\gamma^2)} + v^2(1 - 2(\beta^2+\gamma^2))}}\begin{pmatrix} \alpha - v\sqrt{1 - v^2(\beta^2+\gamma^2)} \\ (1-v^2)\beta \\ (1-v^2)\gamma \end{pmatrix}.$$

This time, if we look at some examples for the vector $\vec{n}^+$, we observe that it takes always the form $\sqrt{1 - v_\perp^2}\,\vec{e} - \vec{v_\perp}$. Again, this will be showed to be true in section 3.3.2. We can now put everything together and calculate the advanced electric field, where we see a similar result as for the retarded electric field.

**Result 3.2: Advanced Electric Field of a Point Particle with Constant Velocity**

$$\vec{E}^{+} = \frac{(\vec{n}^{+} + \vec{v})(1 - \vec{v}^{2})}{\|\vec{x} - \vec{q}^{+}\|^{2}_{\mathbb{R}^3}(1 + \vec{n}^{+} \cdot \vec{v})^{3}} =$$
$$\frac{(1-v^2)^3}{\epsilon^2\left(\sqrt{1 - 2v\alpha\sqrt{1-v^2(\beta^2+\gamma^2)} + v^2(1-2(\beta^2+\gamma^2))} + v\alpha - v^2\sqrt{1-v^2(\beta^2+\gamma^2)}\right)^3} \times$$
$$\begin{pmatrix} \alpha - v\sqrt{1-v^2(\beta^2+\gamma^2)} + v\sqrt{1 - 2v\alpha\sqrt{1-v^2(\beta^2+\gamma^2)} + v^2(1-2(\beta^2+\gamma^2))} \\ (1-v^2)\beta \\ (1-v^2)\gamma \end{pmatrix} \tag{3.14}$$

These results illustrate that even for the simplest case, the equations of the fields are very complicated and there is no way to read off important properties of them. For example, it is not obvious from Results 3.1 and 3.2 that the difference between the two fields is zero. Even worse, the factor $1/\epsilon^2$ in both fields could let us think that the difference of these fields is actually divergent. But actually **these two fields are the same**.

As the reader may see in Appendix A.1, the following equation is true

$$\begin{aligned} M :=& \left(\sqrt{1 - 2v\alpha\sqrt{1-v^2(\beta^2+\gamma^2)} + v^2(1-2(\beta^2+\gamma^2))} + v\alpha - v^2\sqrt{1-v^2(\beta^2+\gamma^2)}\right)^3 \\ =& \left(\sqrt{1 + 2v\alpha\sqrt{1-v^2(\beta^2+\gamma^2)} + v^2(1-2(\beta^2+\gamma^2))} - v\alpha - v^2\sqrt{1-v^2(\beta^2+\gamma^2)}\right)^3 \end{aligned} \tag{3.15}$$

Therefore we can write the radiation electric field as

$$\begin{aligned} \vec{E}_{rad} =& \vec{E}^{-} - \vec{E}^{+} = \frac{(1-v^2)^3}{\epsilon^2 M} \times \\ & \begin{pmatrix} \alpha + v\sqrt{1-v^2(\beta^2+\gamma^2)} - v\sqrt{1 + 2v\alpha\sqrt{1-v^2(\beta^2+\gamma^2)} + v^2(1-2(\beta^2+\gamma^2))} \\ (1-v^2)\beta \\ (1-v^2)\gamma \end{pmatrix} \\ -\frac{(1-v^2)^3}{\epsilon^2 M} & \begin{pmatrix} \alpha - v\sqrt{1-v^2(\beta^2+\gamma^2)} + v\sqrt{1 - 2v\alpha\sqrt{1-v^2(\beta^2+\gamma^2)} + v^2(1-2(\beta^2+\gamma^2))} \\ (1-v^2)\beta \\ (1-v^2)\gamma \end{pmatrix} \\ = \frac{(1-v^2)^3}{\epsilon^2 M} & \begin{pmatrix} 2v\sqrt{1-v^2(\beta^2+\gamma^2)} - v \begin{pmatrix} \sqrt{1 + 2v\alpha\sqrt{1-v^2(\beta^2+\gamma^2)} + v^2(1-2(\beta^2+\gamma^2))} \\ + \sqrt{1 - 2v\alpha\sqrt{1-v^2(\beta^2+\gamma^2)} + v^2(1-2(\beta^2+\gamma^2))} \end{pmatrix} \\ 0 \\ 0 \end{pmatrix} \end{aligned}$$

which, if we use equation (A.4), reduces to

$$= \frac{(1-v^2)^3}{\epsilon^2 M} \begin{pmatrix} 2v\sqrt{1-v^2(\beta^2+\gamma^2)} - 2v\sqrt{1-v^2(\beta^2+\gamma^2)} \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{3.16}$$

In this way, we arrived to the same conclusion as in the last section. For the calculation of the radiation magnetic field we proceed following the same path. Using the equation (3.4) and our Results 3.1 and 3.2 we obtain

$$\begin{aligned} \vec{B}_{rad} &= \vec{B}^{-} - \vec{B}^{+} = \vec{n}^{-} \times \vec{E}^{-} + \vec{n}^{+} \times \vec{E}^{+} \\ &= \frac{(1-v^2)^4 v}{\epsilon^2 M} \begin{pmatrix} 0 \\ -\gamma \\ \beta \end{pmatrix} - \frac{(1-v^2)^4 v}{\epsilon^2 M} \begin{pmatrix} 0 \\ -\gamma \\ \beta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \end{aligned} \tag{3.17}$$

Summarized, our results are

**Main Result 3.3: The Radiation Fields are Zero if the Velocity is Constant**

$$\vec{E}_{rad} = \vec{0},$$

$$\vec{B}_{rad} = \vec{0}.$$

## 3.3 General Formulas for the Expansion

We are ready to explore a more general scenario. In this section we develop formulas for the retarded and advanced times $t^\pm$ in the neighborhood of the particle's position. Then, we use these formulas to show that the normal vectors converge in a suitable sense. These results play an important role in the study of a particle moving with constant acceleration.

### 3.3.1 Expansion of the Retarded and Advanced Times

Looking back on our results of the second chapter, namely the continuity of $t^\pm$ which was proven in Lemma 2.12 and their differentiability, given in Lemma 2.13, we know that it is not possible to perform a Taylor expansion of $t^\pm$ at the particles position. As we know, the gradient of the retarded and advanced times is given by

$$\vec{\nabla} t^\pm_{(t,\vec{x})} = \frac{\pm\vec{n}^\pm}{1 \pm \vec{n}^\pm \cdot \vec{v}^\pm},$$

which is not defined at $\vec{q}_{(t)}$. Beyond the mere continuity of $t^\pm$, we would like to have an expression for $t^\pm$ in the neighborhood of the particle. We begin by examining the equation of the retarded time, which is rewritten as

$$t^-_{(t,\vec{q}_{(t)}+\epsilon\vec{e})} = t - \|\vec{q}_{(t)} + \epsilon\vec{e} - \vec{q}_{(t^-)}\|_{\mathbb{R}^3} := t - \epsilon C_{\epsilon(\vec{e},\vec{v}_{(t)})},$$

where the coefficient $C_{\epsilon(\vec{e},\vec{v}_{(t)})}$ (from now on just written as $C_\epsilon$) may be dependent on parameters like $\epsilon$, the velocity $\vec{v}_{(t)}$ or the direction $\vec{e}$ chosen to take the limit. From this equation we see that

$$C_\epsilon = \left\| \vec{e} + \frac{\vec{q}_{(t)} - \vec{q}_{(t-\epsilon C_\epsilon)}}{\epsilon} \right\|_{\mathbb{R}^3}.$$

In other words, we obtain an implicit formula whose solution provides the desired parameter $C_\epsilon$. We can perform a Taylor expansion of the retarded position $q^-$ around $t$:

$$\vec{q}_{(t-\epsilon C_\epsilon)} = \vec{q}_{(t)} - \epsilon C_\epsilon \vec{v}_{(t)} + \frac{\epsilon^2 C_\epsilon^2}{2}\vec{a}_{(\lambda)}$$

for some $\lambda \in (t - \epsilon C_\epsilon, t)$ (Lagrange form of the remainder). Then, we can insert this Taylor expansion in the defining equation of $C_\epsilon$ and get

$$C_\epsilon = \left\| \vec{e} + C_\epsilon \vec{v}_{(t)} - \frac{\epsilon C_\epsilon^2}{2}\vec{a}_{(\lambda)} \right\|_{\mathbb{R}^3}.$$

To make further progress, we define an orthonormal basis $\{\vec{e}, \vec{f}, \vec{g}\}$ and split all vectors in parallel and perpendicular components to $\vec{e}$ such that we can also rewrite the last equation as [1]

$$C_\epsilon = \left\| (1 + C_\epsilon v_{||} - \frac{\epsilon C_\epsilon^2}{2})\vec{e} + (C_\epsilon v_{\mathrm{f}} - \frac{\epsilon C_\epsilon^2}{2} a_{\mathrm{f}})\vec{f} + (C_\epsilon v_{\mathrm{g}} - \frac{\epsilon C_\epsilon^2}{2} a_{\mathrm{g}})\vec{g} \right\|_{\mathbb{R}^3},$$

where $v_{||}$ and $a_{||}$ denote the scalar product between $\vec{v}$ or $\vec{a}$ with $\vec{e}$. Note that we omitted the times $t$ and $\lambda$ to make the expression readable. To avoid any confusion, note that the velocity is always evaluated at time $t$ and the acceleration at the time $\lambda$.

[1] we use the subscript $||$ to refer to the parallel component of a vector with respect to $\vec{e}$, and the subscripts $f, g$ to denote the components in the direction of $\vec{f}$ and $\vec{g}$, respectively. This approach is adopted because, in the final equation, we only require $v_{||}$, $a_{||}$ and the perpendicular components, e.g. $v_\perp^2 = v_{\mathrm{f}}^2 + v_{\mathrm{g}}^2$, making irrelevant the choice of the vectors $\vec{f}$ and $\vec{g}$.

This statement can be squared and rearranged as a polynomial equation of fourth grade which roots are the desired coefficients $C_\epsilon$. We obtain

$$-\frac{\epsilon^2}{4}\overrightarrow{a}^2 C_\epsilon^4 + \epsilon\overrightarrow{v}\cdot\overrightarrow{a}C_\epsilon^3 + (1-v^2+\epsilon a_{||})C_\epsilon^2 - 2v_{||}C_\epsilon - 1 = 0$$

or after a small rearrange of terms

$$(1-v^2)C_\epsilon^2 - 2v_{||}C_\epsilon - 1 + \epsilon\left(a_{||}C_\epsilon^2 + \overrightarrow{v}\cdot\overrightarrow{a}C_\epsilon^3 - \frac{\epsilon}{4}\overrightarrow{a}^2 C_\epsilon^4\right) = 0. \tag{3.18}$$

The formula above reveals a particularly interesting feature. It resembles the sum of two different polynomials in the form $P_{(C_\epsilon)} + \epsilon Q_{(C_\epsilon)} = 0$. Since we are seeking an expansion of the retarded time around $\overrightarrow{q}_{(t)}$, we assume $\epsilon$ to be small. This suggests that the polynomial $P$ must be corrected by $\epsilon Q$. Consequently, we expect $C_\epsilon$ to be very close to another coefficient, denoted as $C_0$, which corresponds to the roots of equation (3.18) when $\epsilon = 0$. In other words, $C_0$ solves

$$(1-v^2)C_0^2 - 2v_{||}C_0 - 1 = 0$$

which has two solutions

$$C_0 = \frac{v_{||} \pm \sqrt{1-v_\perp^2}}{1-v^2}.$$

This leads us to consider that, as $\epsilon$ becomes smaller, $C_\epsilon$ should converge to $C_0$. Consequently, it seems reasonable to expand $C_\epsilon$ in powers of $\epsilon$. Informally, this can be written as

$$C_\epsilon := C_0 + \epsilon C_1 + \epsilon^2 C_2 + \ldots \quad .$$

### Example 3.2: Numerical Comparison between $C_\epsilon$ and $C_0$

Using Phyton we can compute numerically both $C_\epsilon$ and $C_0$ for some given values of the velocity and acceleration. For this example, we choose the following values

$$v_{||} = \tfrac{1}{2} \qquad v_\perp = \tfrac{1}{2}$$

$$a_{||} = 0.4 \quad a_\perp = 0.3$$

and we get the two square roots $C_0$

$$C_0 = -0.73205 \quad C_0 = 2.73205$$

Now we can compare these results to the numerical solution of $C_\epsilon$

| $\epsilon$ | First Root | Second Root |
|---|---|---|
| 0.1 | -0.72770 | 2.31732 |
| 0.001 | -0.73200 | 2.72623 |
| 0.0001 | -0.73204 | 2.73146 |
| 0.00001 | -0.73205 | 2.73199 |

The last example supports our idea of $C_\epsilon$ being close to $C_0$.

Because we know that the retarded time is unique, we must decide which of the two roots $C_0$ could be used as a good approximation of $C_\epsilon$. Looking back at the equation (3.1), we observe that the right choice is the plus sign. For this reasons, the retarded time takes the following form

$$t^- = t - \epsilon C_0 + \tilde{R}_\epsilon = t - \epsilon\frac{v_{||} + \sqrt{1-v_\perp^2}}{1-v^2} + \tilde{R}_\epsilon,$$

where $\tilde{R}_\epsilon$ denotes the remainder of our approximation. To show that $\tilde{R}_\epsilon$ cpnverges to zero, we first define the polynomial

$$P(X) := (1-v^2)X^2 - 2v_{||}X - 1 + \epsilon\left(a_{||}X^2 + \overrightarrow{v}\cdot\overrightarrow{a}X^3 - \frac{\epsilon}{4}\overrightarrow{a}^2 X^4\right)$$

with $P(C_\epsilon) = 0$. Now we set

$$C_\epsilon := C_0 + \epsilon C_1 + R_\epsilon$$

and insert this expression in $P(X)$. If we define the coefficient

$$C_1 := -\frac{C_0^2}{2}\frac{\overrightarrow{v}\cdot\overrightarrow{a}C_0 + a_{||}}{(1-v^2)C_0 - v_{||}}, \tag{3.19}$$

we obtain the following formula (see Appendix B.1)

$$\begin{aligned} P(C_0 + \epsilon C_1 + R_\epsilon) =& 2(1-v^2)C_0R_\epsilon + (1-v^2)(\epsilon C_1 + R_\epsilon)^2 + \\ & \epsilon\Big( a_{||}(2C_0(\epsilon C_1 + R_\epsilon) + (\epsilon C_1 + R_\epsilon)^2) + \\ & \quad \overrightarrow{v}\cdot\overrightarrow{a}(3C_0^2(\epsilon C_1 + R_\epsilon) + 3C_0(\epsilon C_1 + R_\epsilon)^2 + (\epsilon C_1 + R_\epsilon)^3) - \\ & \quad \frac{\epsilon}{4}a^2(C_0^4 + 4C_0^3(\epsilon C_1 + R_\epsilon) + 6C_0^2(\epsilon C_1 + R_\epsilon)^2 + \\ & \quad 4C_0(\epsilon C_1 + R_\epsilon)^3 + (\epsilon C_1 + R_\epsilon)^4)\Big) = 0. \end{aligned}$$

In order to show that $R_\epsilon$ tends to zero we take the limit $\epsilon \to 0$ and use the continuity of the polynomials to obtain

$$\lim_{\epsilon\to 0} P(C_\epsilon) = 2(1-v^2)C_0 \lim_{\epsilon\to 0} R_\epsilon + (1-v^2)(\lim_{\epsilon\to 0} R_\epsilon)^2 \overset{!}{=} 0.$$

The last equation hast two solutions, namely

$$\lim_{\epsilon\to 0} R_\epsilon = -2C_0$$

and

$$\lim_{\epsilon\to 0} R_\epsilon = 0.$$

The last one is the only possible solution, since we can also take the limit $\epsilon \to 0$ on equation (3.18) and see that $\lim_{\epsilon\to 0} C_\epsilon = C_0$. If we assume that $\lim_{\epsilon\to 0} R_\epsilon = -2C_0$, this would imply $\lim_{\epsilon\to 0} C_\epsilon = -C_0$, which an incorrect result.

Inserting the expansion of $C_\epsilon$ in the defining equation of the retarded time we obtain

$$t^- = t - \epsilon C_\epsilon = t - \epsilon C_0 - \epsilon^2 C_1 - \epsilon R_\epsilon.$$

## Result 3.4: Expansion of $t^-$

The retarded time in the neighborhood of the particle's position can be expressed as

$$t^-_{(t,\overrightarrow{q}_{(t)}+\epsilon\overrightarrow{e})} = t - \epsilon C_0 + \epsilon^2 \frac{C_0^2}{2}\frac{\overrightarrow{v}\cdot\overrightarrow{a}C_0 + a_{||}}{(1-v^2)C_0 - v_{||}} + \tilde{R}_\epsilon, \tag{3.20}$$

where $C_0$ is given by

$$C_0 = \frac{v_{||} + \sqrt{1-v_\perp^2}}{1-v^2}.$$

*Remark.* We introduced the second-order approximation because it will be necessary when calculating the normal vectors. See, for instance, section 3.3.2.

We turn now our attention to the advanced time $t^+$. The expressions for the advanced time are very similar to those for the retarded time. Therefore, we have the advantage that the calculations are made in the same way as before. We rewrite $t^+$ as

$$t^+_{(t,\overrightarrow{q}_{(t)}+\epsilon\overrightarrow{e})} = t + \left\|\overrightarrow{q}_{(t)} + \epsilon\overrightarrow{e} - \overrightarrow{q}_{(t^+)}\right\|_{\mathbb{R}^3} := t + \epsilon B_{\epsilon(\overrightarrow{e},\overrightarrow{v}_{(t)})},$$

with a new coefficient $B_\epsilon$. Repeating all steps from the last page we are able to obtain the following coefficients

$$\begin{aligned} B_\epsilon &= B_0 + \epsilon B_1 + R^*_\epsilon, \\ B_0 &= \frac{-v_{||} + \sqrt{1-v_\perp^2}}{1-v^2}, \\ B_1 &= \frac{B_0^2}{2}\frac{\overrightarrow{v}\cdot\overrightarrow{a}B_0 - a_{||}}{(1-v^2)B_0 + v_{||}}, \end{aligned}$$

and

$$\lim_{\epsilon\to 0} R^*_\epsilon = 0.$$

Finally we get the expansion of $t^+$.

**Result 3.5: Expansion of $t^+$**

The advanced time in the neighborhood of the particle can be expressed as

$$t^{+}_{(t,\vec{q}_{(t)}+\epsilon\vec{e})} = t + \epsilon B_0 + \epsilon^2 \frac{B_0^2}{2} \frac{\vec{v}\cdot\vec{a}B_0 - a_{||}}{(1-v^2)B_0 + v_{||}} + \tilde{R}^*_\epsilon, \tag{3.21}$$

where $B_0$ is given by

$$B_0 = \frac{-v_{||} + \sqrt{1-v_\perp^2}}{1-v^2}.$$

### 3.3.2 Convergence of the Normal Vectors

Now we are equipped with enough knowledge to study how the normal vectors behave in the neighborhood of the point particle. The defining equation of the retarded and advanced normal vectors is given by

$$\vec{n}^{\pm} = \frac{\vec{x} - \vec{q}^{\pm}}{\|\vec{x} - \vec{q}^{\pm}\|_{\mathbb{R}^3}}.$$

We make the computation first for the retarded normal vector. Inserting $\vec{x} = \vec{q}_{(t)} + \epsilon\vec{e}$ in the last formula yields

$$\vec{n}^{-} = \frac{\vec{e} + \frac{\vec{q}_{(t)} - \vec{q}_{(t^-)}}{\epsilon}}{C_\epsilon}.$$

If we make the replacement $t^- = t - \epsilon C_\epsilon$, we can Taylor-expand the retarded position to obtain $\vec{q}_{(t^-)} = \vec{q}_{(t)} - \epsilon C_0 \vec{v}_{(t)} + R$. Inserting this expression in the equation of the normal vector, we arrive at

$$\vec{n}^{-} = \frac{\vec{e}}{C_\epsilon} + \vec{v} + \frac{R}{\epsilon C_\epsilon},$$

where the last term is well defined since the remainder is of order $\epsilon^2$. Now we set our expansion of $C_\epsilon$ and perform a Taylor-expansion on the denominator

$$\frac{1}{C_\epsilon} = \frac{1}{C_0 + \epsilon C_1 + R_\epsilon} = \frac{1}{C_0(1 + \frac{\epsilon C_1}{C_0} + \frac{R_\epsilon}{C_0})}$$
$$= \frac{1}{C_0}\left(1 - \frac{\epsilon C_1}{C_0} + \mathcal{O}(\epsilon^2)\right).$$

We can divide the statement by $C_0$ since $C_0 = 0$ implies $v^2 = 1$, condition that is excluded in our scenario. With this result, the normal vector takes the form

$$\vec{n}^{-} = \frac{\vec{e}}{C_0} + \vec{v} + \mathcal{O}(\epsilon).$$

**Result 3.6: Convergence of the retarded normal vector**

The retarded normal vector converges at the particle's position with the limit

$$\lim_{\epsilon\to 0} \vec{n}_{(t^-_{(t,\vec{q}_{(t)}+\epsilon\vec{e})},\vec{q}_{(t)}+\epsilon\vec{e})} = \frac{\vec{e}}{C_{0(t)}} + \vec{v}_{(t)}. \tag{3.22}$$

If we split the velocity in parallel and perpendicular components to $\vec{e}$, then the limit can be written as

$$\lim_{\epsilon\to 0} \vec{n}_{(t^-_{(t,\vec{q}_{(t)}+\epsilon\vec{e})},\vec{q}_{(t)}+\epsilon\vec{e})} = \sqrt{1 - v^2_{\perp(t)}}\,\vec{e} + \vec{v}_{\perp(t)}. \tag{3.23}$$

This limit is not unique since it depends on the direction used to approach the particle.

The same procedure can be done to compute the limit of the advanced normal vector. As calculated in the Result 3.5, we can use the coefficient $B_\epsilon$ and expand the expression in powers of $\epsilon$. Repeating all steps we obtain

$$\vec{n}^{+} = \frac{\vec{e} + \frac{\vec{q}_{(t)} - \vec{q}_{(t^+)}}{\epsilon}}{B_\epsilon}.$$

Here, we recall that $t^+ = t + \epsilon B_\epsilon$. Again, the Taylor expansion yields

$$\vec{n}^+ = \frac{\vec{e}}{B_\epsilon} - \vec{v} - \frac{R}{\epsilon B_\epsilon}.$$

We can notice the similarity of this expression with that of $\vec{n}^-$. Expanding the coefficient $B_\epsilon$ we get

$$\vec{n}^+ = \frac{\vec{e}}{B_0} - \vec{v} + \mathcal{O}(\epsilon).$$

**Result 3.7: Convergence of the advanced normal vector**

The advanced normal vector converges at the particle's position with the limit

$$\lim_{\epsilon \to 0} \vec{n}_{(t^+_{(t,\vec{q}(t)+\epsilon\vec{e})}, \vec{q}(t)+\epsilon\vec{e})} = \frac{\vec{e}}{B_{0(t)}} - \vec{v}_{(t)}. \tag{3.24}$$

If we split the velocity in parallel and perpendicular components to $\vec{e}$, then the limit can be written as

$$\lim_{\epsilon \to 0} \vec{n}_{(t^+_{(t,\vec{q}(t)+\epsilon\vec{e})}, \vec{q}(t)+\epsilon\vec{e})} = \sqrt{1 - v^2_{\perp(t)}}\,\vec{e} - \vec{v}_{\perp(t)}. \tag{3.25}$$

## 3.4 A Particle Moving with Constant Acceleration

In the previous section, we derived general formulas and demonstrated that the normal vectors can be meaningfully expanded, being well-defined at the particle's position. With this foundational understanding, we can now proceed to analyze a particle moving with constant acceleration in space. Surprisingly, we will obtain an expression for the Coulomb component of the radiation field that exhibits a divergence. Specifically, we explicitly calculate that the Coulomb part of the radiation field diverges as $1/\epsilon$.

### 3.4.1 The Coulomb Radiation Fields

Since our focus is on studying the electromagnetic fields near the particle, we now address the question of whether the Coulomb (or near) fields contribute to the emission of radiation, or if they can be neglected in favor of focusing solely on the far fields.

In this section, we provide an explicit example by analyzing the case of a particle moving along a straight line with constant acceleration. For this case, we derive the following simple formulas

$$\begin{aligned}
\vec{q}^{\,-} &= \vec{q} - \epsilon C_\epsilon \vec{v} + \frac{\epsilon^2}{2} C_\epsilon^2 \vec{a}, \\
\vec{n}^- &= \frac{\vec{e}}{C_\epsilon} + \vec{v} - \frac{\epsilon}{2} C_\epsilon \vec{a}, \\
\vec{v}^- &= \vec{v} - \epsilon C_\epsilon \vec{a}.
\end{aligned}$$

Notice that the coefficients $C_\epsilon$ theoretically contain an infinite number of terms. However, because the position does not have more than two derivatives, the expressions become more compact. We insert these terms in the equation of the retarded Coulomb field

$$\vec{E}^-_C = \frac{(\vec{n}^- - \vec{q}^{\,-})(1 - (\vec{v}^-)^2)}{\|\vec{x} - \vec{q}^{\,-}\|^2_{\mathbb{R}^3}(1 - \vec{n}^- \cdot \vec{v}^-)^3}$$

and obtain

**Result 3.8: The Retarded electric Coulomb Field**

$$\begin{aligned}
\Rightarrow \vec{E}^-_C = \frac{1}{\epsilon^2 A^3}\Big( &(1 - v^2)\vec{e} + 2\epsilon C_\epsilon a v \vec{e} - \epsilon^2 C_\epsilon^2 a^2 \vec{e} \\
&+ (1 - v^2)\frac{\epsilon}{2} C_\epsilon^2 \vec{a} + \epsilon^2 C_\epsilon^3 a v \vec{a} - \frac{\epsilon^3}{2} C_\epsilon^4 a^2 \vec{a}\Big),
\end{aligned} \tag{3.26}$$

where we used the short hand notation

$$A=(1-v^2)C_\epsilon - v_{||} + \epsilon C_\epsilon a_{||} + \frac{3}{2}\epsilon C_\epsilon^2 va - \frac{\epsilon^2}{2}C_\epsilon^3 a^2.$$

For the advanced Coulomb field we get in a similar way

$$\begin{aligned}\overrightarrow{q}^+ &= \overrightarrow{q} + \epsilon B_\epsilon \overrightarrow{v} + \frac{\epsilon^2}{2}B_\epsilon^2 \overrightarrow{a}\\ \overrightarrow{n}^+ &= \frac{\overrightarrow{e}}{B_\epsilon} - \overrightarrow{v} - \frac{\epsilon}{2}B_\epsilon \overrightarrow{a}\\ \overrightarrow{v}^+ &= \overrightarrow{v} + \epsilon B_\epsilon \overrightarrow{a},\end{aligned}$$

which can be inserted in

$$\overrightarrow{E}_C^+ = \frac{(\overrightarrow{n}^+ + \overrightarrow{q}^+)(1-(\overrightarrow{v}^+)^2)}{\|\overrightarrow{x} - \overrightarrow{q}^+\|_{\mathbb{R}^3}^2 (1+\overrightarrow{n}^+ \cdot \overrightarrow{v}^+)^3}$$

and gives us the following expression

**Result 3.9: The Advanced Electric Coulomb Field**

$$\begin{aligned}\Rightarrow \overrightarrow{E}_C^+ = \frac{1}{\epsilon^2 \tilde{A}^3}\Big((1-v^2)\overrightarrow{e} - 2\epsilon B_\epsilon av\overrightarrow{e} - \epsilon^2 B_\epsilon^2 a^2 \overrightarrow{e}\\ + (1-v^2)\frac{\epsilon}{2}B_\epsilon^2 \overrightarrow{a} - \epsilon^2 B_\epsilon^3 av\overrightarrow{a} - \frac{\epsilon^3}{2}B_\epsilon^4 a^2 \overrightarrow{a}\Big),\end{aligned} \tag{3.27}$$

again with the notation

$$\tilde{A} = (1-v^2)B_\epsilon + v_{||} + \epsilon B_\epsilon a_{||} - \frac{3}{2}\epsilon B_\epsilon^2 va - \frac{\epsilon^2}{2}B_\epsilon^3 a^2.$$

### Computation to $\epsilon^{-2}$th Order

We see from the last two results 3.8 and 3.9 that only the fist term is of order $\epsilon^{-2}$. If we wish to calculate the first term in the expansion of the radiation electric field, we have to compute

$$\overrightarrow{E}_{Rad,C} = \overrightarrow{E}_C^- - \overrightarrow{E}_C^+ = \frac{1}{\epsilon^2}(1-v^2)\overrightarrow{e}\left(\frac{1}{A^3} - \frac{1}{\tilde{A}^3}\right),$$

with the coefficients $1/A$ and $1/\tilde{A}$ expanded to the zeroth order. We then have

$$A = C_0(1-v^2) - v_{||} = \sqrt{1-v_\perp^2} = B_0(1-v^2) + v_{||} = \tilde{A},$$

which shows that the desired term vanishes.

**Main Result 3.10: The Coulomb Part of the Radiation Electric Field**

The $\epsilon^{-2}$ term of the Coulomb radiation field is zero

$$\overrightarrow{E}_{Rad,C} = 0 + \mathcal{O}(\epsilon^{-1}). \tag{3.28}$$

### Computation to $\epsilon^{-1}$th Order

Now that we know the first term disappears, we may wonder what happens next. For this reason, we must expand the coefficients $A$ and $\tilde{A}$ to the first order and we must take into consideration all terms in the retarded and advanced coulomb fields up to order $\epsilon$. In this case we obtain

$$\begin{aligned}A^3 &= (1-v_\perp^2)^{3/2} + 3\epsilon(1-v_\perp^2)\left(C_1(1-v^2) + C_0 a_{||} + \frac{3}{2}C_0^2 va\right) + \mathcal{O}(\epsilon^2),\\ \tilde{A}^3 &= (1-v_\perp^2)^{3/2} + 3\epsilon(1-v_\perp^2)\left(B_1(1-v^2) + B_0 a_{||} - \frac{3}{2}B_0^2 va\right) + \mathcal{O}(\epsilon^2).\end{aligned} \tag{3.29}$$

For the terms of order $\epsilon^{-2}$, we must use these expressions, while for the terms of order $\epsilon^{-1}$ we can simplify the calculation by using the expressions up to the zeroth order. Combining everything, we are able to write

$$\begin{aligned}\vec{E}_{Rad,C} = \vec{E}_C^- - \vec{E}_C^+ =& \frac{(1-v^2)\vec{e}}{\epsilon^2}\times \\ &\left(\frac{1}{(1-v_\perp^2)^{3/2}+3\epsilon(1-v_\perp^2)\left(C_1(1-v^2)+C_0a_{||}+\frac{3}{2}C_0^2va\right)+\mathcal{O}(\epsilon^2)}-\right. \\ &\left.\frac{1}{(1-v_\perp^2)^{3/2}+3\epsilon(1-v_\perp^2)\left(B_1(1-v^2)+B_0a_{||}-\frac{3}{2}B_0^2va\right)+\mathcal{O}(\epsilon^2)}\right)+ \\ &\frac{2va(C_0+B_0)}{\epsilon(1-v_\perp^2)^{3/2}}\vec{e}+\frac{(1-v^2)(C_0^2-B_0^2)}{2\epsilon(1-v_\perp^2)^{3/2}}\vec{a}+\mathcal{O}(\epsilon^0).\end{aligned}$$

The big therm in the parenthesis can be expanded and we get

$$\begin{aligned}\vec{E}_C^- - \vec{E}_C^+ =& \frac{(1-v^2)\vec{e}}{\epsilon^2(1-v_\perp^2)^{3/2}}\left[1-1-\right. \\ &\frac{3\epsilon}{(1-v_\perp^2)^{1/2}}\left(C_1(1-v^2)+C_0a_{||}+\frac{3}{2}C_0^2va\right)+ \\ &\left.\frac{3\epsilon}{(1-v_\perp^2)^{1/2}}\left(B_1(1-v^2)+B_0a_{||}-\frac{3}{2}B_0^2va\right)\right]+ \\ &\frac{2va(C_0+B_0)}{\epsilon(1-v_\perp^2)^{3/2}}\vec{e}+\frac{(1-v^2)(C_0^2-B_0^2)}{2\epsilon(1-v_\perp^2)^{3/2}}\vec{a}+\mathcal{O}(\epsilon^0).\end{aligned}$$

Here we see again that the first term in the expansion cancels out, so that in the worst case this expression diverges as $1/\epsilon$. We can simplify the last expression using the formulas of Appendix C. After a rather tedious calculation, we obtain

$$\begin{aligned}\vec{E}_C^- - \vec{E}_C^+ =& \frac{3(1-v^2)\vec{e}}{\epsilon(1-v_\perp^2)^2}\left(\frac{va(1-v^2)(B_0^3+C_0^3)}{2(1-v_\perp^2)^{1/2}}-\frac{3va(1-v_\perp^2+v_{||}^2)}{(1-v^2)^2}\right)+ \\ &\frac{4va}{\epsilon(1-v^2)(1-v_\perp^2)}\vec{e}+\frac{2v_{||}}{\epsilon(1-v^2)(1-v_\perp^2)}\vec{a}+\mathcal{O}(\epsilon^0) \\ =&\frac{-2va}{\epsilon(1-v^2)(1-v_\perp^2)}\vec{e}+\frac{2v_{||}^2}{\epsilon(1-v^2)(1-v_\perp^2)}\vec{a}+\mathcal{O}(\epsilon^0) \\ =&\frac{2a_\perp}{\epsilon(1-v^2)(1-v_\perp^2)}\left(v_{||}\vec{e}_\perp-v_\perp\vec{e}\right)+\mathcal{O}(\epsilon^0).\end{aligned}$$

For the magnetic fields we can use

$$\vec{B}_C^\pm = \mp\vec{n}^\pm\times\vec{E}_C^\pm$$

to compute them. As a result of an extensive calculation we have

$$\vec{B}_C^- - \vec{B}_C^+ = \frac{-2vav_\perp}{\epsilon(1-v^2)(1-v_\perp^2)}\vec{e}_\perp\times\vec{e}-\frac{2a_\perp}{\epsilon(1-v_\perp^2)}\vec{e}_\perp\times\vec{e}+\mathcal{O}(\epsilon^0).$$

And so we get the important and unexpected result

**Main Result 3.11: The Divergence of the Coulomb Radiation Field**

The difference between the retarded and advanced Coulomb fields diverges as $1/\epsilon$ for all points outside of the trajectory. This holds for both the electric and magnetic radiation fields.

$$\vec{E}_{Rad,C} = \frac{2a_\perp}{\epsilon(1-v^2)(1-v_\perp^2)}\left(v_{||}\vec{e}_\perp-v_\perp\vec{e}\right)+\mathcal{O}(\epsilon^0). \tag{3.30}$$

$$\vec{B}_{Rad,C} = \frac{-2vav_\perp}{\epsilon(1-v^2)(1-v_\perp^2)}\vec{e}_\perp\times\vec{e}-\frac{2a_\perp}{\epsilon(1-v_\perp^2)}\vec{e}_\perp\times\vec{e}+\mathcal{O}(\epsilon^0). \tag{3.31}$$

### 3.4.2 The Far Radiation Fields

The immediate question is how the far fields behave in the vicinity of the particle. We now show that they diverge in such a way that they precisely cancel the divergence of the Coulomb fields.

The defining equation of the far fields were given as

$$\overrightarrow{E}^{\pm}_{f} = \frac{\overrightarrow{n}^{\pm} \times [(\overrightarrow{n}^{\pm} \pm \overrightarrow{v}^{\pm}) \times \overrightarrow{a}]}{\|\overrightarrow{x} - \overrightarrow{q}^{\pm}\|_{\mathbb{R}^3}(1 \pm \overrightarrow{n}^{\pm} \cdot \overrightarrow{v}^{\pm})^3},$$

and

$$\overrightarrow{B}^{\pm}_{f} = \mp \overrightarrow{n}^{\pm} \times \overrightarrow{E}^{\pm}_{f},$$

where the acceleration was kept constant. From this equations we can see that the far fields can in the worst case diverge as $1/\epsilon$, which makes the computation easier since it is possible to keep just the zeroth terms of $A$ and $\tilde{A}$. Bringing everything together, we obtain

$$\begin{aligned}\overrightarrow{E}_{Rad,f} = \overrightarrow{E}^{-}_{f} - \overrightarrow{E}^{+}_{f} =& \frac{C_0 a_\perp}{\epsilon(1-v_\perp^2)^{3/2}}(v_\perp \overrightarrow{e} - v_{||}\overrightarrow{e_\perp} - \frac{1}{C_0}\overrightarrow{e_\perp}) - \\ & \frac{B_0 a_\perp}{\epsilon(1-v_\perp^2)^{3/2}}(v_{||} e_\perp - v_\perp \overrightarrow{e} - \frac{1}{B_0}\overrightarrow{e_\perp}) + \mathcal{O}(\epsilon^0),\end{aligned}$$

which after some algebra it reduces to

$$\overrightarrow{E}_{Rad,f} = \frac{-2a_\perp}{\epsilon(1-v^2)(1-v_\perp^2)}\,(v_{||}\overrightarrow{e_\perp} - v_\perp \overrightarrow{e}) + \mathcal{O}(\epsilon^0). \tag{3.32}$$

This is exactly the opposite of the expression obtained for the Coulomb fields. For the far magnetic field we can repeat the whole process and at the end get in the same way the following result

$$\begin{aligned}\overrightarrow{B}_{Rad,f} = \overrightarrow{B}^{-}_{f} - \overrightarrow{B}^{+}_{f} =& \frac{v^2 a_\perp (C_0 + B_0)}{\epsilon(1-v_\perp^2)^{3/2}}\overrightarrow{e_\perp} \times \overrightarrow{e} - \\ & \frac{a_\perp}{\epsilon(1-v_\perp^2)^{3/2}}\left(\frac{1}{C_0} + \frac{1}{B_0}\right)\overrightarrow{e} \times \overrightarrow{e_\perp} + \mathcal{O}(\epsilon^0),\end{aligned}$$

which again can be simplified using the equations given in Appendix C to achieve

$$\overrightarrow{B}^{-}_{f} - \overrightarrow{B}^{+}_{f} = \frac{2vav_\perp}{\epsilon(1-v^2)(1-v_\perp^2)}\overrightarrow{e_\perp} \times \overrightarrow{e} + \frac{2a_\perp}{\epsilon(1-v_\perp^2)}\overrightarrow{e_\perp} \times \overrightarrow{e} + \mathcal{O}(\epsilon^0). \tag{3.33}$$

These calculations demonstrate that the radiation fields do not contain divergent terms. At this stage, it is not possible to compute the terms of order $\epsilon^0$ since we kept everything to order $\epsilon^{-1}$. If one wishes to calculate the next order, the computations become exceedingly tedious, as it is necessary to compute the coefficients $C_2$ and $B_2$ while carefully tracking all terms in the expansion.

While the approach used in this chapter is straightforward and easy to apply, it is not practical for obtaining more detailed information for general trajectories. From Dirac's perspective, we are left with cumbersome expressions, as we are not working in a relativistic framework. The calculation of the general expression for the radiation field will be the focus of the next chapter, where we utilize the field tensor rather than the electromagnetic fields. This is the optimal approach, as the insights gained in this chapter reveal that one must compute with the full fields when discussing radiation.

# Chapter 4

# Dirac's Paper on Radiation Reaction

This chapter is again written in mathematical form and it is devoted to the profound study of Dirac's original derivation of the radiation field tensor. In contrast to the previous chapter, where we dealt with separate expressions for the electric and magnetic fields, we will use the formulas for the field tensors

$$F^{\mu\nu-}_{(x)} = \frac{e}{\dot{q}_{(\tau^-)} \cdot (x - q_{(\tau^-)})} \left[ \frac{\mathrm{d}}{\mathrm{d}\tau} \frac{\dot{q}^{\nu}_{(\tau)}(x^{\mu} - q^{\mu}_{(\tau)}) - \dot{q}^{\mu}_{(\tau)}(x^{\nu} - q^{\nu}_{(\tau)})}{\dot{q}_{(\tau)} \cdot (x - q_{(\tau)})} \right] \Bigg|_{\tau=\tau^-}, \tag{4.1}$$

and

$$F^{\mu\nu+}_{(x)} = \frac{-e}{\dot{q}_{(\tau^+)} \cdot (x - q_{(\tau^+)})} \left[ \frac{\mathrm{d}}{\mathrm{d}\tau} \frac{\dot{q}^{\nu}_{(\tau)}(x^{\mu} - q^{\mu}_{(\tau)}) - \dot{q}^{\mu}_{(\tau)}(x^{\nu} - q^{\nu}_{(\tau)})}{\dot{q}_{(\tau)} \cdot (x - q_{(\tau)})} \right] \Bigg|_{\tau=\tau^+}. \tag{4.2}$$

These expressions are derived in Appendix E. We will also need an extension of the radiation field, which will be studied in this chapter:

**Definition 4.1. (Radiation Field)** Given a point $x$ in space-time, we define the extension of the radiation field $F^{\mu\nu}_{Rad} : \mathbb{M} \to \mathbb{R}^6$ at that point as

$$\overline{F}^{\mu\nu}_{Rad(x)} := \begin{cases} F^{\mu\nu-}_{(x)} - F^{\mu\nu+}_{(x)} & \forall x \in \mathbb{M} \setminus q(\mathbb{R}), \\ \\ \frac{4e}{3} \left( \dddot{q}^{\mu}_{(\tau)} \dot{q}^{\nu}_{(\tau)} - \dddot{q}^{\nu}_{(\tau)} \dot{q}^{\mu}_{(\tau)} \right) & \forall x = q_{(\tau)} \in q(\mathbb{R}). \end{cases} \tag{4.3}$$

The main goal of this chapter is to show the following result:

**Theorem 4.2. (Point-wise Convergence of The Radiation Field)** Let $q$ be a world-line as given in Definition 2.6 and let $(x_n)_{n\in\mathbb{N}}$ be a sequence of points outside the world-line which converge to the point $q_{(\tau^*)} \in q(\mathbb{R})$. Then,

1. outside the world-line it is possible to write
$$F^{\mu\nu}_{Rad(x_n)} = \frac{4e}{3} \left( \dddot{q}^{\mu}_{(\tau_n)} \dot{q}^{\nu}_{(\tau_n)} - \dddot{q}^{\nu}_{(\tau_n)} \dot{q}^{\mu}_{(\tau_n)} \right) + \mathcal{R}^{\mu\nu}_{(x_n)}, \tag{4.4}$$
$R^{\mu\nu}$ is the error term.

2. For the remainder, there exists a constant $C$ such that for all $n \in \mathbb{N}$ with $\|x_n - q_{(\tau^*)}\|_{\mathbb{R}^4} < 1$, it holds that
$$\left| \mathcal{R}^{\mu\nu}_{(x_n)} \right| \leq C \|x_n - q_{(\tau^*)}\|_{\mathbb{R}^4}. \tag{4.5}$$
The constant $C$ depends only on the supremum norm of $q$ and its first five derivatives over a closed ball of radius $r > 0$.

*Remark.* Some facts that are important to notice here:

1. The result is independent of the sequence $(x_n)_{n\in\mathbb{N}}$ chosen to approach the world-line.
2. We will estimate the convergence rate, explicitly showing that $\overline{F}_{Rad(x_n)}$ converges point-wise to $\overline{F}_{Rad(q_{(\tau^*)})}$.
3. This means that $\overline{F}_{Rad(x)}$ is the continuous extension of $F_{Rad(x)}$.

## Summary of this Chapter

As a summary, we would like to provide a sketch of the proof of the Theorem 4.2 stating what was done by Dirac and what is our contribution. In order to expand the Lienard-Wiechert field tensors, we need to demonstrate first the following results:

1. Dirac mentioned a point on the world-line at which $\dot{q} \cdot \gamma = 0$ must hold. We provide a proof of its existence in the Lemma 4.3. This point, which we call "Dirac's choice" and denote as $q_{(\tau_n)}$, simplifies remarkably the expansion of the fields. Thus, we avoid the large computations made in the previous chapter.

2. We demonstrate that the parameter $\tau_n$ converges to $\tau^*$ in Corollary 4.4.

Then, if we shift the advanced $\tau^+_{(x_n)} = \tau_{(x_n)} + \delta_{(x_n)}$ and retarded $\tau^-_{(x_n)} = \tau_{(x_n)} - \sigma_{(x_n)}$ parameters as Dirac did, it is possible to perform the expansion of the field tensors in the same way as in [Dir38]. Lemma 4.7 repeats this calculation.
As we are also interested in the rate of convergence, we calculate the remainder of the expansion and derive some important bounds (equations (4.19), (4.22) and (4.27)), which will allow us to show that the error term vanishes in the limit $x_n \to q_{(\tau^*)}$. This is done in the last part of this chapter in which we provide the main result of this work, namely the convergence of the radiation field (see subsection 4.2.1).

## 4.1 Rigorous Proof of Dirac's Formula

We begin this section with an important proof: the existence of a special point along the particle's trajectory where the computation of the field tensors becomes significantly simplified. In this context, we will utilize the formal definitions provided in the second chapter, as well as some of the results presented there.

**Reminder:** To ensure clarity for the reader, we will now summarize the definitions provided in the second chapter:

1. World Line, Definition 2.6: the map

$$q : \begin{cases} \mathbb{R} \longrightarrow \mathbb{M} \\ \tau \longmapsto q_{(\tau)} := \begin{pmatrix} \tilde{t}_{(\tau)} \\ \vec{q}_{(\tilde{t}_{(\tau)})} \end{pmatrix}, \end{cases}$$

   is called a world-line, $\tilde{t}_{(\tau)}$ is the time at world-line parameter $\tau \in \mathbb{R}$, $\vec{q}_{(\tilde{t}_{(\tau)})}$ is the position at time $\tilde{t}_{(\tau)} \in \mathbb{R}$ and $\vec{v}_{(\tilde{t})} := \dfrac{\mathrm{d}\vec{q}}{\mathrm{d}\tilde{t}}$ is the velocity of the particle, if the following properties are satisfied:

   (a) $q \in \mathcal{C}^\infty_{(\mathbb{R},\mathbb{M})}$

   (b) $\tilde{t} : \mathbb{R} \to \mathbb{R}$ is bijective

   (c) The four-velocity is time-like and positive oriented, i.e $\forall \tau \in \mathbb{R} : \dot{\tilde{t}}_{(\tau)} > \|\dot{\vec{q}}_{(\tilde{t}_{(\tau)})}\|_{\mathbb{R}^3} \geq 0$.

   (d) There exist a maximal velocity smaller than the speed of light, i.e

$$\forall \tilde{t} \in \mathbb{R} : \|\vec{v}\|_{\mathbb{R}^3} = \left\| \frac{\mathrm{d}\vec{q}}{\mathrm{d}\tilde{t}} \right\|_{\mathbb{R}^3} \leq v_{max} < 1.$$

   for some $v_{max} \in [0, 1)$.

2. The parameter $\tau$, Definition 2.14: we call the maps

$$\tau^{\pm} : \begin{cases} \mathbb{M} \to \mathbb{R} \\ x \mapsto \tau^{\pm}_{(x)} = \tilde{t}^{-1}_{(t \pm \|\vec{x} - \vec{q}_{(\tilde{t}_{(x)})}\|_{\mathbb{R}^3})} \end{cases}$$

   the advanced (with plus sign) and retarded (with minus sign) parameters.

The main point to notice here is the use of a parameter $\tau_s$ at which the equation

$$\dot{q}_{(\tau_s)} \cdot (x - q_{(\tau_s)}) = 0$$

should hold. This four-position will be denoted here as "Dirac's choice" and it is very useful because the term $\dot{q} \cdot (x - q)$ appears quite often in the Taylor-expansion of the field tensors. Here we give a full proof of the existence of at least one such points at the world-line since this fact was merely postulated in the original work [Dir38, p. 164].

**Lemma 4.3. (Existence of Dirac's choice)** Let $q$ be a world-line as given in Definition 2.6 and let $x$ be any point in $\mathbb{M} \setminus q(\mathbb{R})$ in the Minkowski-spacetime. Then, it is always possible to choose a pair $(\tau_s, \gamma_{(\tau_s)})$ where $\tau_s \in \mathbb{R}$ and $\gamma_{(\tau_s)} \in \mathbb{M}$, with the following properties

1. $\gamma_{(\tau_s)} = x - q_{(\tau_s)}$
2. $\gamma_{(\tau_s)} \cdot \dot{q}_{(\tau_s)} = 0$
3. $\tau_s \in (\tau^-, \tau^+)$, where $\tau^\pm$ are the advanced and retarded parameters as given in Definition 2.14.

*Proof.* Given the four-vector $x \in \mathbb{M}$, we define the maps

$$\gamma : \begin{cases} \mathbb{R} \longrightarrow \mathbb{M} \\ \tau \longmapsto \gamma_{(\tau)} := x - q_{(\tau)} \end{cases} \tag{4.6}$$

and

$$f : \begin{cases} \mathbb{R} \longrightarrow \mathbb{R} \\ \tau \longmapsto f_{(\tau)} := \dot{q}_{(\tau)} \cdot \gamma_{(\tau)}, \end{cases} \tag{4.7}$$

and study the behavior of the function $f$ at different values of $\tau$.

The first point to note is that at $\tau^\pm$, the particle's position, denoted as $q^\pm = q_{(\tau^\pm)}$ for brevity, belongs to the light cone centered at $x$, i.e $q^\pm \in L_x$. Right at these two points, the four-vector $\gamma$ is light-like by Definition 2.14 and for this reason, $f$ can not be zero. For all $\tau \notin [\tau^-, \tau^+]$, the vector $\gamma$ must be time like, since the point $x$ lies in the interior of every light cone centered at $q_{(\tau)}$.

Because of the third property in Definition 2.6, i.e. $\forall \tau \in \mathbb{R} : \dot{q}^0 > \|\vec{\dot{q}}\|_{\mathbb{R}^3} \geq 0$, we have to make a distinction between two different cases:

1. $\forall \tau < \tau^-$ it holds that
$$\gamma^0 > \|\vec{\gamma}\|_{\mathbb{R}^3} \geq 0,$$
and therefore
$$\begin{aligned} \dot{q}_{(\tau)} \cdot \gamma_{(\tau)} &= \dot{q}^0\gamma^0 - \vec{\dot{q}} \cdot \vec{\gamma} \geq \dot{q}^0\gamma^0 - \|\vec{\dot{q}}\|_{\mathbb{R}^3}\|\vec{\gamma}\|_{\mathbb{R}^3} \\ &> \dot{q}^0\gamma^0 - \dot{q}^0\gamma^0 = 0, \end{aligned}$$
where the last inequality is strictly bigger than zero since $\gamma^0$ is also bigger than $\|\vec{\gamma}\|_{\mathbb{R}^3}$. Summarized, we obtain
$$f_{(\tau)} > 0 \quad \forall \quad \tau < \tau^-.$$

2. $\forall \tau > \tau^+$ it holds that
$$\gamma^0 < 0$$
with
$$|\gamma^0| > \|\vec{\gamma}\|_{\mathbb{R}^3} \geq 0$$
In this case, we get
$$\dot{q}_{(\tau)} \cdot \gamma_{(\tau)} = \dot{q}^0\gamma^0 - \vec{\dot{q}} \cdot \vec{\gamma} = -\dot{q}^0|\gamma^0| - \vec{\dot{q}} \cdot \vec{\gamma}$$
$$\leq -\dot{q}^0|\gamma^0| + \|\vec{\dot{q}}\|_{\mathbb{R}^3}\|\vec{\gamma}\|_{\mathbb{R}^3} < -\dot{q}^0|\gamma^0| + \dot{q}^0|\gamma^0| = 0,$$
where the last inequality follows the same way of thinking as in the last case. Therefore
$$f_{(\tau)} < 0 \quad \forall \quad \tau > \tau^+.$$
It follows from the intermediate value theorem and the fact that $f_{(\tau)}$ is continuous, that there exists at least one point $\tau_s \in (\tau^-, \tau^+)$ such that $f_{(\tau_s)} = 0$.

□

**Corollary 4.4.** Let $q$ be a world-line as given in Definition 2.6 and let $(x_n)_{n\in\mathbb{N}}$ be a sequence with $x_n \in \mathbb{M} \setminus q(\mathbb{R})$ for all $n \in \mathbb{N}$ such that the sequence converges to a point of the trajectory $q_{(\tau^*)}$ with $\tau^* \in \mathbb{R}$ and let the four-vectors $\gamma_n \in \mathbb{M}$ be chosen as in Lemma 4.3, i.e.

$$x_n = q_{(\tau_n)} + \gamma_{(\tau_n)} \quad \forall n \in \mathbb{N},$$

with the special property

$$\gamma_{(\tau_n)} \cdot \dot{q}_{(\tau_n)} = 0 \quad \forall n \in \mathbb{N}.$$

Then it follows that

$$\lim_{n\to\infty} \tau_n = \tau^* \tag{4.8}$$

and

$$\lim_{n\to\infty} \gamma^\mu_{(\tau_n)} = 0, \tag{4.9}$$

i.e. the four-vector $\gamma$ converges component-wise to zero.

*Proof.* From Lemma 4.3 we know that

$$\tau_n^- \leq \tau_n \leq \tau_n^+ \quad \forall n \in \mathbb{N},$$

where we use the notation

$$\tau_n^\pm := \tilde{t}^{-1}_{(t^\pm_{(x_n)})}.$$

Because of the squeeze theorem we obtain

$$\lim_{n\to\infty} \tau_n^- \leq \lim_{n\to\infty} \tau_n \leq \lim_{n\to\infty} \tau_n^+,$$

or

$$\tau^* \leq \lim_{n\to\infty} \tau_n \leq \tau^* \quad \Rightarrow \quad \lim_{n\to\infty} \tau_n = \tau^*.$$

Then we look back to the definition of the four-vector $\gamma_{(\tau_n)}$ and take again the limit $n \to \infty$

$$\begin{aligned}
&\gamma^\mu_{(\tau_n)} = x^\mu_n - q^\mu_{(\tau_n)} \\
\Rightarrow &\lim_{n\to\infty} \gamma^\mu_{(\tau_n)} = \lim_{n\to\infty} x^\mu_n - \lim_{n\to\infty} q^\mu_{(\tau_n)} \\
\Rightarrow &\lim_{n\to\infty} \gamma^\mu_{(\tau_n)} = q^\mu_{(\tau^*)} - q^\mu_{(\lim_{n\to\infty} \tau_n)} = 0,
\end{aligned}$$

where in the last step we used the continuity of $q$. □

Now we can repeat the steps made by Dirac but in a rigorous way. We define two new sequences $(\sigma_n)_{n\in\mathbb{N}}$ and $(\delta_n)_{n\in\mathbb{N}}$ of real numbers as

$$\sigma_n := \tau_n - \tau_n^- \in \mathbb{R} \quad \text{and} \quad \delta_n := \tau_n^+ - \tau_n \in \mathbb{R}, \tag{4.10}$$

where, by Lemma 4.3, we know that $\sigma_n > 0$ and $\delta_n > 0$ for all $n \in \mathbb{N}$. From the Corollaries 2.15 and 4.4 we get

$$\lim_{n\to\infty} \sigma_n = \lim_{n\to\infty} \delta_n = 0. \tag{4.11}$$

These new sequences together with Dirac's choice for the four-vectors $\gamma_n \in \mathbb{M}$ play a central role in the derivation of the radiation term.

### About the Divergence of the Lienard-Wiechert Fields

Before we continue our investigation through the different components of the radiation field, we will show a result of general sequences of four-vectors.

**Lemma 4.5.** Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of four-vectors $a_n \in \mathbb{M}$ with components $(a^0_n, \vec{a}_n)$ for all $n \in \mathbb{N}$. Let the sequence $(a^2_n)_{n\in\mathbb{N}}$, with $a^2_n = a^\mu_n a_{n\mu}$, diverge. Then one of the following scenarios is true

1.
$$\lim_{n\to\infty} a^2_n = \infty \quad \Rightarrow \quad \lim_{n\to\infty} a^0_n = \infty \tag{4.12}$$

2.
$$\lim_{n\to\infty} a_n^2 = -\infty \quad \Rightarrow \quad \lim_{n\to\infty} \|\vec{a}_n\|_{\mathbb{R}^3} = \infty \tag{4.13}$$

*Proof.* First suppose $a_n^2 \to \infty$ as $n \to \infty$. For all $n \in \mathbb{N}$ it is true that

$$a_n^2 = (a_n^0)^2 - (\vec{a}_n)^2$$

Therefore

$$(a_n^0)^2 = a_n^2 + (\vec{a}_n)^2 \geq a_n^2$$

So we get

$$|a_n^0| \geq \sqrt{a_n^2} \quad \forall n \in \mathbb{N}.$$

This shows that the components $a_n^0$ must tend to infinity in the limit since they are bounded from below by the divergent four-norm $\sqrt{a_n^2}$.

Now suppose $a_n^2 \to -\infty$ as $n \to \infty$. For all $n \in \mathbb{N}$ we now have

$$(\vec{a}_n)^2 = (a_n^0)^2 - a_n^2 \geq -a_n^2$$

which, again, diverges to infinity because $\|\vec{a}_n\|_{\mathbb{R}^3}^2$ is bounded from below by $-a_n^2$ which diverges to positive infinity. □

**Lemma 4.6. (Relation between $\sigma_n$ and $\gamma_n$)** Let $q$ be a world-line as given in Definition 2.6 and let $(x_n)_{n\in\mathbb{N}}$ be a sequence with $x_n \in \mathbb{M} \setminus q(\mathbb{R})$ for all $n \in \mathbb{N}$. Let this sequence converge to a point of the trajectory $q_{(\tau^*)}$ with $\tau^* \in \mathbb{R}$ and let the four-vectors $\gamma_n \in \mathbb{M}$ be chosen as in Lemma 4.3, i.e.

$$x_n = q_{(\tau_n)} + \gamma_{(\tau_n)}, \quad \gamma_{(\tau_n)} \cdot \dot{q}_{(\tau_n)} = 0 \quad \forall n \in \mathbb{N},$$

with

$$\tau_n \in (\tau_n^-, \tau_n^+) \quad \forall n \in \mathbb{N}.$$

Further, let the sequences of proper retarded and advanced parameters be described as $\tau_n^- = \tau_n - \sigma_n$ and $\tau_n^+ = \tau_n + \delta_n$ .Then

$$\lim_{n\to\infty} \frac{\|\vec{\gamma}_{(\tau_n)}\|_{\mathbb{R}^3}}{(\sigma_n)^k} = \lim_{n\to\infty} \frac{\|\vec{\gamma}_{(\tau_n)}\|_{\mathbb{R}^3}}{(\delta_n)^k} = \infty$$

if $k > 1$.

*Proof.* 1. For the retarded time at the retarded position we have the following equation

$$(x_n - q_{(\tau_n^-)})^2 = 0.$$

Inserting $\tau_n^- = \tau_n - \sigma_n$ and performing a Taylor expansion of the trajectory around $\tau_n$ yields

$$\gamma_{(\tau_n)} \cdot \gamma_{(\tau_n)} + \sigma_n^2 - \sigma_n^2(\ddot{q}_{(\tau_n)} \cdot \gamma_{(\tau_n)}) + \frac{\sigma_n^3}{3}(\dddot{q}_{(\tau_n)} \cdot \gamma_{(\tau_n)}) - \frac{\sigma_n^4}{12}(\ddot{q}_{(\tau_n)})^2 + \mathcal{O}(\sigma_n^5) = 0.$$

Dividing this equation by $\sigma_n^k$ transforms the last formula into

$$\frac{\gamma_{(\tau_n)} \cdot \gamma_{(\tau_n)}}{\sigma_n^k} + \sigma_n^{2-k}(1 - \ddot{q}_{(\tau_n)} \cdot \gamma_{(\tau_n)}) + \mathcal{O}(\sigma_n^{3-k}) = 0 \tag{4.14}$$

If $k = 2$ we obtain the limit

$$\lim_{n\to\infty} \left(\frac{\gamma_{(\tau_n)}}{\sigma_n}\right)^2 = -1.$$

For $k > 2$, equation (4.14) diverges as $-1/\sigma_n^{k-2}$. Therefore , replacing $k/2$ with $k$ we get for $k > 1$

$$\lim_{n\to\infty} \left(\frac{\gamma_{(\tau_n)}}{\sigma_n^k} \cdot \frac{\gamma_{(\tau_n)}}{\sigma_n^k}\right)^2 = -\infty.$$

From the last Lemma we obtain the conjecture.

2. For the advanced times we have

$$(x_n - q_{(\tau_n^+)})^2 = 0.$$

Setting $\tau_n^+ = \tau_n + \delta_n$ and performing again the Taylor expansion

$$\gamma_{(\tau_n)} \cdot \gamma_{(\tau_n)} + \delta_n^2 - \delta_n^2(\ddot{q}_{(\tau_n)} \cdot \gamma_{(\tau_n)}) - \frac{\delta_n^3}{3}(\dddot{q}_{(\tau_n)} \cdot \gamma_{(\tau_n)}) - \frac{\delta_n^4}{12}(\ddot{q}_{(\tau_n)})^2 + \mathcal{O}(\delta_n^5) = 0,$$

which, up to some signs, yields the same conclusion as the equation (4.14).

□

**Lemma 4.7. (Taylor expansion of the field tensors)** Let $q$ be a world-line as given in Definition 2.6 and let $(x_n)_{n\in\mathbb{N}}$ be a sequence with $x_n \in \mathbb{M} \setminus q(\mathbb{R})$ for all $n \in \mathbb{N}$. Let this sequence converge to a point of the trajectory $q_{(\tau^*)}$ with $\tau^* \in \mathbb{R}$ and let the four-vectors $\gamma_n \in \mathbb{M}$ be chosen as in Lemma 4.3, i.e.

$$x_n = q_{(\tau_n)} + \gamma_{(\tau_n)}, \quad \gamma_{(\tau_n)} \cdot \dot{q}_{(\tau_n)} = 0 \quad \forall n \in \mathbb{N}$$

with

$$\tau_n \in (\tau_n^-, \tau_n^+) \quad \forall n \in \mathbb{N}.$$

Then the sequence of advanced and retarded field tensors induced by $(x_n)_{n\in\mathbb{N}}$ is in general divergent in the limit $n \to \infty$.

*Proof.* 1. **The Retarded Field Tensor**

We are going to perform a Taylor expansion of the following expression

$$F^{\mu\nu-}_{(t,\vec{x})} = \frac{e}{\dot{q}_{(\tau^-)} \cdot (x - q_{(\tau^-)})} \left[ \frac{d}{d\tau} \frac{\dot{q}^{\nu}_{(\tau)}(x^\mu - q^\mu_{(\tau)}) - \dot{q}^{\mu}_{(\tau)}(x^\nu - q^\nu_{(\tau)})}{\dot{q}_{(\tau)} \cdot (x - q_{(\tau)})} \right] \Bigg|_{\tau=\tau^-} \tag{4.15}$$

For every $n \in \mathbb{N}$ it is possible to expand the four-position around every $\tau_n$

$$q^\mu_{(\tau_n - \sigma_n)} = q^\mu_{(\tau_n)} - \sigma_n \dot{q}^\mu_{(\tau_n)} + \frac{\sigma_n^2}{2}\ddot{q}^\mu_{(\tau_n)} - \frac{\sigma_n^3}{6}\dddot{q}^\mu_{(\tau_n)} + \mathcal{O}(\sigma_n^4),$$

and the four-velocity

$$\dot{q}^\mu_{(\tau_n - \sigma_n)} = \dot{q}^\mu_{(\tau_n)} - \sigma_n \ddot{q}^\mu_{(\tau_n)} + \frac{\sigma_n^2}{2}\dddot{q}^\mu_{(\tau_n)} + \mathcal{O}(\sigma_n^3).$$

Here it is important to notice, that we are not at the retarded positions. From now on, we will use the short notation $q^\mu_{(\tau_n)} = q^\mu_n$. The remaining calculations are lengthy, with the special property $\gamma_n \cdot \dot{q}_n = 0$ being used repeatedly. **The reader should pay attention where the Einstein summation convention is used and where not. Greek indices denote the components of the four-vectors and the letter $n$ denotes the index of the sequence.**

Therefore, we can compute

$$\dot{q}^\lambda_{(\tau_n-\sigma_n)}\left(x_{n\lambda} - q_{\lambda(\tau_n-\sigma_n)}\right) = \dot{q}^\lambda_n \gamma_{n\lambda} + \sigma_n - \sigma_n \ddot{q}^\lambda_n \gamma_{n\lambda} + \frac{\sigma_n^2}{2}\dddot{q}^\lambda_n \gamma_{n\lambda} - \frac{\sigma_n^3}{6}\ddot{q}^\lambda_n \ddot{q}_{n\lambda} + \mathcal{O}(\sigma_n^4).$$

And, we obtain

$$\begin{aligned}\frac{1}{\dot{q}^\lambda_{(\tau_n-\sigma_n)}\left(x_{n\lambda} - q_{\lambda(\tau_n-\sigma_n)}\right)} &= \frac{1}{\dot{q}^\lambda_n \gamma_{n\lambda} + \sigma_n - \sigma_n \ddot{q}^\lambda_n \gamma_{n\lambda} + \frac{\sigma_n^2}{2}\dddot{q}^\lambda_n \gamma_{n\lambda} - \frac{\sigma_n^3}{6}\ddot{q}^\lambda_n \ddot{q}_{n\lambda} + \mathcal{O}(\sigma_n^4)} \\ &= \frac{1}{\sigma_n(1 - \ddot{q}^\lambda_n \gamma_{n\lambda})} \frac{1}{1 + \frac{\dot{q}^\lambda_n \gamma_{n\lambda}}{\sigma_n(1-\ddot{q}^\lambda_n \gamma_{n\lambda})} + \frac{\sigma_n \dddot{q}^\lambda_n \gamma_{n\lambda}}{2(1-\ddot{q}^\lambda_n \gamma_{n\lambda})} - \frac{\sigma_n^2 \ddot{q}^\lambda_n \ddot{q}_{n\lambda}}{6(1-\ddot{q}^\lambda_n \gamma_{n\lambda})} + \mathcal{O}(\sigma_n^3)}.\end{aligned}$$

Because we are expanding around the point in which $\dot{q}_n \cdot \gamma_n = 0$ holds, we finally get

$$\begin{aligned}\frac{1}{\dot{q}_{(\tau_n-\sigma_n)} \cdot \left(x_n - q_{(\tau_n-\sigma_n)}\right)} &= \frac{1}{\sigma_n(1 - \ddot{q}_n \cdot \gamma_n)} \frac{1}{1 + \frac{\sigma_n \dddot{q}_n \cdot \gamma_n}{2(1-\ddot{q}_n \cdot \gamma_n)} - \frac{\sigma_n^2 \ddot{q}_n^2}{6(1-\ddot{q}_n \cdot \gamma_n)} + \mathcal{O}(\sigma_n^3)} \\ &= \frac{1}{\sigma_n(1 - \ddot{q}_n \cdot \gamma_n)} \left(1 - \frac{\sigma_n \dddot{q}_n \cdot \gamma_n}{2(1 - \ddot{q}_n \cdot \gamma_n)} + \frac{\sigma_n^2 \ddot{q}^2}{6(1 - \ddot{q}_n \cdot \gamma_n)} + \mathcal{O}(\sigma_n^3)\right)\end{aligned}$$

Now we continue with the next big term in the parenthesis in (4.15) and in order to save writing, we denote with $\mu \leftrightarrow \nu$ the same terms of the written expression but with the indices exchanged. The result is then

$$\begin{aligned}&\dot{q}^{\nu}_{(\tau_n-\sigma_n)}\left(x^{\mu}-q^{\mu}_{(\tau_n-\sigma_n)}\right)-(\mu\leftrightarrow\nu)\\ &=\dot{q}^{\nu}_n\gamma^{\mu}_n-\sigma_n\ddot{q}^{\nu}_n\gamma^{\mu}_n-\frac{\sigma_n^2}{2}\ddot{q}^{\nu}_n\dot{q}^{\mu}_n+\frac{\sigma_n^2}{2}\dddot{q}^{\nu}_n\gamma^{\mu}_n+\frac{\sigma_n^3}{3}\dddot{q}^{\nu}_n\dot{q}^{\mu}_n-\big(\mu\leftrightarrow\nu\big)+\mathcal{O}(\sigma_n^4).\end{aligned}$$

These two results can be multiplied to obtain the following expression

$$\begin{aligned}\frac{\dot{q}^{\nu}_{(\tau_n-\sigma_n)}\left(x^{\mu}-q^{\mu}_{(\tau_n-\sigma_n)}\right)-(\mu\leftrightarrow\nu)}{\dot{q}_{(\tau_n-\sigma_n)}\cdot\big(x_n-q_{n(\tau_n-\sigma_n)}\big)}&=\frac{1}{1-\ddot{q}_n\cdot\gamma_n}\Bigg[\frac{\dot{q}^{\nu}_n\gamma^{\mu}_n}{\sigma_n}-\ddot{q}^{\nu}_n\gamma^{\mu}_n-\frac{\dddot{q}_n\cdot\gamma_n}{2(1-\ddot{q}_n\cdot\gamma_n)}\dot{q}^{\nu}_n\gamma^{\mu}_n\\ &-\frac{\sigma_n}{2}\ddot{q}^{\nu}_n\dot{q}^{\mu}_n+\frac{\sigma_n}{2}\dddot{q}^{\nu}_n\gamma^{\mu}_n+\frac{\sigma_n}{6}\frac{\ddot{q}_n^2}{(1-\ddot{q}_n\cdot\gamma_n)}\dot{q}^{\nu}_n\gamma^{\mu}_n+\frac{\sigma_n^2}{3}\dddot{q}^{\nu}_n\dot{q}^{\mu}_n-\big(\mu\leftrightarrow\nu\big)+\mathcal{O}(\sigma_n^3)\Bigg].\end{aligned}$$

For the next step we have to differentiate this expression with respect to $\tau_n$. Because we chose $\tau=\tau_n-\sigma_n$ with $\tau_n$ fixed, the derivative with respect to $\tau$ is equivalent to minus the derivative with respect to $\sigma_n$. This results in

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\tau}\frac{\dot{q}^{\nu}_{(\tau_n-\sigma_n)}\left(x^{\mu}_n-q^{\mu}_{(\tau_n-\sigma_n)}\right)-(\mu\leftrightarrow\nu)}{\dot{q}^{\lambda}_{(\tau_n-\sigma_n)}\big(x_{\lambda}-q_{\lambda(\tau_n-\sigma_n)}\big)}&=\frac{-1}{1-\ddot{q}_n\cdot\gamma_n}\Bigg[\frac{-\dot{q}^{\nu}_n\gamma^{\mu}_n}{\sigma_n^2}-\frac{1}{2}\ddot{q}^{\nu}_n\dot{q}^{\mu}_n\\ &+\frac{1}{2}\dddot{q}^{\nu}_n\gamma^{\mu}_n+\frac{1}{6}\frac{\ddot{q}_n^2}{(1-\ddot{q}_n\cdot\gamma_n)}\dot{q}^{\nu}_n\gamma^{\mu}_n+\frac{2\sigma_n}{3}\dddot{q}^{\nu}_n\dot{q}^{\mu}_n-\big(\mu\leftrightarrow\nu\big)+\mathcal{O}(\sigma_n^2)\Bigg].\end{aligned}$$

For the last part, we multiply this expression with

$$\frac{e}{\dot{q}_{(\tau^-)}\cdot(x-q_{(\tau^-)})},$$

which can be expanded in the same way as we did before. The equation of the retarded field tensor becomes

$$\begin{aligned}F^{\mu\nu-}_n=\frac{e}{(1-\ddot{q}^{\lambda}_n\gamma_{n\lambda})^2}\Bigg(&\frac{\dot{q}^{\nu}_n\gamma^{\mu}_n}{\sigma_n^3}+\frac{\ddot{q}^{\nu}_n\dot{q}^{\mu}_n}{2\sigma_n}-\frac{\dddot{q}^{\nu}_n\gamma^{\mu}_n}{2\sigma_n}-\frac{\dddot{q}^{\lambda}_n\gamma_{n\lambda}}{2}\frac{\dot{q}^{\nu}_n\gamma^{\mu}_n}{\sigma_n^2}\\ &-\frac{2}{3}\dddot{q}^{\nu}_n\dot{q}^{\mu}_n-\big(\mu\leftrightarrow\nu\big)+\mathcal{O}(\sigma_n)\Bigg).\end{aligned}\tag{4.16}$$

Because of Lemma 4.6 we conclude that at least the spatial components of $F^{\mu\nu-}_n$ in the last expression diverge when $n\to\infty$.

2. **The Advanced Field Tensor**

   In this case, the expression reads

$$F^{\mu\nu+}_{(t,\vec{x})}=\frac{-e}{\dot{q}^{\gamma}_{(\tilde{t}_{(\tau^+)})}(x_{\gamma}-q_{\gamma(\tilde{t}_{(\tau^+)})})}\left[\frac{\mathrm{d}}{\mathrm{d}\tau}\frac{\dot{q}^{\nu}_{(\tilde{t}_{(\tau)})}(x^{\mu}-q^{\mu}_{(\tilde{t}_{(\tau)})})-\dot{q}^{\mu}_{(\tilde{t}_{(\tau)})}(x^{\nu}-q^{\nu}_{(\tilde{t}_{(\tau)})})}{\dot{q}^{\gamma}_{(\tilde{t}_{(\tau)})}(x_{\gamma}-q_{\gamma(\tilde{t}_{(\tau)})})}\right]\Bigg|_{\tau=\tau^+}.\tag{4.17}$$

Here we see that the computations are almost the same as those for the retarded fields. The only difference relays in the fact that we set $\tau=\tau_n+\delta_n$. The final expression reads

$$\begin{aligned}F^{\mu\nu+}_n=\frac{e}{(1-\ddot{q}^{\lambda}_n\gamma_{n\lambda})^2}\Bigg(&\frac{\dot{q}^{\nu}_n\gamma^{\mu}_n}{\delta_n^3}+\frac{\ddot{q}^{\nu}_n\dot{q}^{\mu}_n}{2\delta_n}-\frac{\dddot{q}^{\nu}_n\gamma^{\mu}_n}{2\delta_n}+\frac{\dddot{q}^{\lambda}_n\gamma_{n\lambda}}{2}\frac{\dot{q}^{\nu}_n\gamma^{\mu}_n}{\delta_n^2}\\ &+\frac{2}{3}\dddot{q}^{\nu}_n\dot{q}^{\mu}_n-\big(\mu\leftrightarrow\nu\big)+\mathcal{O}(\delta_n)\Bigg),\end{aligned}\tag{4.18}$$

This expression is also divergent by virtue of Lemma 4.6.

□

*Remark.* It is important to notice that we mixed up two different parameters in one notation, namely $\tau$ from the big term in the parenthesis in the equation of $F^{\mu\nu}$ before the derivative and $\tau_n^-$ outside the parenthesis. In that sense, the expressions for the fields above are not complete, since they do not have any details on the relationship between $\sigma_n$, $\delta_n$ and the four-vector $\gamma_n$. Only when we use the equation of the light-cone (as in Lemma 4.6), we are able to get additional information. The last part of this chapter is devoted to these relations.

## 4.2 Estimation of the Remainder and Main Theorem

We are ready to study the radiation field around a particle. We will first show how can we rewrite $\sigma_n$ and $\delta_n$ in terms of $\gamma_n$. Then, we will calculate the radiation field around a particle.

*Remark.* Instead of demanding the world-line to be globally differentiable, we can take our arguments locally. Let $\mathcal{B}_r(q_{(\tau^*)})$ be a ball of radius $r$ centered at $q_{(\tau^*)}$. Then, the following facts are true:

1. Because we study the sequence $(x_n)_{n\in\mathbb{N}}$ that is convergent, we know

$$\forall r > 0 \quad \exists N \in \mathbb{N} : \forall n > N \Rightarrow x_n \in \overline{\mathcal{B}}_r(q_{(\tau^*)}).$$

2. Because $\overline{\mathcal{B}}_r(q_{(\tau^*)})$ is a compact set and $t^\pm$ are continuous, we are sure that

$$\exists \quad T^+ := \sup_{x\in\overline{\mathcal{B}}_r(q_{(\tau^*)})} t^+_{(x)} < \infty$$

and

$$\exists \quad T^- := \inf_{x\in\overline{\mathcal{B}}_r(q_{(\tau^*)})} t^-_{(x)} < \infty.$$

3. Therefore, all continuous functions that take values in $[T^-, T^+]$ or $[\tau_{(T^-)}, \tau_{(T^+)}]$ take also maxima and minima in this interval. For the next derivations, we will not need more than the fifth derivative of the world-line and we assume that they are defined in the compact set $\overline{\mathcal{B}}_r(q_{(\tau^*)})$.

**Lemma 4.8.** Let $q$ be a world-line as given in Definition 2.6 and let $(x_n)_{n\in\mathbb{N}}$ be a sequence with $x_n \in \mathbb{M} \setminus q(\mathbb{R})$ for all $n \in \mathbb{N}$. Let this sequence converge to a point of the trajectory $q_{(\tau^*)}$ with $\tau^* \in \mathbb{R}$ and let the four-vectors $\gamma_n \in \mathbb{M}$ be chosen as in Lemma 4.3. Then

$$|\sigma_n| \leq \frac{4}{1 - v_{max}} \sup_{t\in[T^-,T^+]} \left(\frac{d\tau}{dt}\right) \cdot \|x_n - q_{(t^*)}\|_{\mathbb{R}^4}. \tag{4.19}$$

The same inequality is true for $\delta_n$.

*Proof.* Per definition we have

$$\sigma_n = \tau_n - \tau_n^-.$$

Therefore

$$|\sigma_n| = |\tau_n - \tau_n^-| \leq |\tau_n^+ - \tau_n^-| = \left|\tau_{(t^+_{(x_n)})} - \tau_{(t^-_{(x_n)})}\right|$$

$$\leq \sup_{t\in[T^-,T^+]} \left(\frac{d\tau}{dt}\right) \left|t^+_{(x_n)} - t^-_{(x_n)}\right| \leq \sup_{t\in[T^-,T^+]} \left(\frac{d\tau}{dt}\right) \left(\left|t^+_{(x_n)} - t^*\right| + \left|t^* - t^-_{(x_n)}\right|\right)$$

and, if we use the results given in (2.11), we get

$$|\sigma_n| \leq \frac{2}{1 - v_{max}} \sup_{t\in[T^-,T^+]} \left(\frac{d\tau}{dt}\right) \cdot \left(|t_n - t^*| + \|\vec{x}_n - \vec{q}_{(t^*)}\|_{\mathbb{R}^3}\right).$$

If we notice that $|t_n - t^*| \leq \|x_n - q_{(\tau^*)}\|_{\mathbb{R}^4}$ and $\|\vec{x}_n - \vec{q}_{(t^*)}\|_{\mathbb{R}^3} \leq \|x_n - q_{(\tau^*)}\|_{\mathbb{R}^4}$ we obtain finally

$$|\sigma_n| \leq \frac{4}{1 - v_{max}} \sup_{t\in[T^-,T^+]} \left(\frac{d\tau}{dt}\right) \cdot \|x_n - q_{(\tau^*)}\|_{\mathbb{R}^4}.$$

This inequality is also valid for $\delta_n$ because $|\delta_n| = |\tau_n^+ - \tau_n| \leq |\tau_n^+ - \tau_n^-|$. $\square$

*Remark.* Because $\gamma_n^2$ is space-like and is given by the Minkowski-metric, we also have the inequality

$$\begin{aligned}
\epsilon_n :=& \sqrt{-\gamma_n^2} = \sqrt{-(\gamma_n^0)^2 + \vec{\gamma}_n^2} \leq \sqrt{(\gamma_n^0)^2 + \vec{\gamma}_n^2} = \|\gamma_n\|_{\mathbb{R}^4} = \|x_n - q_{(\tau_n)}\|_{\mathbb{R}^4} \\
&\leq \|x_n - q_{(\tau^*)}\|_{\mathbb{R}^4} + \|q_{(\tau_n)} - q_{(\tau^*)}\|_{\mathbb{R}^4} \leq \|x_n - q_{(\tau^*)}\|_{\mathbb{R}^4} + \left(\sup_{\tau\in[\tau_{(T^-)},\tau_{(T^+)}]} \dot{q}_{(\tau)}\right) |\tau_n - \tau^*| \\
&\leq \|x_n - q_{(\tau^*)}\|_{\mathbb{R}^4} + \left(\sup_{\tau\in[\tau_{(T^-)},\tau_{(T^+)}]} \dot{q}_{(\tau)}\right) |\tau^+ - \tau^-| \\
&\leq \|x_n - q_{(\tau^*)}\|_{\mathbb{R}^4} + \left(\sup_{\tau\in[\tau_{(T^-)},\tau_{(T^+)}]} \dot{q}_{(\tau)}\right) \frac{4}{1 - v_{max}} \sup_{t\in[T^-,T^+]} \left(\frac{d\tau}{dt}\right) \cdot \|x_n - q_{(\tau^*)}\|_{\mathbb{R}^4}
\end{aligned}$$

So we are able to obtain the following estimation

$$\epsilon_n \leq \left(1 + \frac{4}{1 - v_{max}} \sup_{t\in[T^-,T^+]} \left(\frac{d\tau}{dt}\right) \left(\sup_{\tau\in[\tau_{(T^-)},\tau_{(T^+)}]} \dot{q}_{(\tau)}\right)\right) \|x_n - q_{(\tau^*)}\|_{\mathbb{R}^4} =: C_1 \|x_n - q_{(\tau^*)}\|_{\mathbb{R}^4}. \tag{4.20}$$

*Remark.* We can also get a very important lower bound for $\epsilon_n$. At the point of Dirac's choice, $\gamma_n \cdot \dot{q}_n = 0$, we have

$$\begin{aligned}
\gamma_n \cdot \dot{q}_n = \gamma_n^0 \dot{q}_n^0 - \vec{\gamma}_n \cdot (\dot{q}_n^0 \vec{v}) = 0 \\
\Rightarrow |\gamma_n^0| \leq v_{max} \|\vec{\gamma}_n\|_{\mathbb{R}^3}.
\end{aligned}$$

This holds under the assumption $\dot{q}_n^0 \neq 0$, which is valid if one chooses the parameter $\tau$ for example as the proper time. From this last inequality we obtain

$$\begin{aligned}
\epsilon_n^2 = \vec{\gamma}_n^2 - (\gamma_n^0)^2 \geq (1 - v_{max}^2)\|\vec{\gamma}_n\|_{\mathbb{R}^3}^2 = \frac{1}{2}(1 - v_{max}^2)(\|\vec{\gamma}_n\|_{\mathbb{R}^3}^2 + \|\vec{\gamma}_n\|_{\mathbb{R}^3}^2) \\
\Rightarrow \epsilon_n^2 \geq \frac{1}{2}(1 - v_{max}^2)(\|\vec{\gamma}_n\|_{\mathbb{R}^3}^2 + (\gamma_n^0)^2) = \frac{1}{2}(1 - v_{max}^2)\|\gamma_n\|_{\mathbb{R}^4}^2.
\end{aligned}$$

So we end up with the following inequality

$$\frac{1}{\sqrt{2}} \sqrt{1 - v_{max}^2} \|\gamma_n\|_{\mathbb{R}^4} \leq \epsilon_n \leq \|\gamma_n\|_{\mathbb{R}^4}. \tag{4.21}$$

This also means, that we can estimate an upper boundary of each component of $\gamma_n$ because

$$|\gamma_n^\mu| \leq \|\gamma_n\|_{\mathbb{R}^4} \leq \frac{\sqrt{2}}{\sqrt{1 - v_{max}^2}} \epsilon_n. \tag{4.22}$$

**Lemma 4.9. (Relation between $\sigma_n$ and $\sqrt{-\gamma_n^2}$)** Let $q$ be a world-line as given in Definition 2.6 and let $(x_n)_{n\in\mathbb{N}}$ be a sequence with $x_n \in \mathbb{M} \setminus q(\mathbb{R})$ for all $n \in \mathbb{N}$. Let this sequence converge to a point of the trajectory $q_{(\tau^*)}$ with $\tau^* \in \mathbb{R}$ and let the four-vectors $\gamma_n \in \mathbb{M}$ be chosen as in Lemma 4.3. Then, as a first approximation, we have

$$\sigma_n = \mathcal{O}(\sqrt{-\gamma_n^2}) \tag{4.23}$$

and

$$\delta_n = \mathcal{O}(\sqrt{-\gamma_n^2}). \tag{4.24}$$

*Proof.* We perform a Taylor-expansion of the retarded position up to the third order and obtain

$$q_{(\tau_n^-)} = q_{(\tau_n - \sigma_n)} = q_n - \sigma_n \dot{q}_n + \frac{\sigma_n^2}{2} \ddot{q}_n + R_n,$$

where we have used the notation $q_n := q_{(\tau_n)}$ and $R_n$ is the remainder of the expansion which is of order $\mathcal{O}(\sigma_n^3)$. Inserting this in the equation of the light-cone yields

$$(x_n - q_{(\tau_n^-)})^2 = 0 \Rightarrow (q_n + \gamma_n - q_n + \sigma_n \dot{q}_n - \frac{\sigma_n^2}{2} \ddot{q}_n - R_n)^2 = 0.$$

Expanding the last equation and using the property that $\gamma_n \cdot \dot{q}_n = 0$ results in

$$\gamma_n^2 + \sigma_n^2 - \sigma_n^2 \gamma_n \cdot \ddot{q}_n - 2\gamma_n \cdot R_n - 2\sigma_n \dot{q}_n \cdot R_n + \frac{\sigma_n^4}{4} \ddot{q}_n^2 + \sigma_n^2 \ddot{q}_n \cdot R_n + R_n^2 = 0.$$

We rename the last five terms as

$$\tilde{R}_n := -2\gamma_n \cdot R_n - 2\sigma_n \dot{q}_n \cdot R_n + \frac{\sigma_n^4}{4}\ddot{q}_n^2 + \sigma_n^2 \ddot{q}_n \cdot R_n + R_n^2,$$

which is again a remainder of order $\mathcal{O}(\sigma_n^3)$. So we obtain

$$\gamma_n^2 + \sigma_n^2 - \sigma_n^2 \gamma_n \cdot \ddot{q}_n + \tilde{R}_n = 0.$$

After a rearrange of this equation we get

$$\sigma_n = \frac{\sqrt{-\gamma_n^2}}{\sqrt{1 - \gamma_n \cdot \ddot{q}_n + \frac{\tilde{R}_n}{\sigma_n^2}}}.$$

From this last equation, we see that

$$\lim_{n\to\infty} \frac{\sigma_n}{\sqrt{-\gamma_n^2}} = 1 < \infty,$$

which implies $\sigma_n = \mathcal{O}(\sqrt{-\gamma_n^2})$. The proof for $\delta_n$ follows the same path. Here we notice just a change in some signs because

$$q_{(\tau_n^+)} = q_{(\tau_n + \delta_n)} = q_n + \delta_n \dot{q}_n + \frac{\delta_n^2}{2}\ddot{q}_n + R_n.$$

$\square$

The last lemma shows that, as a fist approximation, we can write $\sigma_n = \delta_n = \sqrt{-\gamma_n^2}$. It is important to improve this last equation since in general $\sigma_n$ is not equal to $\delta_n$. In order to do so, we are going to make a clever use of our first approximation. The next lemma resembles Dirac's expansion made in the original paper [Dir38, p. 166] but we are going to take care of the remainder explicitly. This will show that, if we want to compute the error term in the expansion, we need at least the fifth derivative of the world-line to be locally defined.

**Lemma 4.10. (Dirac's approximation of $\sigma_n$ and $\delta_n$)** In the situation as given in the previous lemma, there exists a $n \in \mathbb{N}$ such that we can approximate both $\sigma_n$ and $\delta_n$ as

$$\sigma_n = \frac{\epsilon_n}{\sqrt{1 - \gamma_n \cdot \ddot{q}_n}} \cdot \left(1 - \frac{\gamma_n \cdot \dddot{q}_n}{6} \frac{\epsilon_n}{(1 - \gamma_n \cdot \ddot{q}_n)^{3/2}} + \frac{\ddot{q}_n^2}{24} \frac{\epsilon_n^2}{(1 - \gamma_n \cdot \ddot{q}_n)^2}\right) + \Sigma_n \tag{4.25}$$

and

$$\delta_n = \frac{\epsilon_n}{\sqrt{1 - \gamma_n \cdot \ddot{q}_n}} \cdot \left(1 + \frac{\gamma_n \cdot \dddot{q}_n}{6} \frac{\epsilon_n}{(1 - \gamma_n \cdot \ddot{q}_n)^{3/2}} + \frac{\ddot{q}_n^2}{24} \frac{\epsilon_n^2}{(1 - \gamma_n \cdot \ddot{q}_n)^2}\right) + \Delta_n, \tag{4.26}$$

where we used the definition $\epsilon_n := \sqrt{-\gamma_n^2}$. Here $\Sigma_n$ and $\Delta_n$ are remainder of order grater than $\mathcal{O}(\epsilon_n^3)$.

*Proof.* The proof of this lemma is very similar as the last one. The main point is that we are going to solve the equation of the light cone by iterations making the degree of accuracy better. We begin by choosing a $k_1 \in \mathbb{N}$ such that for all $\tau \in \mathbb{R}$ with $q_{(\tau)} \in \mathcal{B}(\|q_{(\tau^*)} - x_{k_1}\|_{\mathbb{R}^4})$, we assume $q_{(\tau)} \in \mathcal{C}^5_{(\mathbb{R},\mathbb{M})}$. In particular, this assumption gives us local maxima of the first five derivatives in $\mathcal{B}(\|q_{(\tau^*)} - x_{k_1}\|_{\mathbb{R}^4})$. Now we are going to iterate over the light-cone equation to develop higher accuracy in each step.

1. Approximation of $\sigma_n$ until second order:

   Here we perform the Taylor-expansion of the retarded position until the third order and obtain
   $$q_{(\tau_n^-)} = q_{(\tau_n - \sigma_n)} = q_n - \sigma_n \dot{q}_n + \frac{\sigma_n^2}{2}\ddot{q}_n - \frac{\sigma_n^3}{6}\dddot{q}_n + R_n.$$

   Now the remainder is of order $\mathcal{O}(\sigma_n^4)$. Inserting this expression in the equation of the light-cone and simplifying everything we finally get

   $$\gamma_n^2 + \sigma_n^2 - \sigma_n^2 \gamma_n \cdot \ddot{q}_n + \frac{\sigma_n^3}{3}\gamma_n \cdot \dddot{q}_n + \tilde{R}_n = 0,$$

where, as in the previous calculation, $\tilde{R}_n$ is a well defined remainder but now of order $\mathcal{O}(\sigma_n^4)$. We recall at this point that in the last lemma we showed the following equality (with $\epsilon_n := \sqrt{-\gamma_n^2}$)

$$\sigma_n = \frac{\epsilon_n}{\sqrt{1 - \gamma_n \cdot \ddot{q}_n + \Sigma_n}} = \frac{\epsilon_n}{\sqrt{1 - \gamma_n \cdot \ddot{q}_n}} + \mathcal{O}(\epsilon_n^2),$$

where from the last lemma we had that $\Sigma_n$ is a remainder of order $\mathcal{O}(\sigma_n) = \mathcal{O}(\epsilon_n)$. We can use this expression to calculate $\sigma_n^3$ and insert it in our last result, achiving in this way

$$\sigma_n^3 = \left( \frac{\epsilon_n}{\sqrt{1 - \gamma_n \cdot \ddot{q}_n}} + \mathcal{O}(\epsilon_n^2) \right)^3 = \frac{\epsilon_n^3}{(1 - \gamma_n \cdot \ddot{q}_n)^{3/2}} + \mathcal{O}(\epsilon_n^4)$$

and

$$-\epsilon_n^2 + \sigma_n^2 - \sigma_n^2 \gamma_n \cdot \ddot{q}_n + \frac{\gamma_n \cdot \dddot{q}_n}{3} \frac{\epsilon_n^3}{(1 - \gamma_n \cdot \ddot{q}_n)^{3/2}} + \tilde{R}_n = 0.$$

This step is possible since the leading term of $\sigma_n$ is $\epsilon_n$ and therefore any higher power will lead to higher orders of magnitude in the expansion. We can now solve the last equation for $\sigma_n$ and get

$$\sigma_n = \frac{\epsilon_n}{\sqrt{1 - \gamma_n \cdot \ddot{q}_n}} \sqrt{1 - \frac{\gamma_n \cdot \dddot{q}_n}{3} \frac{\epsilon_n}{(1 - \gamma_n \cdot \ddot{q}_n)^{3/2}} - \frac{\tilde{R}_n}{\epsilon_n^2}}.$$

Since we already showed that $\epsilon_n := \sqrt{-\gamma_n^2} \leq C_{\epsilon_n} \|x_n - q_{(\tau^*)}\|_{\mathbb{R}^4}$, we can choose a $k_2 \in \mathbb{N}$, such that for all $n \geq k_2$ we can expand the square root on the right side and obtain

$$\sigma_n = \frac{\epsilon_n}{\sqrt{1 - \gamma_n \cdot \ddot{q}_n}} \left( 1 - \frac{\gamma_n \cdot \dddot{q}_n}{6} \frac{\epsilon_n}{(1 - \gamma_n \cdot \ddot{q}_n)^{3/2}} \right) + \mathcal{O}(\epsilon_n^3).$$

2. Expansion until the fourth order:

   Because later we are going to need a higher accuracy, we derive one order more

$$q_{(\tau_n^-)} = q_{(\tau_n - \sigma_n)} = q_n - \sigma_n \dot{q}_n + \frac{\sigma_n^2}{2} \ddot{q}_n - \frac{\sigma_n^3}{6} \dddot{q}_n + \frac{\sigma_n^4}{24} \ddddot{q}_n + R_n.$$

   Now the remainder is of order $\mathcal{O}(\sigma_n^5)$. Inserting this expression in the equation of the light-cone and multiplying everything we finally get

$$\gamma_n^2 + \sigma_n^2 - \sigma_n^2 \gamma_n \cdot \ddot{q}_n + \frac{\sigma_n^3}{3} \gamma_n \cdot \dddot{q}_n - \frac{\sigma_n^4}{12} \ddot{q}_n^2 - \frac{\sigma_n^4}{12} \gamma_n \cdot \ddddot{q}_n + \tilde{R}_n = 0.$$

   Then we use our last, more accurate, result instead of $\sigma_n^3$ and $\sigma_n^4$. From the last equation we compute

$$\sigma_n^3 = \frac{\epsilon_n^3}{(1 - \gamma_n \cdot \ddot{q}_n)^{3/2}} - \frac{\gamma_n \cdot \dddot{q}_n}{2} \frac{\epsilon_n^4}{(1 - \gamma_n \cdot \ddot{q}_n)^3} + \mathcal{O}(\epsilon_n^5)$$

   and

$$\sigma_n^4 = \frac{\epsilon_n^4}{(1 - \gamma_n \cdot \ddot{q}_n)^2} + \mathcal{O}(\epsilon_n^5).$$

   Combining everything together we attain

$$\begin{aligned} 0 = &- \epsilon_n^2 + \sigma_n^2 (1 - \gamma_n \cdot \ddot{q}_n) + \frac{\gamma_n \cdot \dddot{q}_n}{3} \frac{\epsilon_n^3}{(1 - \gamma_n \cdot \ddot{q}_n)^{3/2}} - \frac{(\gamma_n \cdot \dddot{q}_n)^2}{6} \frac{\epsilon_n^4}{(1 - \gamma_n \cdot \ddot{q}_n)^3} - \\ & \frac{\ddot{q}_n^2}{12} \frac{\epsilon_n^4}{(1 - \gamma_n \cdot \ddot{q}_n)^2} - \frac{\gamma_n \cdot \ddddot{q}_n}{12} \frac{\epsilon_n^4}{(1 - \gamma_n \cdot \ddot{q}_n)^2} + \tilde{R}_n. \end{aligned}$$

   And if we rearrange the terms, we obtain

$$\begin{aligned} \sigma_n = \frac{\epsilon_n}{\sqrt{1 - \gamma_n \cdot \ddot{q}_n}} \cdot \Bigg[ 1 &- \frac{\gamma_n \cdot \dddot{q}_n}{3} \frac{\epsilon_n}{(1 - \gamma_n \cdot \ddot{q}_n)^{3/2}} + \frac{(\gamma_n \cdot \dddot{q}_n)^2}{6} \frac{\epsilon_n^2}{(1 - \gamma_n \cdot \ddot{q}_n)^3} + \\ & \frac{\ddot{q}_n^2}{12} \frac{\epsilon_n^2}{(1 - \gamma_n \cdot \ddot{q}_n)^2} + \frac{\gamma_n \cdot \ddddot{q}_n}{12} \frac{\epsilon_n^2}{(1 - \gamma_n \cdot \ddot{q}_n)^2} - \frac{\tilde{R}_n}{\epsilon_n^2} \Bigg]^{1/2} \end{aligned}$$

This expression can be Taylor-expanded again for all $n \geq k_2$ to get

$$\sigma_n = \frac{\epsilon_n}{\sqrt{1-\gamma_n \cdot \ddot{q}_n}} \cdot \left(1 - \frac{\gamma_n \cdot \dddot{q}_n}{6} \frac{\epsilon_n}{(1-\gamma_n \cdot \ddot{q}_n)^{3/2}} + \frac{(\gamma_n \cdot \dddot{q}_n)^2}{12} \frac{\epsilon_n^2}{(1-\gamma_n \cdot \ddot{q}_n)^3} + \frac{\ddot{q}_n^2}{24} \frac{\epsilon_n^2}{(1-\gamma_n \cdot \ddot{q}_n)^2} + \frac{\gamma_n \cdot \ddddot{q}_n}{24} \frac{\epsilon_n^2}{(1-\gamma_n \cdot \ddot{q}_n)^2} - \frac{(\gamma_n \cdot \dddot{q}_n)^2}{24} \frac{\epsilon_n^2}{(1-\gamma_n \cdot \ddot{q}_n)^3}\right) + \mathcal{O}(\epsilon_n^4).$$

The last thing we want to mention is that we also showed that for the components of $\gamma_n$ satisfy

$$|\gamma_n^\mu| \leq \frac{\sqrt{2}}{\sqrt{1-v_{max}^2}} \epsilon_n.$$

This means that for an $i$ and $l \in \{0, ..., 5\}$ it is true that

$$\left| \left(\gamma_n \cdot \frac{d^l q}{d\tau^l}\right) \epsilon_n^i \right| \leq \sup_{\tau_{(T-)}, \tau_{(T-)}} \left\| \frac{d^l q}{d\tau^l} \right\|_{max} \cdot \|\gamma_n\|_{max} \, \epsilon_n^i \leq \sup_{\tau_{(T-)}, \tau_{(T-)}} \left\| \frac{d^l q}{d\tau^l} \right\|_{max} \frac{\sqrt{2}}{\sqrt{1-v_{max}^2}} \epsilon_n^{i+1},$$

or in other words, this expressions are of an order less or equal to $\mathcal{O}(\epsilon_n^{i+1})$. Because $\gamma_n$ tends component-wise to zero, it is also true that at some $k_3 \in \mathbb{N}$, this last expression will be smaller than $\epsilon_n^i$. The exact order of these expressions is unknown but we define a remainder

$$\Sigma_n := \frac{\epsilon_n}{\sqrt{1-\gamma_n \cdot \ddot{q}_n}} \left( \frac{(\gamma_n \cdot \dddot{q}_n)^2}{12} \frac{\epsilon_n^2}{(1-\gamma_n \cdot \ddot{q}_n)^3} + \frac{\gamma_n \cdot \ddddot{q}_n}{24} \frac{\epsilon_n^2}{(1-\gamma_n \cdot \ddot{q}_n)^2} - \frac{(\gamma_n \cdot \dddot{q}_n)^2}{24} \frac{\epsilon_n^2}{(1-\gamma_n \cdot \ddot{q}_n)^3} \right) + \mathcal{O}(\epsilon_n^4)$$

which for all $n \geq \max\{k_1, k_2, k_3\}$ is of order superior as $\epsilon_n^3$ and rewriting our result we have

$$\sigma_n = \frac{\epsilon_n}{\sqrt{1-\gamma_n \cdot \ddot{q}_n}} \cdot \left(1 - \frac{\gamma_n \cdot \dddot{q}_n}{6} \frac{\epsilon_n}{(1-\gamma_n \cdot \ddot{q}_n)^{3/2}} + \frac{\ddot{q}_n^2}{24} \frac{\epsilon_n^2}{(1-\gamma_n \cdot \ddot{q}_n)^2}\right) + \Sigma_n.$$

The computations for $\delta_n$ are identical, the final result is then

$$\delta_n = \frac{\epsilon_n}{\sqrt{1-\gamma_n \cdot \ddot{q}_n}} \cdot \left(1 + \frac{\gamma_n \cdot \dddot{q}_n}{6} \frac{\epsilon_n}{(1-\gamma_n \cdot \ddot{q}_n)^{3/2}} + \frac{\ddot{q}_n^2}{24} \frac{\epsilon_n^2}{(1-\gamma_n \cdot \ddot{q}_n)^2}\right) + \Delta_n,$$

where the remainder $\Delta_n$ is defined in the same way as $\Sigma_n$. □

*Remark.* We emphasize here the need of the first five derivatives of $q$. Now we are less restrictive and do not need to impose $q \in \mathcal{C}^\infty$ as we demand that $q \in \mathcal{C}^5$ in a compact ball of radius $r$ centered at $q_{(\tau^*)}$.

**Corollary 4.11.** As a consequence of the last result, we obtain

$$\lim_{n \to \infty} \left(\frac{1}{\sigma_n} - \frac{1}{\delta_n}\right) = 0$$

*Proof.* We use a subtle change of notation and rewrite

$$\sigma_n = \frac{\epsilon_n}{\sqrt{1-\gamma_n \cdot \ddot{q}_n}} \cdot \left(1 - \frac{\gamma_n \cdot \dddot{q}_n}{6} \frac{\epsilon_n}{(1-\gamma_n \cdot \ddot{q}_n)^{3/2}} + \frac{\ddot{q}_n^2}{24} \frac{\epsilon_n^2}{(1-\gamma_n \cdot \ddot{q}_n)^2} + \Sigma_n\right)$$

and

$$\delta_n = \frac{\epsilon_n}{\sqrt{1-\gamma_n \cdot \ddot{q}_n}} \cdot \left(1 + \frac{\gamma_n \cdot \dddot{q}_n}{6} \frac{\epsilon_n}{(1-\gamma_n \cdot \ddot{q}_n)^{3/2}} + \frac{\ddot{q}_n^2}{24} \frac{\epsilon_n^2}{(1-\gamma_n \cdot \ddot{q}_n)^2} + \Delta_n\right)$$

where now $\Sigma_n$ and $\Delta_n$ are of an order greater than $\epsilon_n^2$. We compute then

$$\frac{1}{\sigma_n} - \frac{1}{\delta_n} = \frac{\sqrt{1-\gamma_n \cdot \ddot{q}_n}}{\epsilon_n}\left(\frac{1}{1 - \frac{\gamma_n \cdot \dddot{q}_n}{6}\frac{\epsilon_n}{(1-\gamma_n\cdot\ddot{q}_n)^{3/2}} + \frac{\ddot{q}_n^2}{24}\frac{\epsilon_n^2}{(1-\gamma_n\cdot\ddot{q}_n)^2} + \Sigma_n} - \frac{1}{1 + \frac{\gamma_n \cdot \dddot{q}_n}{6}\frac{\epsilon_n}{(1-\gamma_n\cdot\ddot{q}_n)^{3/2}} + \frac{\ddot{q}_n^2}{24}\frac{\epsilon_n^2}{(1-\gamma_n\cdot\ddot{q}_n)^2} + \Delta_n}\right)$$

$$= \frac{\sqrt{1-\gamma_n \cdot \ddot{q}_n}}{\epsilon_n}\left(\frac{\frac{\gamma_n \cdot \dddot{q}_n}{3}\frac{\epsilon_n}{(1-\gamma_n\cdot\ddot{q}_n)^{3/2}} + \Delta_n - \Sigma_n}{1 + \frac{\ddot{q}_n^2}{12}\frac{\epsilon_n^2}{(1-\gamma_n\cdot\ddot{q}_n)^2} + R_n}\right)$$

$$\Rightarrow \frac{1}{\sigma_n} - \frac{1}{\delta_n} = \frac{\sqrt{1-\gamma_n \cdot \ddot{q}_n}}{1 + \frac{\ddot{q}_n^2}{12}\frac{\epsilon_n^2}{(1-\gamma_n\cdot\ddot{q}_n)^2} + R_n}\left(\frac{\gamma_n \cdot \dddot{q}_n}{3(1-\gamma_n\cdot\ddot{q}_n)^{3/2}} + \frac{\Delta_n - \Sigma_n}{\epsilon_n}\right),$$

where in the last expression $R_n$ is a remainder of order greater that $\epsilon_n^2$ and $\frac{\Delta_n - \Sigma_n}{\epsilon_n}$ is a remainder of order greater than $\epsilon_n$. By choosing $n$ big enough, we can make the following estimation

$$\left|\frac{\sqrt{1-\gamma_n \cdot \ddot{q}_n}}{1 + \frac{\ddot{q}_n^2}{12}\frac{\epsilon_n^2}{(1-\gamma_n\cdot\ddot{q}_n)^2} + R_n}\right| \le 2\sqrt{2}$$

and therefore

$$\left|\frac{1}{\sigma_n} - \frac{1}{\delta_n}\right| \le 2\sqrt{2}\left(\frac{2^{3/2}}{3}|\gamma_n \cdot \dddot{q}_n| + \left|\frac{\Delta_n - \Sigma_n}{\epsilon_n}\right|\right).$$

This expression can also be bounded from above. To see this we notice first that

$$\left|\frac{\Delta_n - \Sigma_n}{\epsilon_n}\right| \le \epsilon_n$$

and

$$|\gamma_n \cdot \dddot{q}_n| = |\gamma_n^0 \dddot{q}_n^0 - \overrightarrow{\gamma}_n \cdot \overrightarrow{\dddot{q}}_n|$$

$$\le |\gamma_n^0 \dddot{q}_n^0| + \|\overrightarrow{\gamma}_n \cdot \overrightarrow{\dddot{q}}_n\|_{\mathbb{R}^3} \le \|\dddot{q}\|_{max}(|\gamma_n^0| + \|\overrightarrow{\gamma}_n\|_{\mathbb{R}^3}) \le 2\|\dddot{q}\|_{max}\|\gamma_n\|_{\mathbb{R}^4}, \tag{4.27}$$

where $\|\dddot{q}\|_{max}$ denotes the maximum norm of $\dddot{q}$ taken in from the map of the interval $[T^-, T^+]$ as given in equation (4.20). Taking this into account we get

$$\left|\frac{1}{\sigma_n} - \frac{1}{\delta_n}\right| \le 2\sqrt{2}\left(\frac{2^{5/2}}{3}\|\dddot{q}\|_{max}\|\gamma_n\|_{\mathbb{R}^4} + \epsilon_n\right) \le 2\sqrt{2}\left(\frac{2^{5/2}}{3}\|\dddot{q}\|_{max} + 1\right) C_1\|x_n - q_{(\tau^*)}\|_{\mathbb{R}^4}.$$

The last expression tends to zero in the limit $n \to \infty$. □

Before presenting the main result of this section, we derive a somewhat surprising finding that reveals a fundamental characteristic of the radiation field: the necessity of considering the Coulomb fields when discussing radiation.

**Corollary 4.12.** In contrast to our previous result, i.e. the convergence of $\frac{1}{\sigma_n} - \frac{1}{\delta_n}$, the expression

$$\frac{1}{\sigma_n^3} - \frac{1}{\delta_n^3} \tag{4.28}$$

is divergent if $\gamma_n \cdot \dddot{q}_n \neq 0$.

*Proof.* In order to show this important fact, we take our expansions of $\sigma_n$ and $\delta_n$ and compute

$$\sigma_n^3 = \frac{\epsilon_n^3}{(1-\gamma_n\cdot\ddot{q}_n)^{3/2}}\left(1 - \frac{\gamma_n \cdot \dddot{q}_n}{2}\frac{\epsilon_n}{(1-\gamma_n\cdot\ddot{q}_n)^{3/2}} + \frac{\ddot{q}_n^2}{8}\frac{\epsilon_n^2}{(1-\gamma_n\cdot\ddot{q}_n)^2} + \Sigma_n\right)$$

and

$$\delta_n^3 = \frac{\epsilon_n^3}{(1-\gamma_n\cdot\ddot{q}_n)^{3/2}}\left(1 + \frac{\gamma_n \cdot \dddot{q}_n}{2}\frac{\epsilon_n}{(1-\gamma_n\cdot\ddot{q}_n)^{3/2}} + \frac{\ddot{q}_n^2}{8}\frac{\epsilon_n^2}{(1-\gamma_n\cdot\ddot{q}_n)^2} + \Delta_n\right),$$

where $\Sigma_n$ and $\Delta_n$ are again remainder of order grater than $\epsilon_n^3$. Now we can compute

$$\frac{1}{\sigma_n^3}-\frac{1}{\delta_n^3}=\frac{(1-\gamma_n\cdot\ddot{q}_n)^{3/2}}{\epsilon_n^3}\left(\frac{1}{1-\frac{\gamma_n\cdot\dddot{q}_n}{2}\frac{\epsilon_n}{(1-\gamma_n\cdot\ddot{q}_n)^{3/2}}+\frac{\ddot{q}_n^2}{8}\frac{\epsilon_n^2}{(1-\gamma_n\cdot\ddot{q}_n)^2}+\Sigma_n}-\right.$$

$$\left.\frac{1}{1+\frac{\gamma_n\cdot\dddot{q}_n}{2}\frac{\epsilon_n}{(1-\gamma_n\cdot\ddot{q}_n)^{3/2}}+\frac{\ddot{q}_n^2}{8}\frac{\epsilon_n^2}{(1-\gamma_n\cdot\ddot{q}_n)^2}+\Delta_n}\right).$$

After simplifying the last expression we obtain

$$\frac{1}{\sigma_n^3}-\frac{1}{\delta_n^3}=\frac{(1-\gamma_n\cdot\ddot{q}_n)^{3/2}}{\epsilon_n^3(1+\frac{\ddot{q}_n^2}{4}\frac{\epsilon_n^2}{(1-\gamma_n\cdot\ddot{q}_n)^2}+R_n)}\left(\frac{\gamma_n\cdot\dddot{q}_n}{(1-\gamma_n\cdot\ddot{q}_n)^{3/2}}\epsilon_n+\Delta_n-\Sigma_n\right).$$

Here we see the appearance of the term

$$\frac{\gamma_n\cdot\dddot{q}_n}{\epsilon_n^2},$$

which can not be controlled by any result obtained since the components of $\gamma_n$ are of order $\epsilon_n$. $\square$

### 4.2.1 Proof of Theorem 4.2

We are now ready to present our main result:

*Proof.* Outside the world-line, $F^{\mu\nu}_{Rad}$ is well-defined, continuous and even differentiable. This comes from the fact that the Lienard-Wiechert fields are differentiable outside the world-line. We just have to take care about the limit of $q(\mathbb{R})$.

Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of points outside the world-line and let this sequence converge to the four-position $q_{(\tau^*)}$ with $\tau^*\in\mathbb{R}$. Furthermore, we choose to represent the points $x_n$ as in Lemma 4.3, such that all our previous results hold and before giving the retarded and advanced fields, we calculate the inverse of some powers of $\sigma_n$ and $\delta_n$. Using again a Taylor expansion, we obtain the following results

1. $$\frac{1}{\sigma_n}=\frac{\sqrt{1-\ddot{q}_n\cdot\gamma_n}}{\epsilon_n}\left(1+\frac{\dddot{q}_n\cdot\gamma_n}{6}\epsilon_n-\frac{\ddot{q}_n^2}{24}\epsilon_n^2+\Sigma_n\right)$$
2. $$\frac{1}{\sigma_n^2}=\frac{1-\ddot{q}_n\cdot\gamma_n}{\epsilon_n^2}\left(1+\frac{\dddot{q}_n\cdot\gamma_n}{3}\epsilon_n-\frac{\ddot{q}_n^2}{12}\epsilon_n^2+\Sigma_n\right)$$
3. $$\frac{1}{\sigma_n^3}=\frac{(1-\ddot{q}_n\cdot\gamma_n)^{3/2}}{\epsilon_n^3}\left(1+\frac{\dddot{q}_n\cdot\gamma_n}{2}\epsilon_n-\frac{\ddot{q}_n^2}{8}\epsilon_n^2+\Sigma_n\right)$$
4. $$\frac{1}{\delta_n}=\frac{\sqrt{1-\ddot{q}_n\cdot\gamma_n}}{\epsilon_n}\left(1-\frac{\dddot{q}_n\cdot\gamma_n}{6}\epsilon_n-\frac{\ddot{q}_n^2}{24}\epsilon_n^2+\Delta_n\right)$$
5. $$\frac{1}{\delta_n^2}=\frac{1-\ddot{q}_n\cdot\gamma_n}{\epsilon_n^2}\left(1-\frac{\dddot{q}_n\cdot\gamma_n}{3}\epsilon_n-\frac{\ddot{q}_n^2}{12}\epsilon_n^2+\Delta_n\right)$$
6. $$\frac{1}{\delta_n^3}=\frac{(1-\ddot{q}_n\cdot\gamma_n)^{3/2}}{\epsilon_n^3}\left(1-\frac{\dddot{q}_n\cdot\gamma_n}{2}\epsilon_n-\frac{\ddot{q}_n^2}{8}\epsilon_n^2+\Delta_n\right)$$

The reader should be aware of the abuse of notation made here. All remainder are called the same but in fact they are very different. We chose not to distinguish between them since all are going to be later packed in a new remainder. Inserting these results in the formula of the retarded field tensor, namely

$$F_n^{\mu\nu-}=\frac{e}{(1-\ddot{q}_n\cdot\gamma_n)^2}\left(\frac{\dot{q}_n^\nu\gamma_n^\mu}{\sigma_n^3}+\frac{\ddot{q}_n^\nu\dot{q}_n^\mu}{2\sigma_n}-\frac{\dddot{q}_n^\nu\gamma_n^\mu}{2\sigma_n}-\frac{\dddot{q}_n^\lambda\gamma_{n\lambda}}{2}\frac{\dot{q}_n^\nu\gamma_n^\mu}{\sigma_n^2}\right.$$
$$\left.-\frac{2}{3}\dddot{q}_n^\nu\dot{q}_n^\mu-(\mu\leftrightarrow\nu)+\mathcal{O}(\sigma_n)\right),$$

we obtain

$$
\begin{aligned}
F_n^{\mu\nu-} =& \frac{e}{(1-\ddot{q}_n\cdot\gamma_n)^2}\Bigg(\dot{q}_n^{\nu}\gamma_n^{\mu}\frac{(1-\ddot{q}_n\cdot\gamma_n)^{3/2}}{\epsilon_n^3}+\dot{q}_n^{\nu}\gamma_n^{\mu}\frac{(1-\ddot{q}_n\cdot\gamma_n)^{3/2}}{\epsilon_n^2}\frac{\dddot{q}_n\cdot\gamma_n}{2}-\\
&\dot{q}_n^{\nu}\gamma_n^{\mu}\frac{(1-\ddot{q}_n\cdot\gamma_n)^{3/2}}{\epsilon_n}\frac{\ddot{q}_n^2}{8}+\frac{(\ddot{q}_n^{\nu}\dot{q}_n^{\mu}-\dddot{q}_n^{\nu}\gamma_n^{\mu})}{2}\cdot\frac{1}{\sigma_n}-\dot{q}_n^{\nu}\gamma_n^{\mu}\frac{\gamma_n\cdot\dddot{q}_n}{2}\frac{1-\ddot{q}_n\cdot\gamma_n}{\epsilon_n^2}-\\
&\dot{q}_n^{\nu}\gamma_n^{\mu}\frac{(\gamma_n\cdot\dddot{q}_n)^2}{6}\frac{1-\ddot{q}_n\cdot\gamma_n}{\epsilon_n}+\dot{q}_n^{\nu}\gamma_n^{\mu}\frac{\gamma_n\cdot\dddot{q}_n}{2}(1-\ddot{q}_n\cdot\gamma_n)\frac{\ddot{q}_n^2}{12}-\frac{2}{3}\dddot{q}_n^{\nu}\dot{q}_n^{\mu}-\big(\mu\leftrightarrow\nu\big)+R_n^{\mu\nu}\Bigg).
\end{aligned}
$$

Here the remainder comes from various terms of order $\epsilon_n$ or higher, such that we can estimate it from above and write $R_n^{\mu\nu}\le C_2\epsilon_n$. The same can be done for the advanced field. Its Taylor-expansion is given by

$$
\begin{aligned}
F_n^{\mu\nu+}=\frac{e}{(1-\ddot{q}_n\cdot\gamma_n)^2}\Bigg(&\frac{\dot{q}_n^{\nu}\gamma_n^{\mu}}{\delta_n^3}+\frac{\ddot{q}_n^{\nu}\dot{q}_n^{\mu}}{2\delta_n}-\frac{\dddot{q}_n^{\nu}\gamma_n^{\mu}}{2\delta_n}+\frac{(\dddot{q}_n\cdot\gamma_n)}{2}\frac{\dot{q}_n^{\nu}\gamma_n^{\mu}}{\delta_n^2}\\
&+\frac{2}{3}\dddot{q}_n^{\nu}\dot{q}_n^{\mu}-\big(\mu\leftrightarrow\nu\big)+\mathcal{O}(\delta_n)\Bigg),
\end{aligned}
$$

and therefore we get in this case

$$
\begin{aligned}
F_n^{\mu\nu+} =& \frac{e}{(1-\ddot{q}_n\cdot\gamma_n)^2}\Bigg(\dot{q}_n^{\nu}\gamma_n^{\mu}\frac{(1-\ddot{q}_n\cdot\gamma_n)^{3/2}}{\epsilon_n^3}-\dot{q}_n^{\nu}\gamma_n^{\mu}\frac{(1-\ddot{q}_n\cdot\gamma_n)^{3/2}}{\epsilon_n^2}\frac{\dddot{q}_n\cdot\gamma_n}{2}-\\
&\dot{q}_n^{\nu}\gamma_n^{\mu}\frac{(1-\ddot{q}_n\cdot\gamma_n)^{3/2}}{\epsilon_n}\frac{\ddot{q}_n^2}{8}+\frac{(\ddot{q}_n^{\nu}\dot{q}_n^{\mu}-\dddot{q}_n^{\nu}\gamma_n^{\mu})}{2}\cdot\frac{1}{\delta_n}+\dot{q}_n^{\nu}\gamma_n^{\mu}\frac{\gamma_n\cdot\dddot{q}_n}{2}\frac{1-\ddot{q}_n\cdot\gamma_n}{\epsilon_n^2}-\\
&\dot{q}_n^{\nu}\gamma_n^{\mu}\frac{(\gamma_n\cdot\dddot{q}_n)^2}{6}\frac{1-\ddot{q}_n\cdot\gamma_n}{\epsilon_n}-\dot{q}_n^{\nu}\gamma_n^{\mu}\frac{\gamma_n\cdot\dddot{q}_n}{2}(1-\ddot{q}_n\cdot\gamma_n)\frac{\ddot{q}_n^2}{12}+\frac{2}{3}\dddot{q}_n^{\nu}\dot{q}_n^{\mu}-\big(\mu\leftrightarrow\nu\big)+R_n^{'\mu\nu}\Bigg),
\end{aligned}
$$

where in this case $R_n^{'\mu\nu}\le C_3\epsilon_n$. Now we subtract both expressions and obtain

$$
\begin{aligned}
F_n^{\mu\nu-}-F_n^{\mu\nu+}=\frac{e}{(1-\ddot{q}_n\cdot\gamma_n)^2}\Bigg(&\dot{q}_n^{\nu}\gamma_n^{\mu}\frac{(1-\ddot{q}_n\cdot\gamma_n)^{3/2}}{\epsilon_n^2}(\dddot{q}_n\cdot\gamma_n)+\frac{(\ddot{q}_n^{\nu}\dot{q}_n^{\mu}-\dddot{q}_n^{\nu}\gamma_n^{\mu})}{2}\left(\frac{1}{\sigma_n}-\frac{1}{\delta_n}\right)-\\
&\dot{q}_n^{\nu}\gamma_n^{\mu}(\gamma_n\cdot\dddot{q}_n)\frac{1-\ddot{q}_n\cdot\gamma_n}{\epsilon_n^2}+\dot{q}_n^{\nu}\gamma_n^{\mu}(\gamma_n\cdot\dddot{q}_n)(1-\ddot{q}_n\cdot\gamma_n)\frac{\ddot{q}_n^2}{12}-\\
&\frac{4}{3}\dddot{q}_n^{\nu}\dot{q}_n^{\mu}-\big(\mu\leftrightarrow\nu\big)+R_n^{\mu\nu}-R_n^{'\mu\nu}\Bigg).
\end{aligned}
$$

We can simplify this expression and rewrite it as

$$
\begin{aligned}
F_n^{\mu\nu-}-F_n^{\mu\nu+}=\frac{e}{(1-\ddot{q}_n\cdot\gamma_n)^2}\Bigg(&\dot{q}_n^{\nu}\gamma_n^{\mu}\frac{(1-\ddot{q}_n\cdot\gamma_n)(\dddot{q}_n\cdot\gamma_n)}{\epsilon_n^2}\left(\sqrt{1-\ddot{q}_n\cdot\gamma_n}-1\right)\\
&-\frac{4}{3}\dddot{q}_n^{\nu}\dot{q}_n^{\mu}-\big(\mu\leftrightarrow\nu\big)+\mathcal{R}_n^{\mu\nu}\Bigg),
\end{aligned}
\tag{4.29}
$$

where we have set

$$
\mathcal{R}_n^{\mu\nu}:=\frac{(\ddot{q}_n^{\nu}\dot{q}_n^{\mu}-\dddot{q}_n^{\nu}\gamma_n^{\mu})}{2}\left(\frac{1}{\sigma_n}-\frac{1}{\delta_n}\right)+\dot{q}_n^{\nu}\gamma_n^{\mu}(\gamma_n\cdot\dddot{q}_n)(1-\ddot{q}_n\cdot\gamma_n)\frac{\ddot{q}_n^2}{12}+R_n^{\mu\nu}-R_n^{'\mu\nu}.
$$

Notice that, thanks to the corollary 4.11, $\mathcal{R}_n^{\mu\nu}$ is a well defined remainder since every single term tends to zero in the limit $n\to\infty$. In order to get an upper boundary of the remainder, we follow the same path as at the beginning of this section (see section 4.2), we choose a ball of radius $r$ centered at $q_{(\tau^*)}$, defining then the interval $[T^-,T^+]$ (or $[\tau_{(T^-)},\tau_{(T^+)}]$) over which we can take the supremum of $q$ and its derivatives. The estimate is given by the following long inequality

$$
\begin{aligned}
|\mathcal{R}_n^{\mu\nu}| \leq& \frac{1}{2}(|\ddot{q}_n^\nu||\dot{q}_n^\mu| + |\dddot{q}_n^\nu||\gamma_n^\mu|)\left|\frac{1}{\sigma_n} - \frac{1}{\delta_n}\right| + |\dot{q}_n^\nu||\gamma_n^\mu||\gamma_n \cdot \dddot{q}_n||1-\ddot{q}_n\cdot\gamma_n|\frac{|\ddot{q}_n^2|}{12} + |R_n^{\mu\nu}| + |R_n'^{\mu\nu}| \\
\leq& \frac{1}{2}\left(\sup_{\tau\in[\tau_{(T-)},\tau_{(T+)}]}|\ddot{q}_{(\tau)}^\nu|\sup_{\tau\in[\tau_{(T-)},\tau_{(T+)}]}|\dot{q}_{(\tau)}^\mu| + \sup_{\tau\in[\tau_{(T-)},\tau_{(T+)}]}|\dddot{q}_{(\tau)}^\nu|\frac{\sqrt{2}}{\sqrt{1-v_{max}^2}}\epsilon_n\right)2\sqrt{2}C_1\times \\
&\left(1+\frac{2^{5/2}}{3}\sup_{\tau\in[\tau_{(T-)},\tau_{(T+)}]}\|\dddot{q}_{(\tau)}\|_{max}\right)\|x_n - q_{(\tau^*)}\|_{\mathbb{R}^4} + \\
&\sup_{\tau\in[\tau_{(T-)},\tau_{(T+)}]}|\dot{q}_{(\tau)}^\nu|\frac{\sqrt{2}}{\sqrt{1-v_{max}^2}}\epsilon_n\cdot 2\sup_{\tau\in[\tau_{(T-)},\tau_{(T+)}]}\|\dddot{q}_{(\tau)}\|_{max}\|\gamma_n\|_{\mathbb{R}^4}\times \\
&\left(1+2\sup_{\tau\in[\tau_{(T-)},\tau_{(T+)}]}\|\ddot{q}_{(\tau)}\|_{max}\|\gamma_n\|_{\mathbb{R}^4}\right)\frac{\sup_{\tau\in[\tau_{(T-)},\tau_{(T+)}]}|\ddot{q}_{(\tau)}^2|}{12} + C_2\epsilon_n + C_3\epsilon_n,
\end{aligned}
$$

where we have use in various instances the inequalities derived in this section (equations (4.19), (4.22) and (4.27)). If we use the fact that $\epsilon_n \leq C_1\|x_n - q_{(\tau^*)\|_{\mathbb{R}^4}}$ and choose $n$ big enough such that $\epsilon_n \leq 1$, then we can group all the terms in the last inequality as a single constant times the euclidean distance, i.e

$$
|\mathcal{R}_n^{\mu\nu}| \leq C^{\mu\nu}\|x_n - q_{(\tau^*)}\|_{\mathbb{R}^4},
$$

where the coefficients $C^{\mu\nu}$ are independent of $n$. Taking the maximum over $(\mu,\nu)\in\{0,...,3\}^2$, we can also define a global constant

$$
C := \max_{(\mu,\nu)\in\{0,\ldots,3\}^2} C^{\mu\nu}
$$

and state

$$
|\mathcal{R}_n^{\mu\nu}| \leq C\|x_n - q_{(\tau^*)}\|_{\mathbb{R}^4}.
$$

The only question that remains open is what happens with the fist term in equation (4.29) since it goes as $1/\epsilon_n^2$. We notice, that choosing $n$ big enough, we can expand the square root and obtain

$$
\sqrt{1-\ddot{q}_n\cdot\gamma_n} - 1 = -\frac{1}{2}\ddot{q}_n\cdot\gamma_n + \mathcal{O}((\ddot{q}_n\cdot\gamma_n)^2).
$$

This allows us to show that the fist term in our expression is in fact convergent (recall equation (4.22)) and we get

$$
\begin{aligned}
&\left|\dot{q}_n^\nu\gamma_n^\mu\frac{(1-\ddot{q}_n\cdot\gamma_n)(\dddot{q}_n\cdot\gamma_n)}{\epsilon_n^2}\left(\sqrt{1-\ddot{q}_n\cdot\gamma_n}-1\right)\right| \\
&\leq \frac{(2)^{5/2}}{(1-v_{max}^2)^{3/2}}\sup_{\tau\in[\tau_{(T-)},\tau_{(T+)}]}\|\dot{q}_{(\tau)}\|_{max}\sup_{\tau\in[\tau_{(T-)},\tau_{(T+)}]}\|\ddot{q}_{(\tau)}\|_{max}\times \\
&\sup_{\tau\in[\tau_{(T-)},\tau_{(T+)}]}\|\dddot{q}_{(\tau)}\|_{max}\left(1+2\sup_{\tau\in[\tau_{(T-)},\tau_{(T+)}]}\|\ddot{q}_{(\tau)}\|_{max}\frac{\sqrt{2}}{\sqrt{1-v_{max}^2}}\epsilon_n\right)\epsilon_n \\
&\leq \tilde{C}\|x_n - q_{(\tau^*)}\|_{\mathbb{R}^4}.
\end{aligned}
$$

With this last inequality, we have shown that it is in fact possible to control every term in the expansion of the radiation field. This allows us to compute the desired result

$$
\begin{aligned}
&\left|F^{\mu\nu}_{Rad(q_{(\tau^*)})} - F^{\mu\nu}_{Rad(x_n)}\right| = \left|\frac{4e}{3}\left(\dddot{q}^\mu_{(\tau)}\dot{q}^\nu_{(\tau)} - \dddot{q}^\nu_{(\tau)}\dot{q}^\mu_{(\tau)}\right) - \right. \\
&\left.\frac{e}{(1-\ddot{q}_n\cdot\gamma_n)^2}\left(\dot{q}_n^\nu\gamma_n^\mu\frac{(1-\ddot{q}_n\cdot\gamma_n)(\dddot{q}_n\cdot\gamma_n)}{\epsilon_n^2}\left(\sqrt{1-\ddot{q}_n\cdot\gamma_n}-1\right) - \frac{4}{3}\dddot{q}_n^\nu\dot{q}_n^\mu - (\mu\leftrightarrow\nu) + \mathcal{R}_n^{\mu\nu}\right)\right| \\
\leq& \frac{4e}{3}\left|\left(\dddot{q}^\mu_{(\tau)}\dot{q}^\nu_{(\tau)} - \dddot{q}^\nu_{(\tau)}\dot{q}^\mu_{(\tau)}\right) - \frac{1}{(1-\ddot{q}_{(\tau_n)}\cdot\gamma_{(\tau_n)})^2}\left(\dddot{q}^\mu_{(\tau_n)}\dot{q}^\nu_{(\tau_n)} - \dddot{q}^\nu_{(\tau_n)}\dot{q}^\mu_{(\tau_n)}\right)\right| + \tilde{C}\epsilon_n + |\mathcal{R}_n^{\mu\nu}| \\
\leq& \frac{4e}{3}\left|\left(\dddot{q}^\mu_{(\tau)}\dot{q}^\nu_{(\tau)} - \dddot{q}^\nu_{(\tau)}\dot{q}^\mu_{(\tau)}\right) - \frac{1}{(1-\ddot{q}_{(\tau_n)}\cdot\gamma_{(\tau_n)})^2}\left(\dddot{q}^\mu_{(\tau_n)}\dot{q}^\nu_{(\tau_n)} - \dddot{q}^\nu_{(\tau_n)}\dot{q}^\mu_{(\tau_n)}\right)\right| + \\
&(\tilde{C}+C)\|x_n - q_{(\tau^*)}\|_{\mathbb{R}^4} \xrightarrow{n\to\infty} 0
\end{aligned}
$$

where in the last step we used the explicit estimation of the remainder. $\square$

# Chapter 5

# Conclusions

We have taken an initial step in the careful treatment of radiation reaction, where we proved that $\overline{F}^{\mu\nu}_{Rad}$, as given in Definition 4.1, is the continuous extension of the radiation field and supplied an upper bound of the error term (4.5).

The next step is to study the system of equations

$$m\ddot{q}^{\mu}_{(\tau)} = \dot{q}^{\nu}_{(\tau)} F^{\mu}_{(q_{(\tau)})\nu}, \tag{5.1}$$

$$\Box A^{\mu}_{(x)} = 0, \tag{5.2}$$

$$F^{\mu\nu}_{(x)} = \partial^{\mu} A^{\nu}_{(x)} - \partial^{\nu} A^{\mu}_{(x)}, \tag{5.3}$$

for an initial value $F^{\mu\nu}_{(x_0)} = \overline{F}^{\mu\nu}_{Rad(x_0)}$. The understanding of $\overline{F}^{\mu\nu}_{Rad(x)}$ is a prerequisite if one wishes to study in detail the equations above. A solution of this system will also be a solution of Dirac's original formula (G.1) but the approach proposed here needs as initial values $q_{(\tau_0)}$, $\dot{q}_{(\tau_0)}$ and $A^{\mu}_{(x_0)}$ instead of $q_{(\tau_0)}$, $\dot{q}_{(\tau_0)}$ and $\ddot{q}_{(\tau_0)}$ as Dirac needed. In this sense, the research in this topic may provide new light about the dynamics with radiation reaction.

# Appendix A

# Computations for the Radiation Fields at Constant Velocity

## A.1 Proof that the Two "Big Terms" are Equal

We begin by calculating the following subtraction

$$\begin{aligned}&\left(\sqrt{1-2v\alpha\sqrt{1-v^2(\beta^2+\gamma^2)}+v^2(1-2(\beta^2+\gamma^2))}+v\alpha-v^2\sqrt{1-v^2(\beta^2+\gamma^2)}\right)^3-\\&\left(\sqrt{1+2v\alpha\sqrt{1-v^2(\beta^2+\gamma^2)}+v^2(1-2(\beta^2+\gamma^2))}-v\alpha-v^2\sqrt{1-v^2(\beta^2+\gamma^2)}\right)^3\end{aligned} \tag{A.1}$$

$$\begin{aligned}=&2\alpha^3v^3+3\alpha^2v^2\sqrt{1-2v\alpha\sqrt{1-v^2(\beta^2+\gamma^2)}+v^2(1-2(\beta^2+\gamma^2))}-\\
&3\alpha^2v^2\sqrt{1+2v\alpha\sqrt{1-v^2(\beta^2+\gamma^2)}+v^2(1-2(\beta^2+\gamma^2))}-6\alpha\beta^2v^7-12\alpha\beta^2v^5-\\
&12\alpha\beta^2v^3-6\alpha\gamma^2v^7-12\alpha\gamma^2v^5-12\alpha\gamma^2v^3+6\alpha v^5-\\
&6\alpha v^3\sqrt{1-v^2(\beta^2+\gamma^2)}\sqrt{1-2v\alpha\sqrt{1-v^2(\beta^2+\gamma^2)}+v^2(1-2(\beta^2+\gamma^2))}-\\
&6\alpha v^3\sqrt{1-v^2(\beta^2+\gamma^2)}\sqrt{1+2v\alpha\sqrt{1-v^2(\beta^2+\gamma^2)}+v^2(1-2(\beta^2+\gamma^2))}+18\alpha v^3-\\
&2\alpha v\sqrt{1-v^2(\beta^2+\gamma^2)}\sqrt{1-2v\alpha\sqrt{1-v^2(\beta^2+\gamma^2)}+v^2(1-2(\beta^2+\gamma^2))}-\\
&2\alpha v\sqrt{1-v^2(\beta^2+\gamma^2)}\sqrt{1+2v\alpha\sqrt{1-v^2(\beta^2+\gamma^2)}+v^2(1-2(\beta^2+\gamma^2))}+6\alpha v-\\
&3\beta^2v^6\sqrt{1-2v\alpha\sqrt{1-v^2(\beta^2+\gamma^2)}+v^2(1-2(\beta^2+\gamma^2))}+\\
&3\beta^2v^6\sqrt{1+2v\alpha\sqrt{1-v^2(\beta^2+\gamma^2)}+v^2(1-2(\beta^2+\gamma^2))}-\\
&2\beta^2v^2\sqrt{1-2v\alpha\sqrt{1-v^2(\beta^2+\gamma^2)}+v^2(1-2(\beta^2+\gamma^2))}+\\
&2\beta^2v^2\sqrt{1+2v\alpha\sqrt{1-v^2(\beta^2+\gamma^2)}+v^2(1-2(\beta^2+\gamma^2))}-\\
&3\gamma^2v^6\sqrt{1-2v\alpha\sqrt{1-v^2(\beta^2+\gamma^2)}+v^2(1-2(\beta^2+\gamma^2))}+\\
&3\gamma^2v^6\sqrt{1+2v\alpha\sqrt{1-v^2(\beta^2+\gamma^2)}+v^2(1-2(\beta^2+\gamma^2))}-\\
&2\gamma^2v^2\sqrt{1-2v\alpha\sqrt{1-v^2(\beta^2+\gamma^2)}+v^2(1-2(\beta^2+\gamma^2))}+\\
&2\gamma^2v^2\sqrt{1+2v\alpha\sqrt{1-v^2(\beta^2+\gamma^2)}+v^2(1-2(\beta^2+\gamma^2))}+\\
&3v^4\sqrt{1-2v\alpha\sqrt{1-v^2(\beta^2+\gamma^2)}+v^2(1-2(\beta^2+\gamma^2))}-\\
&3v^4\sqrt{1+2v\alpha\sqrt{1-v^2(\beta^2+\gamma^2)}+v^2(1-2(\beta^2+\gamma^2))}+\end{aligned}$$

$$
\begin{aligned}
&+ v^2\sqrt{1-2v\alpha\sqrt{1-v^2(\beta^2+\gamma^2)}+v^2(1-2(\beta^2+\gamma^2))}-\\
&v^2\sqrt{1+2v\alpha\sqrt{1-v^2(\beta^2+\gamma^2)}+v^2(1-2(\beta^2+\gamma^2))}+\\
&\sqrt{1-2v\alpha\sqrt{1-v^2(\beta^2+\gamma^2)}+v^2(1-2(\beta^2+\gamma^2))}-\\
&\sqrt{1+2v\alpha\sqrt{1-v^2(\beta^2+\gamma^2)}+v^2(1-2(\beta^2+\gamma^2))}
\end{aligned}
$$

Now we rename the big square roots

$$
\begin{aligned}
D &:= \sqrt{1-2v\alpha\sqrt{1-v^2(\beta^2+\gamma^2)}+v^2(1-2(\beta^2+\gamma^2))}\\
J &:= \sqrt{1+2v\alpha\sqrt{1-v^2(\beta^2+\gamma^2)}+v^2(1-2(\beta^2+\gamma^2))}
\end{aligned}
\tag{A.2}
$$

and get the following equation

$$
\begin{aligned}
&\left(D+v\alpha-v^2\sqrt{1-v^2(\beta^2+\gamma^2)}\right)^3-\left(J-v\alpha-v^2\sqrt{1-v^2(\beta^2+\gamma^2)}\right)^3\\
&=2\alpha^3v^3+3\alpha^2v^2(D-J)-6\alpha\beta^2v^7-12\alpha\beta^2v^5-12\alpha\beta^2v^3-6\alpha\gamma^2v^7\\
&-12\alpha\gamma^2v^5-12\alpha\gamma^2v^3+6\alpha v^5-6\alpha v^3\sqrt{1-v^2(\beta^2+\gamma^2)}(D+J)\\
&-2\alpha v\sqrt{1-v^2(\beta^2+\gamma^2)}(D+J)+18\alpha v^3+6\alpha v-3\beta^2v^6(D-J)\\
&-2\beta^2v^2(D-J)-3\gamma^2v^6(D-J)-2\gamma^2v^2(D-J)+3v^4(D-J)\\
&+v^2(D-J)+(D-J).
\end{aligned}
\tag{A.3}
$$

In order to simplify this expression, we need to compute $D \pm J$ and a trick may help us here

$$
\begin{aligned}
(D\pm J)^2 &= 1-2\alpha v\sqrt{1-v^2(\beta^2+\gamma^2)}+v^2(1-2(\beta^2+\gamma^2))\\
&\pm2\sqrt{(1-2v\alpha\sqrt{1-v^2(\beta^2+\gamma^2)}+v^2(1-2(\beta^2+\gamma^2)))(1+2v\alpha\sqrt{1-v^2(\beta^2+\gamma^2)}+v^2(1-2(\beta^2+\gamma^2)))}\\
&+1+2\alpha v\sqrt{1-v^2(\beta^2+\gamma^2)}+v^2(1-2(\beta^2+\gamma^2))\\
&=2+2v^2(1-2(\beta^2+\gamma^2))\\
&\pm2\sqrt{1+2v^2-4(\alpha^2+\beta^2+\gamma^2)v^2+v^4+4v^4(\beta^2+\gamma^2)(\alpha^2-1)+4v^4(\beta^2+\gamma^2)^2}\\
&=2+2v^2-4v^2(\beta^2+\gamma^2)\pm2\sqrt{1-2v^2+v^4}\\
&=2+2v^2-4v^2(\beta^2+\gamma^2)\pm2(1-v^2),
\end{aligned}
$$

where for this calculation we have use the fact that $\alpha^2+\beta^2+\gamma^2=1$. Simplifying yields

$$
(D+J)^2=4(1-v^2(\beta^2+\gamma^2)) \Rightarrow D+J=2\sqrt{1-v^2(\beta^2+\gamma^2)}
\tag{A.4}
$$

with + as the only possible solution since the sum of two square roots is always positive. We also get

$$
(D-J)^2=4v^2(1-\beta^2+\gamma^2)=4\alpha^2v^2 \Rightarrow D-J=\pm2\alpha v
\tag{A.5}
$$

To decide which sign is the right one, we look at the definitions of $D$ and $J$ and notice that $J > D$ so that we can state that the minus sign is to be chosen. Putting everything together we obtain

$$
\begin{aligned}
&\left(D+v\alpha-v^2\sqrt{1-v^2(\beta^2+\gamma^2)}\right)^3-\left(J-v\alpha-v^2\sqrt{1-v^2(\beta^2+\gamma^2)}\right)^3\\
=&2\alpha^3v^3-6\alpha^3v^3-6\alpha\beta^2v^7-12\alpha\beta^2v^5-12\alpha\beta^2v^3-6\alpha\gamma^2v^7\\
&-12\alpha\gamma^2v^5-12\alpha\gamma^2v^3+6\alpha v^5-12\alpha v^3+12\alpha\beta^2v^5+12\alpha\gamma^2v^5\\
&-4\alpha v+4\alpha\beta^2v^3+4\alpha\gamma^2v^3+18\alpha v^3+6\alpha v+6\alpha\beta^2v^7+4\alpha\beta^2v^3\\
&+6\alpha\gamma^2v^7+4\alpha\gamma^2v^3-6\alpha v^5-2\alpha v^3-2\alpha v=0
\end{aligned}
\tag{A.6}
$$

We summarize this appendix as

## Main Result A.1: The Two "Big Terms" are Equal

$$
\begin{aligned}
&\left(\sqrt{1-2v\alpha\sqrt{1-v^2(\beta^2+\gamma^2)}+v^2(1-2(\beta^2+\gamma^2))}+v\alpha-v^2\sqrt{1-v^2(\beta^2+\gamma^2)}\right)^3\\
&-\left(\sqrt{1+2v\alpha\sqrt{1-v^2(\beta^2+\gamma^2)}+v^2(1-2(\beta^2+\gamma^2))}-v\alpha-v^2\sqrt{1-v^2(\beta^2+\gamma^2)}\right)^3\\
&=0
\end{aligned}
\tag{A.7}
$$

# Appendix B

# Proofs for the General Taylor Expansion

## B.1 Missing computation of the expansion of $C_\epsilon$

To show that the error $R_\epsilon$ is small we begin with equation (3.18) and use the fact that

$$C_\epsilon = C_0 + \epsilon C_1 + R_\epsilon,$$

where $C_0$ is given as

$$C_0 = \frac{v_{||} + \sqrt{1 - v_\perp^2}}{1 - v^2} \tag{B.1}$$

and $C_1$ is

$$C_1 = -\frac{C_0^2}{2} \frac{\overrightarrow{v} \cdot \overrightarrow{a} C_0 + a_{||}}{(1 - v^2)C_0 - v_{||}} \tag{B.2}$$

Now we insert $C_\epsilon$ in the expression (3.18) and obtain

$$\begin{aligned}
&(1 - v^2)C_\epsilon^2 - 2v_{||}C_\epsilon - 1 + \epsilon\left(a_{||}C_\epsilon^2 + \overrightarrow{v} \cdot \overrightarrow{a} C_\epsilon^3 - \frac{\epsilon}{4}\overrightarrow{a}^2 C_\epsilon^4\right) \\
&= (1 - v^2)(C_0 + \epsilon C_1 + R_\epsilon)^2 - 2v_{||}(C_0 + \epsilon C_1 + R_\epsilon) - 1 \\
&+ \epsilon\left(a_{||}(C_0 + \epsilon C_1 + R_\epsilon)^2 + \overrightarrow{v} \cdot \overrightarrow{a}(C_0 + \epsilon C_1 + R_\epsilon)^3 - \frac{\epsilon}{4}\overrightarrow{a}^2(C_0 + \epsilon C_1 + R_\epsilon)^4\right) \\
&= \left[(1 - v^2)C_0^2 - 2v_{||}C_0 - 1\right] + (1 - v^2)(2C_0(\epsilon C_1 + R_\epsilon) + (\epsilon C_1 + R_\epsilon)^2) - 2v_{||}(\epsilon C_1 + R_\epsilon) \\
&+ \epsilon\Bigg(a_{||}(C_0^2 + 2C_0(\epsilon C_1 + R_\epsilon) + (\epsilon C_1 + R_\epsilon)^2) \\
&+ \overrightarrow{v} \cdot \overrightarrow{a}(C_0^3 + 3C_0^2(\epsilon C_1 + R_\epsilon) + 3C_0(\epsilon C_1 + R_\epsilon)^2 + (\epsilon C_1 + R_\epsilon)^3) \\
&- \frac{\epsilon}{4}a^2(C_0^4 + 4C_0^3(\epsilon C_1 + R_\epsilon) + 6C_0^2(\epsilon C_1 + R_\epsilon)^2 + 4C_0(\epsilon C_1 + R_\epsilon)^3 + (\epsilon C_1 + R_\epsilon)^4)\Bigg).
\end{aligned} \tag{B.3}$$

The first term in square brackets of the last equation becomes zero when we insert equation (B.1). If we rearrange further we get

$$\begin{aligned}
&= \epsilon\left[2(1 - v^2)C_0C_1 - 2v_{||}C_1 + a_{||}C_0^2 + \overrightarrow{v} \cdot \overrightarrow{a} C_0^3\right] + 2(1 - v^2)C_0R_\epsilon + (1 - v^2)(\epsilon C_1 + R_\epsilon)^2 \\
&+ \epsilon\Bigg(a_{||}(2C_0(\epsilon C_1 + R_\epsilon) + (\epsilon C_1 + R_\epsilon)^2) + \overrightarrow{v} \cdot \overrightarrow{a}(3C_0^2(\epsilon C_1 + R_\epsilon) + 3C_0(\epsilon C_1 + R_\epsilon)^2 + (\epsilon C_1 + R_\epsilon)^3) \\
&- \frac{\epsilon}{4}a^2(C_0^4 + 4C_0^3(\epsilon C_1 + R_\epsilon) + 6C_0^2(\epsilon C_1 + R_\epsilon)^2 + 4C_0(\epsilon C_1 + R_\epsilon)^3 + (\epsilon C_1 + R_\epsilon)^4)\Bigg).
\end{aligned} \tag{B.4}$$

Here we see again that the first term in square brackets is zero if we insert equation (B.2). At the end we obtain the desired result.

# Appendix C

# Useful Identities of the Coefficients $C_i$ and $B_i$

In this appendix we list a number of useful relations between the coefficients $C_0$, $B_0$, $C_1$, $B_1$, $C_2$ and $B_2$. These equations are widely used when studying the fields of a particle moving with a given trajectory. The coefficients are

$$C_0 = \frac{v_\parallel + \sqrt{1-v_\perp^2}}{1-v^2}, \qquad B_0 = \frac{-v_\parallel + \sqrt{1-v_\perp^2}}{1-v^2},$$

$$C_1 = -\frac{C_0^2}{2}\frac{\overrightarrow{v}\cdot\overrightarrow{a}C_0 + a_\parallel}{\sqrt{1-v_\perp^2}}, \quad B_1 = \frac{B_0^2}{2}\frac{\overrightarrow{v}\cdot\overrightarrow{a}B_0 - a_\parallel}{\sqrt{1-v_\perp^2}},$$

$$C_2 = \frac{\frac{a^2}{4}C_0^4 - 3\overrightarrow{v}\cdot\overrightarrow{a}C_0^2C_1 - 2a_\parallel C_0C_1 - (1-v^2)C_1^2}{2\sqrt{1-v_\perp^2}},$$

$$B_2 = \frac{\frac{a^2}{4}B_0^4 + 3\overrightarrow{v}\cdot\overrightarrow{a}B_0^2B_1 - 2a_\parallel B_0B_1 - (1-v^2)B_1^2}{2\sqrt{1-v_\perp^2}}.$$

Therefore we obtain

1. $$C_0 + B_0 = \frac{2\sqrt{1-v_\perp^2}}{1-v^2}$$

2. $$C_0 - B_0 = \frac{2v_\parallel}{1-v^2}$$

3. $$C_0^2 + B_0^2 = \frac{2\left(1+v_\parallel^2 - v_\perp^2\right)}{\left(1-v^2\right)^2}$$

4. $$C_0^2 - B_0^2 = \frac{4v_\parallel\sqrt{1-v_\perp^2}}{\left(1-v^2\right)^2}$$

5. $$C_0^3 + B_0^3 = \frac{2\left(1-v_\perp^2\right)^{3/2} + 6v_\parallel^2\sqrt{1-v_\perp^2}}{\left(1-v^2\right)^3}$$

6. $$C_0^3 - B_0^3 = \frac{2v_\parallel^3 + 6v_\parallel(1-v_\perp^2)}{\left(1-v^2\right)^3}$$

7. $$C_0^4 + B_0^4 = \frac{2v_\parallel^4 + 12v_\parallel^2(1-v_\perp^2) + 2(1-v_\perp^2)^2}{\left(1-v^2\right)^4}$$

8.

$$C_0^4 - B_0^4 = \frac{8v_\parallel^3(1-v_\perp^2)^{1/2} + 8v_\parallel(1-v_\perp^2)^{3/2}}{(1-v^2)^4}$$

9.

$$C_1 + B_1 = \frac{\overrightarrow{v}\cdot\overrightarrow{a}(B_0^3 - C_0^3) - a_\parallel(C_0^2 + B_0^2)}{2\sqrt{1-v_\perp^2}}$$

10.

$$C_1 - B_1 = \frac{-\overrightarrow{v}\cdot\overrightarrow{a}(B_0^3 + C_0^3) + a_\parallel(B_0^2 - C_0^2)}{2\sqrt{1-v_\perp^2}}$$

11.

$$C_2 - B_2 = \frac{\frac{a^2}{4}(C_0^4 - B_0^4) - 3\overrightarrow{v}\cdot\overrightarrow{a}(C_0^2C_1 + B_0^2B_1) + 2a_\parallel(B_0B_1 - C_0C_1) + (1-v^2)(B_1^2 - C_1^2)}{2\sqrt{1-v_\perp^2}}$$

# Appendix D

# About the Differentiability of the Retarded/Advanced Times

The goal of this appendix is to study the behavior of the remainder term in equation (2.15).Using the error term formula given by Lagrange we can write for a $\lambda$ in $x$ and $x+\epsilon$

$$\begin{aligned} R = \frac{1}{2}\Bigg[ &\left(t^{\pm}_{(t,x+\epsilon,y,z)} - t^{\pm}_{(t,x,y,z)}\right)^2 \left(\frac{\partial^2}{\partial s^2} \left\| \begin{pmatrix} x \\ y \\ z \end{pmatrix} - \overrightarrow{q}_{(s)} \right\|_{\mathbb{R}^3}\right)\Bigg|_{s=\lambda} \\ &+(x+\epsilon-x)\left(t^{\pm}_{(t,x+\epsilon,y,z)} - t^{\pm}_{(t,x,y,z)}\right)\left(\frac{\partial^2}{\partial s \partial s'} \left\| \begin{pmatrix} s' \\ y \\ z \end{pmatrix} - \overrightarrow{q}_{(s)} \right\|_{\mathbb{R}^3}\right)\Bigg|_{(t^{\pm},s')=(\lambda,\lambda')} \\ &+(x+\epsilon-x)^2 \left(\frac{\partial^2}{\partial s'^2} \left\| \begin{pmatrix} s' \\ y \\ z \end{pmatrix} - \overrightarrow{q}_{(t^{\pm}_{(x)})} \right\|_{\mathbb{R}^3}\right)\Bigg|_{s'=\lambda'}\Bigg]. \end{aligned} \tag{D.1}$$

The partial derivatives can be computed explicitly, so we obtain

$$C^1_\epsilon := \frac{\partial^2}{\partial s^2} \left\| \begin{pmatrix} x \\ y \\ z \end{pmatrix} - \overrightarrow{q}_{(s)} \right\|_{\mathbb{R}^3} = -\overrightarrow{a}_{(s)} \cdot \overrightarrow{n}_{(s,\overrightarrow{x})} + \frac{\overrightarrow{v}^2_{(s)} - (\overrightarrow{v}_{(s)} \cdot \overrightarrow{n}_{(s,\overrightarrow{x})})^2}{\|\overrightarrow{x} - \overrightarrow{q}_{(s)}\|_{\mathbb{R}^3}},$$

$$C^2_\epsilon := \frac{\partial^2}{\partial s \partial s'} \left\| \begin{pmatrix} s' \\ y \\ z \end{pmatrix} - \overrightarrow{q}_{(s)} \right\|_{\mathbb{R}^3} = \frac{-v^1_{(s)} + (\overrightarrow{v}_{(s)} \cdot \overrightarrow{n}_{(s,\overrightarrow{x})}) n^1_{(s,\overrightarrow{x})}}{\|\overrightarrow{x} - \overrightarrow{q}_{(s)}\|_{\mathbb{R}^3}}$$

and

$$C^3_\epsilon := \frac{\partial^2}{\partial s'^2} \left\| \begin{pmatrix} s' \\ y \\ z \end{pmatrix} - \overrightarrow{q}_{(t^{\pm}_{(x)})} \right\|_{\mathbb{R}^3} = \frac{1 - (n^1_{(s,\overrightarrow{x})})^2}{\|\overrightarrow{x} - \overrightarrow{q}_{(s)}\|_{\mathbb{R}^3}}.$$

All these three derivatives can be bounded from above, namely

$$|C^1_\epsilon| \leq a_{max} + \frac{2v^2_{max}}{\min_{s \in \left(t^{\pm}_{(t,\overrightarrow{x})}, t^{\pm}_{(t+\epsilon,\overrightarrow{x})}\right)} \|\overrightarrow{x} - \overrightarrow{q}_{(s)}\|_{\mathbb{R}^3}},$$

$$|C^2_\epsilon| \leq \frac{2v_{max}}{\min_{s \in \left(t^{\pm}_{(t,\overrightarrow{x})}, t^{\pm}_{(t+\epsilon,\overrightarrow{x})}\right)} \|\overrightarrow{x} - \overrightarrow{q}_{(s)}\|_{\mathbb{R}^3}}$$

and

$$|C^3_\epsilon| \leq \frac{2}{\min_{s \in \left(t^{\pm}_{(t,\overrightarrow{x})}, t^{\pm}_{(t+\epsilon,\overrightarrow{x})}\right)} \|\overrightarrow{x} - \overrightarrow{q}_{(s)}\|_{\mathbb{R}^3}}.$$

We rewrite now the remainder as

$$R = \frac{1}{2}\left[\left(t^{\pm}_{(t,x+\epsilon,y,z)} - t^{\pm}_{(t,x,y,z)}\right)^2 C^1_\epsilon + \epsilon \left(t^{\pm}_{(t,x+\epsilon,y,z)} - t^{\pm}_{(t,x,y,z)}\right) C^2_\epsilon + \epsilon^2 C^3_\epsilon\right].$$

Now we can divide this expression by $\epsilon$ and use the equation (2.15) to solve for $R/\epsilon$. We get

$$\frac{R}{\epsilon} = \frac{1}{2}\left[\left(t^{\pm}_{(t,x+\epsilon,y,z)} - t^{\pm}_{(t,x,y,z)}\right) C^1_\epsilon \frac{\left(t^{\pm}_{(t,x+\epsilon,y,z)} - t^{\pm}_{(t,x,y,z)}\right)}{\epsilon} + \left(t^{\pm}_{(t,x+\epsilon,y,z)} - t^{\pm}_{(t,x,y,z)}\right) C^2_\epsilon + \epsilon C^3_\epsilon\right].$$

Therefore we are able to state

$$\frac{R}{\epsilon}\left(2 \mp \frac{C^1_\epsilon\left(t^{\pm}_{(t,x+\epsilon,y,z)} - t^{\pm}_{(t,x,y,z)}\right)}{1 \pm \overrightarrow{v}_{(t^{\pm}_{(t,\overrightarrow{x})})} \cdot \overrightarrow{n}_{(t^{\pm}_{(t,\overrightarrow{x})},\overrightarrow{x})}}\right) =$$

$$\frac{\pm n^1_{(t^{\pm}_{(t,\overrightarrow{x})},\overrightarrow{x})}\left(t^{\pm}_{(t,x+\epsilon,y,z)} - t^{\pm}_{(t,x,y,z)}\right) C^1_\epsilon}{1 \pm \overrightarrow{v}_{(t^{\pm}_{(t,\overrightarrow{x})})} \cdot \overrightarrow{n}_{(t^{\pm}_{(t,\overrightarrow{x})},\overrightarrow{x})}} + \left(t^{\pm}_{(t,x+\epsilon,y,z)} - t^{\pm}_{(t,x,y,z)}\right) C^2_\epsilon + \epsilon C^3_\epsilon.$$

Taking the limit in both sides of the equation yields

$$\lim_{\epsilon\to 0}\frac{2R}{\epsilon} = 0. \tag{D.2}$$

# Appendix E

# Deriving the Formulas of the Field Tensor

In this Appendix we want to derive the formulas (4.1) and (4.2). First, we show that it is possible to rewrite the four-potential in a very useful form and for this purpose, we recall the explicit computation made in the example of the first chapter, see example 1.1. We derived there the following formulas for the four-potentials of a point particle

$$A^{\mu-}_{(t,\overrightarrow{x})} = \frac{\mu_0 e}{4\pi} \frac{1}{\|\overrightarrow{x} - \overrightarrow{q}^-\|_{\mathbb{R}^3}\left(1 - \frac{\overrightarrow{v}^- \cdot \overrightarrow{n}^-}{c}\right)} \begin{pmatrix} c \\ \overrightarrow{v}^- \end{pmatrix}, \tag{E.1}$$

and

$$A^{\mu+}_{(t,\overrightarrow{x})} = \frac{\mu_0 e}{4\pi} \frac{1}{\|\overrightarrow{x} - \overrightarrow{q}^+\|_{\mathbb{R}^3}\left(1 + \frac{\overrightarrow{v}^+ \cdot \overrightarrow{n}^+}{c}\right)} \begin{pmatrix} c \\ \overrightarrow{v}^+ \end{pmatrix}. \tag{E.2}$$

Given a point particle and a point outside the world-line $x \neq q(\mathbb{R})$, it is possible to formulate the four-vector potentials as

$$A^{\mu-}_{(x)} = 2e \int_{-\infty}^{\lambda} \dot{q}^{\mu}_{(\tau)} \delta\left((x - q_{(\tau)})^2\right) d\tau \quad \text{for every} \quad \lambda \in (\tau^-, \tau^+) \tag{E.3}$$

and

$$A^{\mu+}_{(x)} = 2e \int_{\lambda}^{\infty} \dot{q}^{\mu}_{(\tau)} \delta\left((x - q_{(\tau)})^2\right) d\tau \quad \text{for every} \quad \lambda \in (\tau^-, \tau^+). \tag{E.4}$$

We can prove that the last equations are actually true if we compute them explicitly and show that they are equal to the four-potentials derived in the example 1.1. We first notice that the Dirac's delta function in the integral has two zeros, namely where

$$|t - \tilde{t}_{(\tau)}| = \pm \|\overrightarrow{x} - \overrightarrow{q}_{(\tilde{t}_{(\tau)})}\|_{\mathbb{R}^3}$$

holds. But as we are not integrating over the whole real numbers, we artificially exclude one of the zeros of the delta function in each case. So for example, for the first integral we have

$$A^{\mu-}_{(x)} = 2e \int_{-\infty}^{\lambda} \dot{q}^{\mu}_{(\tau)} \delta\left((x - q_{(\tau)})^2\right) d\tau$$

$$= e \int_{-\infty}^{\lambda} \dot{q}^{\mu}_{(\tau)} \frac{\delta(\tau - \tau^-)}{\dot{q}_{(\tau^-)} \cdot (x - q_{(\tau^-)})} d\tau = \frac{e \dot{q}^{\mu}_{(\tau^-)}}{\dot{q}_{(\tau^-)} \cdot (x - q_{(\tau^-)})},$$

where the existence of the zero $\tau^-$ is given thanks to the Lemma 2.9 and the bijectivity of $\tilde{t}_{(\tau)}$. The last equation is the exact same result as the potential calculated in the example of the first chapter upon the constants that determine the units. The computation of the advanced four-potential is exactly the same but with an extra minus sign, because

$$\dot{q}_{(\tau^+)} \cdot (x - q_{(\tau^+)}) = \dot{q}^0_{(\tau^+)} \left(t - t^+_{(x)}\right) - \dot{\overrightarrow{q}}_{(\tau^+)} \cdot \left(\overrightarrow{x} - \overrightarrow{q}_{(\tau^+)}\right)$$

$$= \dot{q}^0_{(\tau^+)} \left[(t - t^+_{(x)}) - \overrightarrow{v}_{(t^+_{(x)})} \cdot \left(\overrightarrow{x} - \overrightarrow{q}_{(t^+_{(x)})}\right)\right]$$

$$= -\dot{q}^0_{(\tau^+)} \left\| \vec{x} - \vec{q}_{(t^+_{(x)})} \right\|_{\mathbb{R}^3} \left[ 1 + \vec{v}_{(t^+_{(x)})} \cdot \vec{n}_{(t^+_{(x)}, \vec{x})} \right],$$

which is negative, meaning that the following equality holds in our domain of integration

$$\delta\left((x - q_{(\tau)})^2\right) = \frac{\delta(\tau - \tau^+)}{2|\dot{q}_{(\tau^+)} \cdot (x - q_{(\tau^+)})|} = \frac{-\delta(\tau - \tau^+)}{2\dot{q}_{(\tau^+)} \cdot (x - q_{(\tau^+)})}. \tag{E.5}$$

These equations allow us to get the useful formulas for the field tensors and in order to provide them, we follow the same path as in [Jac14, p. 765] and compute the partial derivatives of the four-potential written as integrals over the parameter $\tau$. For the retarded four-potential we get

$$\begin{aligned} \partial^\mu A^{\nu-}_{(x)} &= 2e \int_{-\infty}^{\lambda} \dot{q}^\nu_{(\tau)} \partial^\mu \delta\left((x - q_{(\tau)})^2\right) d\tau \\ &= 2e \int_{-\infty}^{\lambda} \dot{q}^\nu_{(\tau)} \eta^{\mu\beta} \partial_\beta \delta\left((x^\omega - q^\omega_{(\tau)})(x_\omega - q_{\omega(\tau)})\right) d\tau. \end{aligned}$$

Using the chain rule we can rewrite the partial derivative of the Dirac's delta function as

$$\partial_\beta \delta\left((x^\omega - q^\omega_{(\tau)})(x_\omega - q_{\omega(\tau)})\right) = \frac{x_\beta - q_{\beta(\tau)}}{\dot{q}_{(\tau)} \cdot (x - q_{(\tau)})} \frac{d}{d\tau} \delta\left((x - q_{(\tau)})^2\right).$$

Hence we attain

$$= 2e \int_{-\infty}^{\lambda} \dot{q}^\nu_{(\tau)} \eta^{\mu\beta} \frac{x_\beta - q_{\beta(\tau)}}{\dot{q}_{(\tau)} \cdot (x - q_{(\tau)})} \frac{d}{d\tau} \delta\left((x - q_{(\tau)})^2\right) d\tau,$$

where the last equation can be reformulated with partial integration in the form

$$\partial^\mu A^{\nu-}_{(x)} = 2e \int_{-\infty}^{\lambda} \frac{\mathrm{d}}{\mathrm{d}\tau} \left[ \dot{q}^\nu_{(\tau)} \frac{x^\mu - q^\mu_{(\tau)}}{\dot{q}_{(\tau)} \cdot (x - q_{(\tau)})} \right] \delta\left((x - q_{(\tau)})^2\right) d\tau,$$

and therefore

$$\begin{aligned} \partial^\mu A^{\nu-}_{(x)} &= e \int_{-\infty}^{\lambda} \frac{\mathrm{d}}{\mathrm{d}\tau} \left[ \dot{q}^\nu_{(\tau)} \frac{x^\mu - q^\nu_{(\tau)}}{\dot{q}_{(\tau)} \cdot (x - q_{(\tau)})} \right] \frac{\delta(\tau - \tau^-)}{\dot{q}_{(\tau^-)} \cdot (x - q_{(\tau^-)})} d\tau \\ &= \frac{e}{\dot{q}_{(\tau^-)} \cdot (x - q_{(\tau^-)})} \left[ \frac{\mathrm{d}}{\mathrm{d}\tau} \frac{\dot{q}^\nu_{(\tau)}(x^\mu - q^\mu_{(\tau)})}{\dot{q}_{(\tau)} \cdot (x - q_{(\tau)})} \right] \Bigg|_{\tau = \tau^-}. \end{aligned}$$

Finally, using the linearity of the derivative, we get

**Result E.1: Retarded Electromagnetic Field Tensor**

For a point outside the world-line of a particle, i.e. $x \notin q(\mathbb{R})$, it is possible to write the retarded electromagnetic field tensor as

$$F^{\mu\nu-}_{(x)} = \frac{e}{\dot{q}_{(\tau^-)} \cdot (x - q_{(\tau^-)})} \left[ \frac{\mathrm{d}}{\mathrm{d}\tau} \frac{\dot{q}^\nu_{(\tau)}(x^\mu - q^\mu_{(\tau)}) - \dot{q}^\mu_{(\tau)}(x^\nu - q^\nu_{(\tau)})}{\dot{q}_{(\tau)} \cdot (x - q_{(\tau)})} \right] \Bigg|_{\tau = \tau^-} \tag{E.6}$$

where $\tau^-$ is the retarded proper parameter at which the time coordinate of the particle fulfills the equation

$$t^-_{(x)} = t - \|\vec{x} - \vec{q}_{(t^-_{(x)})}\|_{\mathbb{R}^3}.$$

The computations for $\partial^\mu A^{\nu+}$ are exactly the same, we just have to remember that the limits of the integral are from $\tilde{\lambda}$ to $\infty$ and the minus sign comes from the Dirac's delta function as in (E.5). So we obtain

## Result E.2: Advanced Electromagnetic Field Tensor

For a point outside the world-line of a particle, i.e. $x \notin q(\mathbb{R})$, it is possible to write the advanced electromagnetic field tensor as

$$F^{\mu\nu+}_{(x)} = \frac{-e}{\dot{q}_{(\tau^+)} \cdot (x - q_{(\tau^+)})} \left[ \frac{\mathrm{d}}{\mathrm{d}\tau} \frac{\dot{q}^{\nu}_{(\tau)}(x^{\mu} - q^{\mu}_{(\tau)}) - \dot{q}^{\mu}_{(\tau)}(x^{\nu} - q^{\nu}_{(\tau)})}{\dot{q}_{(\tau)} \cdot (x - q_{(\tau)})} \right] \Bigg|_{\tau=\tau^+} \tag{E.7}$$

where $\tau^+$ is the advanced proper parameter at which the time coordinate of the particle fulfills the equation

$$t^{+}_{(x)} = t + \|\overrightarrow{x} - \overrightarrow{q}_{(t^{+}_{(x)})}\|_{\mathbb{R}^3}.$$

# Appendix F

# Derivation of Larmor's Formula

A definition of the power given by a radiating source is found in [FR60, p. 9]. Adapted to a modern type of writing it reads

$$P_{(t^-)} = \lim_{r\to\infty} \int \hat{r} \cdot \overrightarrow{S}_{(\overrightarrow{r},t^-)} r^2 d\Omega \quad \text{by fixed } t^-, \tag{F.1}$$

where $d\Omega$ denotes the solid angle differential, the integral is done over the surface of a sphere of radius $r$ which tends to infinity and $\overrightarrow{S}$ denotes the **Poynting vector** given as

$$\overrightarrow{S}_{(t,\overrightarrow{x})} = \frac{1}{\mu_0} \overrightarrow{E}_{(t,\overrightarrow{x})} \times \overrightarrow{B}_{(t,\overrightarrow{x})}.$$

If we insert equation (1.3) in the definition of power, we can perform a dipole expansion on the current density and obtain the following result in SI units for a point charge

$$P_{(t^-)} = \frac{1}{4\pi\epsilon_0} \frac{2e^2 \|\overrightarrow{a}_{(t^-)}\|^2_{\mathbb{R}^3}}{3c^3}, \tag{F.2}$$

which is known as the *Larmor's formula.* This formula has experimental backup, e.g. in the case of *cyclotron radiation.*

# Appendix G

# Problems with the Dirac-Lorentz-Abraham Force

What happens after one shows the correctness of equation (1.10) and how it is derived from equation (1.8)? Then, it is possible to compute the Lorentz force and introduce it in Newton's second law as an extra term to study the movement of the charged particle. In this section we provide an outlook in Dirac's equations of motion [Dir38, p. 156], namely

$$m\ddot{q}^{\mu} - \frac{2e^2}{3}\dddot{q}^{\mu} - \frac{2e^2}{3}\ddot{q}_{\nu}\ddot{q}^{\nu}\dot{q}^{\mu} = e\dot{q}_{\nu}F^{\mu\nu}_{ext}. \tag{G.1}$$

It is interesting that we gained a new extra term that has the third derivative of the position, sometimes called the "jerk".[1] These equations can be solved exactly for some simple scenarios, creating the following troubles [Dir38, p. 156] :

1. If the external field vanishes, then the solution of these equations is given by

   $$\dot{x} = \sinh(e^{a\tau} + b)$$

   and

   $$\dot{t} = \cosh(e^{a\tau} + b),$$

   with $a = 3m/2e^2$ and $b$ some constant. As the proper time increases from $-\infty$, the particle's velocity approaches the speed of light extremely fast. This is called the "runaway solutions", which appear quite often in the theory.

2. If the interaction is a "pulse" or in other words, a term of the form $\delta(t - x)$, the solution to these equations of motion for the position is given by

   $$\dot{x} = \begin{cases} ce^{a\tau} & \forall \tau < 0, \\ \\ c & \forall \tau > 0. \end{cases}$$

   In other words, the particle must accelerate prior to the interaction and then attain a state of constant velocity. This phenomenon is referred to as the "pre-acceleration problem", presenting a challenge to our conventional understanding of causality.

For all these reasons, it is of outmost importance to understand the theory of charged particles and how they lose energy through radiation. We will not address the problems given by the differential equation, but rather we focus in the expansion of the radiation field.

[1] Here we are ignoring two important facts that are not relevant for the present discussion. First, we do not mention anything about the term $F^- + F^+$ which was also studied by Dirac and led to the "mass renormalization" of the particle. Second, we also ignore a factor of 1/2 that was introduced by Dirac, when splitting the actual field of the particle.

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# Declaration

I hereby declare that this thesis is my own work, and that I have not used any sources and aids other than those stated in the thesis.

München, date of submission

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