Zur Strahlungsdämpfungskraft in klassischer Elektrodynamik

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Christian Bild

(Mat. Nr. 10357248)

geboren am 11.01.1991 in München

Betreuer: Dr. Deckert

Zweitbetreuer: Prof. Dr. Schollwöck

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On radiation reaction in classical electrodynamics

Masterthesis

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Supervisor: Dr. Deckert

Second Supervisor: Prof. Dr. Schollwöck

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1 Introduction

At the heart of classical electrodynamics are the Maxwell equations $\partial_\alpha F^{\alpha\beta} = j^\beta$ and $\partial_\alpha \tilde{F}^{\alpha\beta} = \partial_\alpha \epsilon^{\alpha\beta\gamma\delta} F_{\gamma\delta} = 0$. Together with the Lorentz force law $ma^\alpha = F^{\alpha\beta} j_\beta$ they completely describe the time evolution of continuous charge like a charged fluid or a charged dust. The agreement with experiments is excellent in the classical regime and the theory is, at least in my opinion, very beautiful. So there is now surprise that this 150 years old theory is very successful. Our theory of matter is not continuous but a particle theory. All attempts to construct a theory of a charged particle turned out to be least to some extend unsatisfactory. The most widely, but not fully, accepted equation of motion for a charged particle is the Lorentz-Dirac equation $ma^\alpha = qF^{\alpha\beta} u_\beta + 2/3q^2(\frac{\mathrm{d}a^\alpha}{\mathrm{d}\tau} + a^\beta a_\beta u^\alpha)$. The second part of the equation is the back reaction to the due to its own field, the radiation reaction force. It is easy to see that this equation implies the Lorentz equation in the limit of a continuous charge distribution. This limit means to take $m \to 0$ and $q \to 0$ simultaneously in such a way that q/m stays constant. So the second part of the equation vanishes obviously in this limit and we have the Lorentz equation left over. The really strange thing is that we actually have to go the other way round. It should at least be on the first encounter quite surprising how a simple second order equation can imply a not so simple third order equation. There are basically two ways to arrive at the Lorentz-Dirac equation. The first is to model a charged particle as some continuous charge distribution and evaluate the total Lorentz force of each part of the particle on the rest of it. The second is to model the particle as a point particle and use energy momentum conservation to calculate the back force.
The fist half of my thesis is an overview of some of the scientific literature. Since the problem is old and fundamental the amount of literature is enormous, so it is only possible to present a tiny part of the literature. This half is split in four parts. The first describes some of the extended charge models and their problems, the second some of the works to point particles. Both are without explicit calculations. Some of the calculations to basic concepts are presented in the third part, because they are used later. The last part are some arguments to distrust the work presented above.
The second half is my own work. It is split in three parts. The first are some considerations to uniformly accelerated charge. The second is an attempt to get an improved equation of motion. The third is an analysis of some common mistakes.

2 Introduction

Classical electrodynamics is taught usually in books and lecture as a theory consisting out of two separate challenges. Either the charge distribution is given at all time and one has to calculate the fields via the Maxwell equations or the fields are given at all times and one has to calculate the time evolution of the charge distribution via the Lorentz force. But the actual challenge is to do both at the same time. This can be done in a straight forward way, if the charge distribution can be described by a continuous differentiable function. The usual interpretation of a continuous charge distribution is as a

charged fluid or gas. For such charged fluids classical electrodynamics works perfectly fine, but we know that things like charged particles exists, the electron for example. There is no fully satisfying way incorporate such particles in the framework of classical electrodynamics. The problems arising from shortcomings of the models for charged particles are so severe, that there is still no satisfying equation of motion for a charged particle. That is the aim of this thesis.
The most widely, but not fully, accepted equation of motion is the Abraham-Lorentz-Dirac equation, but this equation posses unphysical solutions and some cases are known were all solutions are unphysical. This equation can be obtained from two fundamentally different models for the charge particle. The first class of models are particles as extended charge distributions. For an arbitrary trajectory of the particle the fields can be calculated and then the Lorentz force on each part of the particle can be integrated to obtain the total force acting on the particle due to its own fields. A Tailor series in the radius of the charge distribution is necessary for the usual models to generate an analytic solution. The result is

$$ma^\alpha = -a^\alpha \frac{q^2}{2r} + 2/3q^2(\frac{\mathrm{d}a^\alpha}{\mathrm{d}\tau} + a^\beta a_\beta u^\alpha) + F^\alpha_{ext} \tag{1}$$

with terms of order $\mathcal{O}(r)$ and higher dropped. The first term plays the role of a mass increase. It's origin is the increase in energy due to the fields. The second term is usually called radiation reaction force. It is independent of the size of the particle and is supposed to ensure energy momentum conservation in radiation processes. The last term is some external force. The second type of model is a point charge model. The force onto itself can be determinate by energy conservation. The fields diverge near the particle. To perform the calculation a cut-off is used. I will show, that a cut-off is equivalent to replacing the point charge by an extended charge, so it is no surprise that the same equation of motion results out of this model. Furthermore I will argue, that in case of runaway solutions terms of order $\mathcal{O}(r)$ cant be neglected. To incorporate those terms I perform the usual calculation without a Tailor series. The resulting equation of motion is my main result. Its equivalent to the Abraham-Lorentz-Dirac equation up to order $\mathcal{O}(r)$, but I am confident, that it is free of unphysical solutions. One of the main differences is, that the equation I obtained is a second order delay-differential equation and not a third order differential equation.
The thesis is structured as follows. The first part is for clarification of notation and units and contain the derivations of some well known standard results, which we need later on. The second part is a short overview of some of the scientific literature. Since the problem is old and fundamental the amount of literature is enormous, so it is only possible to present a tiny part of the literature. This part is split into three sections. The first describes some of the extended charge models and their problems, the second some of the works to point particles. Both are without explicit calculations. The last section are some considerations to runaway solutions and the $\frac{4}{3}$-Problem.
The next part is my own work. It starts with some considerations to uniformly accelerated charge. It continues with the derivation of an equation of motion without the Tailor series. Concluding I examine some of the properties of the new equation of motion.

3 Units and notation

We use Gaussian units with the velocity of light set to one, because this is a common choice in the literature. For the metric $\eta^{\alpha\beta}$ we choose

$$\eta^{\alpha\beta} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{2}$$

Einstein summation convention is used throughout. The field strength tensor $F^{\alpha\beta}$ is connected to the electric field $\vec{E}$ and magnetic field $\vec{B}$ as usually

$$F^{\alpha\beta} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} \tag{3}$$

The connection to the four potential A^α is $F^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha = \partial^{[\alpha} A^{\beta]}$. The square brackets are always used for antisymmetrization and ∂^α stands for $\frac{\partial}{\partial x_\alpha}$. Here is a list some notation.

coordinate time	t
proper time	τ
position vector of the particle	$\vec{z}(t) = \vec{z}$
position four vector of the particle	$z^\alpha(\tau) = z^\alpha$
four velocity	$\frac{dz^\alpha}{d\tau} = u^\alpha(\tau) = u^\alpha$
four acceleration	$\frac{du^\alpha}{d\tau} = a^\alpha(\tau) = a^\alpha$
energy momentum tensor	$T^{\alpha\beta}$
coordinate velocity	$\frac{d\vec{z}}{dt} = \vec{v}(t) = \vec{v}$
Lorentz matrix	$\Lambda^\alpha_\beta(\tau) = \Lambda^\alpha_\beta$
γ	$\frac{1}{\sqrt{1-v^2}}$
four current	j^α
charge	e
Kronecker delta	δ^α_β
Levi-Civita symbol	$\epsilon_{\alpha\beta\gamma\delta}$

If an expression appears in a quantity related to the fields, like $A^\alpha, F^{\alpha\beta}$ or $T^{\alpha\beta}$, the expression depends on the retarded proper time $\tau = \tau_{ret}$. x^α is the four vector describing the position, where a field related quantity is evaluated.

4 Derivation of some basic results

In this part we derive the energy momentum tensor, the Lorentz force, the Liénard-Wiechert-potential, the fields and the energy momentum tensor connected to it and finally Lamor's formula.

4.1 The energy momentum tensor

For the derivation we will use the Lagrangian and Noether's theorem. Lets start with the free Maxwell equations $\partial_\alpha F^{\alpha\beta} = 0$. The Lagrangian has to be a Lorentz scalar and, since we expect the theory to be linear, quadratic in the fields. So there is only one possibility to full file those constrains, since the dual of the field strength tensor allows us only to construct a pseudo scalar

$$L = cF^{\alpha\beta}F_{\alpha\beta} \tag{4}$$

with some constant c. Noether's theorem give us the conserved quantities. Invariance in space time translations give us the canonical energy momentum tensor

$$T^{\alpha\beta}_{can} = \frac{\partial L}{\partial_\alpha A_\gamma}\partial^\beta A_\gamma - \eta^{\alpha\beta}L = c(4F^{\alpha\gamma}\partial^\beta A_\gamma - \eta^{\alpha\beta}F^{\gamma\delta}F_{\gamma\delta}) \tag{5}$$

This energy momentum tensor is not Gauge invariant, but the tensor is not unique, because we can add any quantity with vanishing divergence and still have a conserved tensor. We choose to add $c\partial_\gamma(F^{\alpha\gamma}A^\beta)$. It's divergence with respect to α vanishes, because $F^{\alpha\gamma}$ is antisymmetric and it ensures Gauge invariance. Putting them together give us the usual energy momentum tensor, only c still needs to be determined.

$$T^{\alpha\beta} = -4c(F^{\alpha\gamma}F_\gamma{}^\beta + 1/4\eta^{\alpha\beta}F^{\gamma\delta}F_{\gamma\delta}) \tag{6}$$

4.2 Lorentz force

For the derivation of the energy momentum tensor we used the free Maxwell equations $\partial_\alpha F^{\alpha\beta} = 0$, but now we switch to the general case $\partial_\alpha F^{\alpha\beta} = 4\pi j^\beta$. The tensor is obviously not conserved any more, so lets calculate it's divergence

$$\begin{aligned}\partial_\alpha T^{\alpha\beta} = -4c(\partial_\alpha(F^{\alpha\gamma}F_\gamma{}^\beta) + 1/4\eta^{\alpha\beta}\partial_\alpha(F^{\gamma\delta}F_{\gamma\delta})) = -4c(4\pi j^\gamma F_\gamma{}^\beta + F^{\alpha\gamma}\partial_\alpha F_\gamma{}^\beta \\ + 1/2\eta^{\alpha\beta}F^{\gamma\delta}\partial_\alpha F_{\gamma\delta}) = -4c(4\pi j^\gamma F_\gamma{}^\beta + F^{\gamma\delta}\partial_\gamma F_\delta{}^\beta + 1/2\eta^{\alpha\beta}F^{\gamma\delta}(-\partial_\gamma F_{\delta\alpha} - \partial_\delta F_{\alpha\gamma}) \\ = -4c(4\pi j^\gamma F_\gamma{}^\beta + F^{\gamma\delta}(\partial_\gamma F_\delta{}^\beta + \partial_\gamma F_\beta{}^\delta)) = 4c4\pi F^{\beta\gamma}j_\gamma\end{aligned} \tag{7}$$

We did use $\partial_\alpha F_{\gamma\delta} + \partial_\gamma F_{\delta\alpha} + \partial_\delta F_{\alpha\gamma} = 0$. A comparison with the Lorentz force and the demand of energy momentum conservation fixes $c = \frac{-1}{16\pi}$. In the general case the field strength tensor is composed of some external fields $F^{\alpha\beta}_{ext}$ and the fields of the charge under consideration $F^{\alpha\beta}_{self}$.

$$F^{\alpha\beta} = F^{\alpha\beta}_{ext} + F^{\alpha\beta}_{self} \tag{8}$$

The external fields should have no sources in the area of interest $\partial_\alpha F^{\alpha\beta}_{ext} = 0$. We will later compute the the divergence of the energy momentum tensor of a point particle, but because the fields diverge at the position of the particle, the Lorentz force isn't well defined any more. To get a result anyway we will use Gauss's theorem. The problematic part is completely contained in $F^{\alpha\beta}_{self}$, so there is no reason not to compute the parts, who contain $F^{\alpha\beta}_{ext}$, right away. The energy momentum tensor contains three different types of terms, quadratic terms in $F^{\alpha\beta}_{self}$ and in $F^{\alpha\beta}_{ext}$ and some mixed terms. The quadratic terms in $F^{\alpha\beta}_{self}$ will get evaluated with Gauss's theorem, while the quadratic terms in $F^{\alpha\beta}_{ext}$ have vanishing divergence, because of $\partial_\alpha F^{\alpha\beta}_{ext} = 0$. So the only interesting contribution form the external fields comes from the mixed terms.

$$\frac{1}{4\pi}\partial_\alpha(F^{\alpha\gamma}_{self}F_\gamma{}^\beta{}_{ext} + F^{\alpha\gamma}_{ext}F_\gamma{}^\beta{}_{self} + 1/2\eta^{\alpha\beta}F^{\gamma\delta}_{self}F_{\gamma\delta ext}) = -F^{\beta\gamma}_{ext}j_\gamma \tag{9}$$

The calculation is analogous to 7. In the literature $F^{\alpha\beta}_{ext}$ is usually carried through the whole calculation, but we can already see here, that its contributing only as Lorentz force.

4.3 Liénard-Wiechert-potential

In this part I will give a short derivation for the Liénard-Wiechert-potential. The Maxwell equations for the potential A^α in Lorenz gauge $\partial_\alpha A^\alpha = 0$ are

$$\Box A^\alpha = j^\alpha \tag{10}$$

The general retarded solution to those equations is

$$A^\alpha(t,\vec{x}) = \int_{R^3} \frac{j^\alpha(t-|\vec{y}|, \vec{x}+\vec{y})}{|\vec{y}|} d^3y \tag{11}$$

In the case of a point particle $j^\alpha(\tau)$ is nothing else but $e\int u^\alpha(\tau)\delta^4(z^\alpha - x^\alpha)d\tau$. It is in general possible to choose a coordinate system where $u^\alpha = (1,\vec{0})$ at the retarded time holds. In this coordinate system

$$A^\alpha = e\frac{(1,\vec{0})}{r} \tag{12}$$

is valid, where r is the distance between the point, where we evaluate the potential, and the charge in the co-moving coordinate system at the retarded time. We now have to rewrite this expression with help of four vectors to arrive at an expression, that holds in all coordinate systems. The four vector, which is $(1,\vec{0})$ in the co-moving coordinate system is obviously just u^α. To find four vectors for r we use that $(x^\alpha - z^\alpha)(x_\alpha - z_\alpha) = 0$, which is the condition for the retarded time. This means, that the distance in space and the time difference are the same. Hence we are able to express the distance in space by the 0-component of $x^\alpha - z^\alpha$ in the co-moving system. So we just have to take the scalar product with a vector, which reduces to $(1,\vec{0})$ in the co-moving system. So we end up with

$$A^\alpha = e\frac{u^\alpha}{(x^\beta - z^\beta)u_\beta} = e\frac{u^\alpha}{r} \tag{13}$$

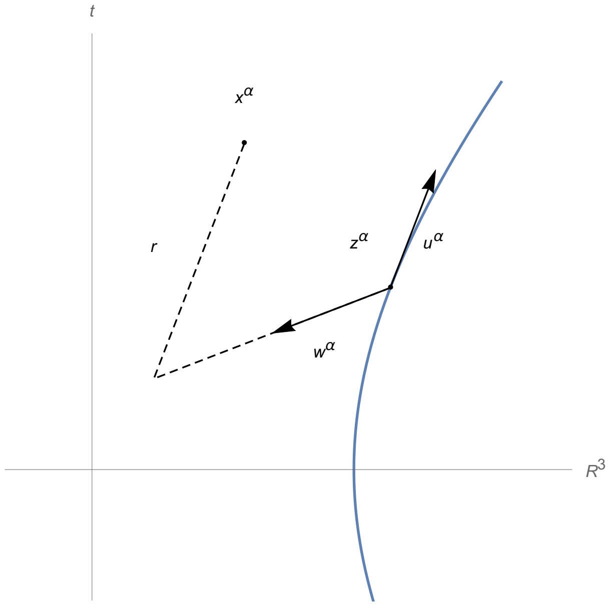

Figure 1: How an arbitrary vector x^α can represented with $z^\alpha, u^\alpha, w^\alpha$ and r

4.4 Field strength tensor for a point particle

In this section we calculate the field strength tensor. It is given by

$$F^{\alpha\beta} = \partial^{[\alpha}A^{\beta]} = e(\frac{a^{[\alpha}}{r}\partial^{\beta]}\tau - \frac{u^{[\alpha}}{r^2}\partial^{\beta]}r) \tag{14}$$

Lets first simplify the second part

$$\partial^\beta r = \partial^\beta((x^\alpha - z^\alpha)u_\alpha) = u^\beta + x^\alpha a_\alpha \partial^\beta \tau - u^\alpha u_\alpha \partial^\beta \tau - z^\alpha a_\alpha \partial^\beta \tau \tag{15}$$

Now only $\partial^\beta \tau$ is missing. To get it we solve $\partial^\beta((x^\alpha - z^\alpha)(x_\alpha - z_\alpha)) = 0$ for it.

$$\partial^\beta(x^\alpha x_\alpha - 2x^\alpha z_\alpha + z^\alpha z_\alpha) = 0 \tag{16}$$

$$2x^\beta - 2z^\beta - 2x^\alpha u_\alpha \partial^\beta \tau + 2z^\alpha u_\alpha \partial^\beta \tau = 0 \tag{17}$$

$$\partial^\beta \tau = \frac{x^\beta - z^\beta}{(x^\alpha - z^\alpha)u_\alpha} = \frac{x^\beta - z^\beta}{r} \tag{18}$$

This result can be simplified considerably be introducing a new four vector w^α, which is defined by $w^\alpha = \frac{x^\alpha - z^\alpha}{r} - u^\alpha$, so that

$$x^\alpha = z^\alpha + r(u^\alpha + w^\alpha) \tag{19}$$

as shown in Figure 1. Form this definition it follows, that w^α is a space-like unit vector $w^\alpha w_\alpha = -1$ and orthogonality with the velocity $w^\alpha u_\alpha = 0$. The purpose of w^α is just to split up $x^\alpha - z^\alpha$ in a time and space part. Additional clarification is gained be realising that w^α is $(0, \vec{e_r})$ in the co-moving coordinate frame at the retarded time, so $w^\alpha = w^\alpha(\tau, \theta, \phi)$. w^α can be written as

$$w^\alpha = \Lambda^\alpha_\beta \begin{pmatrix} 0 \\ \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix} \tag{20}$$

The Lorentz matrix is not just a boost, but also some rotation because of the Thomas precession, but luckily we wont need the exact form the the rotation matrix. Now (18) reduces to

$$\partial^\beta \tau = u^\beta + w^\beta \tag{21}$$

Plugging this into (15) gives

$$\partial^\beta r = u^\beta + (rw^\alpha a_\alpha - 1)\partial^\beta \tau = rw^\alpha a_\alpha u^\beta + (rw^\alpha a_\alpha - 1)w^\beta \tag{22}$$

Finally we just have to collect all terms and put it in (14), which leads to

$$F^{\alpha\beta} = e\frac{1}{r^2}u^{[\alpha}w^{\beta]} + e\frac{1}{r}(a^{[\alpha}w^{\beta]} + a^{[\alpha}u^{\beta]} - w^\gamma a_\gamma u^{[\alpha}w^{\beta]}) \tag{23}$$

This can be even further simplified be introducing $a^\alpha_\perp = a^\alpha + a^\beta w_\beta w^\alpha$.
$a^\alpha_\perp$ is orthogonal to $u^\alpha a_{\perp\alpha} = 0$ and $w^\alpha a_{\perp\alpha} = 0$. The final result is

$$F^{\alpha\beta} = e\frac{1}{r^2}u^{[\alpha}w^{\beta]} + e\frac{1}{r}a^{[\alpha}_\perp(u^{\beta]} + w^{\beta]}) \tag{24}$$

The first part are just the coulomb fields known from electrostatics, while the second part are radiation fields. Those fields appear only if the particle is accelerated. The field strength tensor is a function of the coordinates $(t, x, y, z) = x^{\alpha T}$, but it is much easier to use $(\tau_{ret} = \tau, r, \theta, \phi)$ as coordinates.

4.5 Energy momentum tensor for a point particle

The calculation of the energy momentum tensor is easily done with the results form above. It is given by

$$4\pi T^{\alpha\beta} = F^{\alpha\gamma}F_\gamma{}^\beta - \frac{1}{4}\eta^{\alpha\beta}F^{\gamma\delta}F_{\gamma\delta} \tag{25}$$

Lets start with just the coulomb fields

$$F_c^{\alpha\gamma}F_{c\gamma}{}^\beta = \frac{e^2}{r^4}u^{[\alpha}w^{\gamma]}u_{[\gamma}w^{\beta]} = \frac{e^2}{r^4}(u^\alpha u^\beta - w^\alpha w^\beta) \tag{26}$$

due to the orthogonality of u^α and w^α. Another contraction gives $\frac{2e^2}{r^4}$.The mixed terms contribute with

$$F_c^{\alpha\gamma}F_{r\gamma}{}^\beta = \frac{e^2}{r^3}u^{[\alpha}w^{\gamma]}a_{\perp[\gamma}(u^{\beta]} + w^{\beta]}) = \frac{e^2}{r^3}a^\beta_\perp(u^\alpha + w^\alpha) \tag{27}$$

and the same term term with interchanged indices. A second contraction gives zero. So we have just left the radiation part

$$F_r^{\alpha\gamma} F_{r\gamma}{}^{\beta} = \frac{e^2}{r^2} a_\perp^{[\alpha}(u^{\gamma]} + w^{\gamma]}) a_{\perp[\gamma}(u^{\beta]} + w^{\beta]}) = -\frac{e^2}{r^2} a_\perp^\gamma a_{\perp\gamma}(u^\alpha + w^\alpha)(u^\beta + w^\beta) \quad (28)$$

A second contraction vanishes again. So the final result is

$$\begin{aligned} 4\pi T^{\alpha\beta} &= \frac{e^2}{r^4}(u^\alpha u^\beta - w^\alpha w^\beta - \frac{1}{2}\eta^{\alpha\beta}) \\ &\quad + \frac{e^2}{r^3}(a_\perp^\beta(u^\alpha + w^\alpha) + a_\perp^\alpha(u^\beta + w^\beta)) - \frac{e^2}{r^2} a_\perp^\gamma a_{\perp\gamma}(u^\alpha + w^\alpha)(u^\beta + w^\beta) \end{aligned} \quad (29)$$

4.6 Lamor's formula

In this part I go through an easy derivation of Lamor's formula using what was derived above. The formula describes the energy momentum loss of particle due to radiation, which is the energy momentum flow to infinity. That means we have to calculate the surface integral over the energy momentum tensor over a surface, which is very far away. For big enough distances we can drop the first two terms of the energy momentum tensor derived above. Due to energy conservation we know that it's divergence vanishes there. That means the integral is independent of the form of the surface. It is worth mentioning, that integral over a light cone centred at the particle vanishes, because the normal vector on an light cone is just $u^\alpha + w^\alpha$, a null vector. So we can now choose to integrate over a sphere in the co-moving frame at the retarded time τ centred at the particle. Since we interested on the energy momentum loss at this time we have to integrate at the time $\tau + r$ where r is the radius of the sphere, since the fields need the time r to propagate to the surface. The normal vector on the surface of the sphere is obvious in the co-moving frame $(0, \vec{e_r})$, which is just w^α. Contracting it with the third part of the energy momentum tensor we a left with

$$\int \frac{e^2}{4\pi r^2} a_\perp^\gamma a_{\perp\gamma}(u^\alpha + w^\alpha) d\Omega \quad (30)$$

First we use that $a_\perp^\gamma a_{\perp\gamma} = a^\gamma a_\gamma + (a^\gamma w_\gamma)^2$. Then we sort the terms contributing to the integral. Only w^α dependence on angels and for simplicity we stay in the co-moving frame

$$w^\alpha = \begin{pmatrix} 0 \\ \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix} \quad (31)$$

All odd potencies of w^α give zero for symmetry reasons. Thus there will be only to contributing terms $a^\gamma a_\gamma u^\alpha$ and $(a^\gamma w_\gamma)^2 u^\alpha$. The first gives $4\pi r^2 a^\gamma a_\gamma u^\alpha$ and the second $-\frac{4}{3}\pi r^2 a^\gamma a_\gamma u^\alpha$. Details can be found in page 35.So the final result is

$$\frac{2e^2}{3} a^\gamma a_\gamma u^\alpha \quad (32)$$

The result is independent from r, which already followed from the vanishing divergence of the energy momentum tensor and the vanishing integrals over light cones, since different spheres are connected with light cones. It is nice that the radiation rate is finite, so at the first sight one could still hope to get finite result for the radiation reaction force, although the fields diverge close to the particle. This result cant play the role of force alone, since it is not orthogonal to u^α and there is no reason it should play such a role, because we evaluated the field strength tensor much later and far away. This establishes at least, that the Coulomb field, which where dropped here, play a bigger role than just increasing the mass of the particle. For an extended charge distribution this is obvious anyway.

5 Extended charge models

5.1 Abraham model

One of the obvious and simplest choices one could make for the model of a charged particle is a homogeneously charged sphere. This is of cause a non-relativistic model, since a sphere does not stay a sphere under Lorentz transformations. It was first examined by Abraham, so it is now called the Abraham model*, even before Einstein's paper on special relativity. The charge distribution $\phi(x-\vec{z}(t))$,where $\vec{z}(t)$ is the trajectory of the sphere, is given by

$$\phi(x) = \delta(|\vec{x}| - r)/4\pi r^2 \tag{33}$$

Other choices, like a distribution over the hole volume and not just the surface, are of cause possible, but we will stick to this one. With the Maxwell equations it is now possible to calculate the fields and with the Lorentz force the total force onto itself. The result is

$$2/3e^2 4\pi/(2\pi)^3 \int dk^3 |\phi(\vec{k})|^2 \int d\tau \exp(i\vec{k}\cdot\vec{z}(\tau)) \exp(i\vec{k}\cdot\vec{z}(t-\tau)) \dot{\vec{z}}(t-\tau)\cos(k\tau) \tag{34}$$

where $\phi(\vec{k})$ is the Fourier transform of $\phi(\vec{x})$. This expression can be simplified further, with the approximation $\exp(i\vec{k}\cdot\vec{z}(\tau))\exp(i\vec{k}\cdot\vec{z}(t-\tau)) = 1$. This approximation is valid for small velocities. The result is known as the Sommerfeld-Page equation

$$\frac{e^2}{3r^2}(\dot{\vec{z}}(t-2r) - \dot{\vec{z}}(t)) \tag{35}$$

Such a delay equation allows a quite natural interpretation, since the delay is just the time, which the fields need to propagate from one site of the particle to the other. In part * we will derive our own delay equation. A Tailor expansion of the Sommerfeld-Page equation leads to

$$-4/3\frac{e^2}{2r}\ddot{\vec{z}}(t) + 2/3e^2\dddot{\vec{z}}(t) \tag{36}$$

This can be understood as a non relativistic radiation reaction force. Additionally a 4/3 factor in front of the of the first term appears. This term describes the increase in mass of the charged particle due to energy of its fields. The reasons, why the 4/3 factor is problematic and some of the approaches to solve those problems are described in part *. If we use a Tailor series for the complete term (34) without the approximation, we arrive instead by

$$2/3e^2(\gamma^4(v \cdot \ddot{v})v + 3\gamma^6(v \cdot \dot{v})^2 v + 3\gamma^4(v \cdot \dot{v})\dot{v} + \gamma^2\ddot{v}) \tag{37}$$

This equation was first derived by Abraham. It was later realised by * , that this force can be nicely written in four-vector form

$$2/3e^2(\frac{\mathrm{d}a^\alpha}{\mathrm{d}\tau} + a^\beta a_\beta u^\alpha) \tag{38}$$

* argues,that it is surprising, that the result is a four-vector, since the model is non-relativistic. I disagree, because the force on each part of the particle is a four-vector and so is the integral over the force. It doesn't matter, if the charge distribution chances form in a relativistic way or how it looks at all. What actually is not granted that the total force is orthogonal to four-velocity of the particle, since the force on each part is orthogonal to the four-velocity of that part and not on the four-velocity of the particle. An example for that is a charged sphere, which changes its radius. A detailed description can be found in (spohn yangi abraham schott)

5.2 Lorentz model

The Lorentz model differs from the Abraham model just by the extra condition, that the charge distribution is sphere in its momentary rest system and not in one the laboratory system. That means the charge distribution has to chance form depending on its velocity. A moving charge will looks like an ellipsoid. The self force equation turns out to be equivalent to the one in the Abraham model in the case of vanishing radius of the particle.

5.3 Problems connected to extended charge distributions

The first and most obvious problem is the lack of stability. Since charge of equal singe repel each other, it is to expect, that particles described by the models above would just explode. To make them stable there is some extra force necessary. Accordingly an extended charge distribution with only the Maxwell equations is an insufficient model for charged particle. At least to my knowledge there is no precise mathematical model for such extra forces, which has also a satisfying physical motivation. The lack of such a model will cause us several problems at different points later on.
The next problem is the arbitrary choice we have to make for the form of the particle. Good news thereto is, that in a series expansion of the self force in the radius of a small charge distribution is independent of the form in the zero order term. Feynman* gives the result for non relativistic motion in one dimension

$$F = e^2(\alpha/r\ddot{x} - 2/3\dddot{x} + \gamma r\ddddot{x}) + ... \tag{39}$$

where α and γ are constants of order 1, who depend on the form of the particle. This seems to be a nice result, since the form dependence of the first term doesn't really matter, because this term only participates in form of a mass increase and the third and all higher order terms vanish, if the particle is sufficiently small. But if we look at a three dimensional motion the situation is much more complicated. Charge which is extended longitudinal to the trajectory gives a different contribution to the mass than charge with is extended transversal(lyle). For that reason, if there is some kind of symmetry breaking, mass loses it's scalar character and becomes a tensor instead. Further we will see, that situations exist where the higher order terms cant be dropped.
Also we get into trouble, when we try to use energy momentum conservation, because the unknown stability forces can act as sink or source for energy and momentum. More on that in chapter *.
In addition the Abraham model violates relativity, since it singles out a references frame, while the Lorentz model breaks down for to high acceleration. The reason for this is, that the velocity is bounded be the speed of light, while there is no theoretical limit for the acceleration. So the velocity can change in an arbitrary short time and the particle has to change size accordingly to a Lorentz transformation connected with the velocity change to stay a sphere in the co-moving reference frame. But the particle cant change size faster than with the speed of light. Thus there exists an size dependent upper bound for the acceleration*(spohn).
Now lets just assume we would have a model for the stability forces. Hence we get the possibility of excitations of the particle. It is to expect, that they decay through radiation depending on the oscillation velocity. It follows, that the mass of the particle isn't constant any more, but depends on the inner structure of the particle. That is just one of the reasons, why such a model would probably only led to a very complicated equation of motion, even it is attractive, because it is more complete.
Finally a charged sphere can hardly be considered as a microscopic model, since there is no essential difference between a microscopic and a macroscopic sphere. The physics of those two are the same, one is just smaller.

6 Point charge model

The obvious advantage of such a model is, that the stability problem get circumvented completely. The disadvantage is, that the fields diverge close to the particle. Hence the Lorentz force isn't well defined any more. In contrast to the extended charge models there is no way to calculate the self force directly. The usual way is to apply energy momentum conservation. It is not completely clear, how to do this, since the energy and momentum of fields connected to point particles diverges to. So a very careful treatment of the infinities tougher with a renormalisation of the mass is necessary. Such a proceeder is at least questionable and very liable for mistakes. Chapter 7 discusses such mistakes. But there is some hope nevertheless, because Lamor's formula for the radiation rate of a point charge gives a finite result. So it should be possible to distinct between infinite terms, which increase the mass and come from the infinite energy and

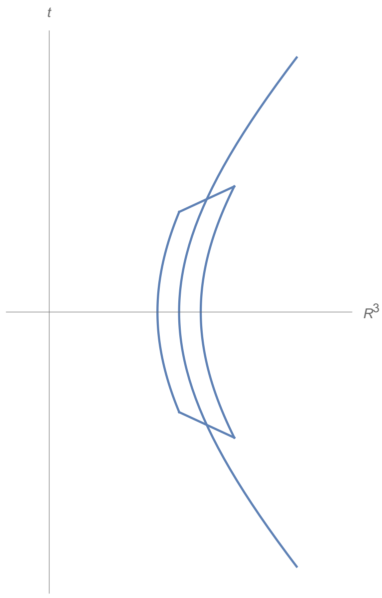

Figure 2: The tube used by Dirac

finite terms, which give the actual radiation reaction and come from the finite radiation rate.

6.1 Dirac's paper

Dirac calculates in [1] the flow of energy and momentum away form a point particle. To do that he embeds the trajectory of the particle in a tube and performs an integration of the energy momentum tensor of the fields over the surface of the tube. First lets make clear how the tube looks exactly. It consist of spheres of radius ϵ in the comoving reference frames. The connection of all those spheres at different points in time give us some three dimensional hyper surface. This surface (Figure 2) looks in a space time diagram like a tube around the trajectory, hence the name. Now lets have a look on the energy momentum conservation argument. Dirac claims, that energy momentum flown through the tube must be equal to the difference of energy momentum in the tube at the ends of the tube. It is reasonable to assume, that he means not only the energy momentum of the fields, but also the energy momentum of the particle. From that, he concludes that the flow of energy momentum through the tube must be a perfect differential. In my opinion, the next step would be to calculate the difference of energy momentum at the ends of the tube, to find out of which expression the flow through the tube is a perfect differential of. But Dirac guesses this expression instead. To motivate his guess we need the expression for the flow of energy momentum through the tube. To calculate it, we start with the Liénard-Wiechert-potential (13)

$$A^\alpha = e\frac{u^\alpha}{\epsilon} \tag{40}$$

To get the fields is a bit challenging, since in general the fields on each point of a sphere, mentioned above, depend on the trajectory at different times. To tackle this problem Dirac develops all involved quantities in a Tailor series in the radius ϵ. The result with higher order terms dropped is

$$F^{\alpha\beta}_{\substack{ret\\adv}} = \frac{2e}{\sqrt{1+\epsilon a^\gamma w_\gamma}}\left(\frac{1}{\epsilon^2}u^{[\alpha}w^{\beta]} - \frac{1}{2\epsilon}u^{[\alpha}a^{\beta]} - \frac{1}{2}a^\gamma w_\gamma u^{[\alpha}a^{\beta]} + \frac{a^2}{8}u^{[\alpha}w^{\beta]} - \frac{1}{2}\dot{a}^{[\alpha}w^{\beta]} \mp \frac{2}{3}\dot{a}^{[\alpha}u^{\beta]}\right) \tag{41}$$

The integral over the energy momentum tensor is with (41)

$$\int T^{\alpha\beta}d^3\sigma_\beta = \int \frac{e^2}{2\epsilon}a^\alpha + eu_\beta f^{\beta\alpha}d\tau \tag{42}$$

where $f^{\alpha\beta} = F^{\alpha\beta}_{ext} + 1/2(F^{\alpha\beta}_{ret} - F^{\alpha\beta}_{adv})$. This expression is orthogonal to the four velocity u^α, so his guess is the simplest possible choice fulfilling this constraint.

$$\dot{B}^\alpha = ka^\alpha \tag{43}$$

After this he identifies k with $\frac{e^2}{2\epsilon} - m$ and finally arrives at the equation of motion

$$ma^\alpha = \frac{2e^2}{3}(\dot{a}^\alpha + a^\beta a_\beta u^\alpha) + eF^{\alpha\beta}_{ext}u_\beta \tag{44}$$

In chapter 7 I show that this guess tougher with the identification of k is wrong for a point particle and only holds for an extended charge distribution.

* dirac parrot rohrlich

6.2 Parrott's calculation

In [7] Parrott repeats Dirac's calculation with two main differences. First he doesn't use a Tailor series, but gives an exact result. The reason why he is able to do so, is the second difference. The tube he uses differs form Dirac's tube and simplifies the calculation significantly. It is defined the following way: Go into the comoving coordinate frame at some time t. Take the light cone from the particle at t. Cut it with a plane of simultaneity at time $t + r$. The result is a sphere of radius r. Connect all spheres from all points in time tougher and you arrive at the tube (Figure 3). This tube is called Bhabha-tube, because it was first suggested by Bhabha, but he never performed the calculation. The reason why this tube simplifies things is, that the fields at each point on a sphere depend on the trajectory at the same time. Accordingly the caps differ form the caps Dirac used. They aren't planes of simultaneity of the comoving frame at τ_1 and τ_2 any more, but given by the light cones at τ_1 and τ_2.

Parrott puts more effort in the energy momentum conservation argument. He starts with the conservation of the total energy momentum tensor. It consist out of the mechanical energy momentum tensor and the electromagnetic energy momentum tensor.

$$\partial_\alpha T^{\alpha\beta}_{em} + \partial_\alpha T^{\alpha\beta}_{mech} = 0 \tag{45}$$

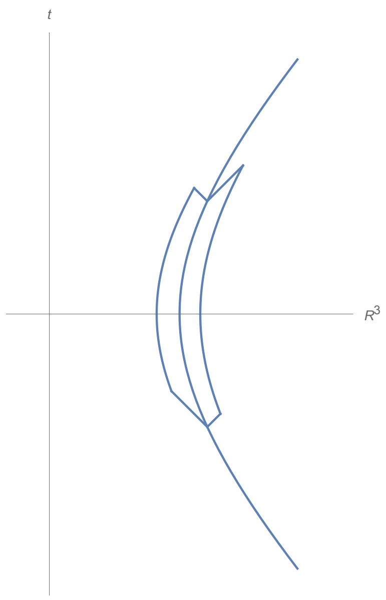

Figure 3: The tube used by Parrott

With (7) this reduces to the usual expression for the Lorentz force $F^{\alpha\beta} j_\beta = ma^\alpha$, but instead of this we use a volume integration over the tube described above. This leads to

$$\int T^{\alpha\beta} d^3\sigma_\beta = mu^\alpha(\tau_1) - mu^\alpha(\tau_2) + \int T^{\alpha\beta}(\tau_1) d^3 c_\beta - \int T^{\alpha\beta}(\tau_2) d^3 c_\beta \tag{46}$$

σ describes the tube, while c describes the caps, which close the tube. Parrott actually calculates the cap integrals so no guessing is necessary. The result is

$$\int T^{\alpha\beta}(\tau_1) d^3 c_\beta = \frac{e^2}{2} u^\alpha(\tau_1)(\frac{1}{\epsilon} - \frac{1}{r}) \tag{47}$$

ϵ is a cut off parameter. If one accepts the cut off, this result matches with Dirac's guess, since the caps just give some mass renormalization. The next step is to absorb the contribution form the caps in the mass. He calls this the first mass renormalization. His result for the flow through the tube is

$$\int T^{\alpha\beta} d^3\sigma_\beta = \frac{e^2}{2r}(u^\alpha(\tau_2) - u^\alpha(\tau_1)) - \frac{2e^2}{3} \int a^\gamma a_\gamma u^\alpha d\tau - e \int F^{\alpha\beta}_{ext} u_\beta d\tau \tag{48}$$

The full equation of motion is now

$$m_{ren} u^\alpha(\tau_1) - m_{ren} u^\alpha(\tau_2) = \frac{e^2}{2r}(u^\alpha(\tau_2) - u^\alpha(\tau_1)) - \frac{2e^2}{3} \int a^\gamma a_\gamma u^\alpha d\tau - e \int F^{\alpha\beta}_{ext} u_\beta d\tau \tag{49}$$

The renormalized mass is $m_{ren} = m + \frac{e^2}{2}(\frac{1}{\epsilon} - \frac{1}{r})$. To get ride of the other terms proportional to the four velocity, he performs a second mass renormalization. This second mass renormalization is completely unproblematic, since all the parts depending on the radius of the tube r just chancel each other. This could have been expected, because Gauss's theorem tells us, that the end result can not depend on the form of the tube. A comparison off Parrott's and Dirac's result of the tube integration should be a surprise, since they differ from each other. While Parrott result is a combination off mass renormalisation plus Lamor's formula plus external Lorentz force Dirac gets an additional term $\dot{a}^\alpha$to the Lamor formula. This term is important, since it makes sure that the radiation reaction force is orthogonal to the four velocity u^α. This is a general constrain on the force, because four acceleration a^α and four velocity u^α are orthogonal, since the four velocity u^α is a unit vector $u^\alpha u_\alpha = 1$. To get an satisfying equation of motion Parrott states next, that the term by which his and Dirac's result differ vanishes when integrated between to times τ_1 and τ_2, for which $a^\alpha(\tau_1) = a^\alpha(\tau_2)$ holds. So he arrives at the same equation of motion as before, but trusting in his own exact calculations, he adds the constrain $a^\alpha(-\infty) = a^\alpha(\infty)$. In chapter 7 it is shown, that though the cut off the point charge is replaced by an extended charge distribution unintentionally, exactly in the same way it happens in Dirac's calculation. Thereby is explained in section 8.1 why the results differ.

6.3 The $\frac{4}{3}$-problem

This problem arises, is one tries to construct an energy momentum four vector for a charged particle. The four vector should not only contain the mechanical energy and momentum, but the total energy and momentum. In our case that is the additionally the energy and momentum contained in it's fields. It seems natural to integrate the energy and the momentum of the fields over the whole space and add it to the mechanical energy momentum four vector. The sum out of those two should still be a four vector. Lets check this explicitly for the electromagnetic part. The easiest case is a point particle at rest at the origin, so there are only the usual Coulomb fields $\vec{E} = e^2\frac{\vec{r}}{r^3}$. The energy density is given by $\frac{1}{8\pi}(E^2 + B^2)$ and the momentum density by $\frac{1}{4\pi}E \times B$. The volume integration leads to the following integral

$$p^0 = \int_0^\infty \frac{e^2}{2r^2} dr \tag{50}$$

This is a diverging integral and to make progress we introduce a cut off ϵ for the lower boundary. So the result is

$$p^\alpha = \begin{pmatrix} \frac{e^2}{2\epsilon} \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{51}$$

Now lets examine, if this object transform like a four vector. To do this, we transform the field strength tensor with a Lorentz boost, so now we have a point particle moving with constant velocity.

$$F^{\alpha\beta}_{mov} = \Lambda^{\alpha}_{\gamma} F^{\gamma\delta}_{rest} \Lambda^{\beta}_{\delta} = \begin{pmatrix} \gamma & 0 & 0 & v\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ v\gamma & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} 0 & -e\frac{x}{r^3} & -e\frac{y}{r^3} & -e\frac{z}{r^3} \\ e\frac{x}{r^3} & 0 & 0 & 0 \\ e\frac{y}{r^3} & 0 & 0 & 0 \\ e\frac{z}{r^3} & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \gamma & 0 & 0 & v\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ v\gamma & 0 & 0 & \gamma \end{pmatrix} = \begin{pmatrix} 0 & -e\frac{x\gamma}{r^3} & -e\frac{y\gamma}{r^3} & -e\frac{z}{r^3} \\ e\frac{x\gamma}{r^3} & 0 & 0 & -e\frac{yv\gamma}{r^3} \\ e\frac{y\gamma}{r^3} & 0 & 0 & -e\frac{xv\gamma}{r^3} \\ e\frac{z}{r^3} & e\frac{yv\gamma}{r^3} & e\frac{xv\gamma}{r^3} & 0 \end{pmatrix} \tag{52}$$

p^3 is with this field strength tensor in lowest order $4/3\frac{e^2}{2\epsilon}v + \mathcal{O}(v^2)$. The factor in front shouldn't be there, if p^α would be a four vector, that is the origin of the name. To understand this result lets go through the Lorentz structure of the objects we used. The energy and momentum density are the zero components of the energy momentum tensor $T^{\alpha 0}$. The integral over the whole space is a three dimensional hyperplane with the normal vector $\begin{pmatrix} 1 \\ \vec{0} \end{pmatrix}$. So p^α is the contraction of the energy momentum tensor with the normal vector of a hyperplane of simultaneity. This actually is a Lorentz vector. What we calculated in the moving coordinate system is the integral over another hyperplane of simultaneity. The integral over the whole space in a moving coordinate system just isn't the same hyperplane as the integral over the whole space in the resting system. Some authors argued, that to avoid this problem one should redefine the energy momentum vector. But the definition as the space integral over the energy momentum density of the fields seems so natural, that every other definition is somehow artificial. There is more a persuading suggestion by Poincare. The condition, that integrating the zero components of some tensor over the whole space results in a Lorentz vector is that the divergence of the tensor vanishes and that the tensor in sufficiently small at infinity. To see this one first connects to hyperplanes of simultaneity at infinity, than perform a surface integration over the flow of the tensor through the surface of the enclosed volume. There is no contribution of the part at infinity, if the tensor is sufficiently small there. The surface integral can be transformed into a volume integral with Gauss's theorem. This volume integral is zero if the divergence of the tensor vanishes, so the flow through both hyperplanes of simultaneity is the same. On first sight it seems that both conditions are fulfilled, but that can't be true. The fields are clearly small enough at infinity to give no contribution, so somewhere in the volume between two hyperplanes of simultaneity the divergence of the energy momentum tensor doesn't vanish. But the energy momentum tensor of a resting point charge should be conserved, so were did we go wrong?

The non vanishing divergence originates because we didn't described a point particle but a charged sphere with our calculation. This is an unintended consequence of the cut off. We dropped a ball with radius ϵ around the charge. But this is correct, if and only if the energy momentum tensor in this ball is zero. So by dropping this ball we solved the problem for an energy momentum tensor with cut off and not the intended energy momentum tensor. This energy momentum tensor with cut off is connected to a field strength tensor with cut off and via the Maxwell equations this field strength tensor is connected to another charge distribution as the original one. In this case the charge distribution is a homogeneously charge sphere with radius ϵ. Now its obvious, who the non vanishing divergence of the energy momentum tensor arises. A charged sphere wouldn't stay stable without an extra force, because the Lorentz force tries to expand the sphere. Poincare's suggestion is to integrate not only over the electromagnetic energy momentum tensor, but also the energy momentum tensor of the stabilising extra force. The sum out of both energy momentum tensors has vanishing divergence and therefore the integral over the whole space of the sum is a Lorentz vector. It is reasonable to incorporate such stabilising forces in the four momentum vector, because they can act as a sink our source of energy and momentum and can make a contribution to the mass. The most important idea of this chapter is, that the correct interpretation of a cut off is as an exchange of the original charge distribution with some other charge distribution.

7 Problems of the Lorentz-Dirac equation

7.1 The existence of runaway solutions

Dirac himself discussed two problems of his equation of motion, the first are runaway solutions and the second is pre-acceleration. I discuss only the first. It is possible to see their appearance in the case of one dimensional motion without an external force. In this case with $m = \frac{2}{3}$ and $e = 1$ the equation of motion takes the following form

$$a^\alpha = \dot{a^\alpha} + a^\beta a_\beta u^\alpha \tag{53}$$

The obvious solution is just u^α is constant, as it is to expect, but there are more solutions. To find those we have to realise first, that in the one dimensional case there is only one four vector orthogonal to u^α. If we normalize this vector it coincides with w^α, if we only evaluate fields at points, which lay in the axis of movement. So we are able to write the acceleration as

$$a^\alpha = Aw^\alpha \tag{54}$$

Since w^α is a unit vector too, we can also write

$$\frac{dw^\alpha}{d\tau} = Bu^\alpha \tag{55}$$

Form $a^\alpha w_\alpha = -u^\alpha \frac{dw_\alpha}{d\tau}$ follows that $A = B$. Plugging this in (53) leaves us with

$$(\frac{dA}{d\tau} - A)w^\alpha = 0 \tag{56}$$

The general solution of this equation is $A = Ce^\tau$, so we see that constant velocity is just the special case $C = 0$ out of a hole family of solutions. All other solutions have

exponentially increasing accelerations and hence exponentially increasing velocities. For that reason they are called runaway solutions. From a mathematical point of view, their existence isn't surprising, because the equation of motion is a third order differential equation, so as inertial conditions one needs position, velocity and acceleration. From a physical point of view those solutions cast at least doubt on the equation of motion, since they are clearly unphysical. Obviously something went wrong in Dirac's derivation. To get ride of those unwanted solutions Dirac states another constrain, which is to pick the acceleration as initial condition, which ensures, that $a^\alpha(\infty) = 0$. This constrain matches with the one Parrott gave, $a^\alpha(-\infty) = a^\alpha(\infty)$, in this case. To make sure, that this constrain save us from unphysical solutions, a proof would be necessary, that it is always possible, to avoid runaways, by choosing the right initial conditions. Sadly it turns out that their exist situations, where all solution are runaway solutions. This is a very serious problem, because in such situations the Lorentz-Dirac equation cant be used. At this point one has to accept, that a better equation of motion is necessary.
In the next part I present such a situation. This was originally realised by *, but what I present can by found in *. I don't present the full proof, but just try to motivate some of the ideas. We start with to particles of opposite sign of charge again in one dimensional motion. For the mass and charge we make the same choices as above. Additionally we restrict the situation to the perfect symmetrical case, with both particles moving towards each other at the begin. The calculation is done for the right particle. If a^α and w^α point in the same direction, $a^\alpha_\perp$ vanishes. This is here the case so interaction comes only form the Coulomb potential. So the equation of motion is

$$a^\alpha = \dot{a^\alpha} + a^\beta a_\beta u^\alpha - \frac{3w^\alpha}{2r^2} \tag{57}$$

What was introduced a the begin can be used here again, so we arrive at

$$\frac{dA}{d\tau} - A = \frac{3}{2r^2} \tag{58}$$

This can be written as

$$\frac{d(e^{-\tau}A)}{d\tau} = \frac{3e^{-\tau}}{2r^2} \tag{59}$$

And finally

$$A = e^\tau(\frac{3}{2}\int_0^\tau \frac{e^{-s}}{r^2} ds + A_0) \tag{60}$$

There are two more relations needed, $\frac{d(v\gamma)}{dt} = \gamma^3 \frac{dv}{dt}$ and $\frac{dv}{dt}$ and their connection to A. The four velocity is given by

$$u^\alpha = \begin{pmatrix} \gamma \\ 0 \\ 0 \\ v\gamma \end{pmatrix} \tag{61}$$

So for the four acceleration holds

$$a^\alpha = \gamma \frac{u^\alpha}{dt} = \begin{pmatrix} \gamma^4 v \frac{dv}{dt} \\ 0 \\ 0 \\ \frac{dv}{dt}\gamma^4 \end{pmatrix} \tag{62}$$

Since

$$w^\alpha = \begin{pmatrix} v\gamma \\ 0 \\ 0 \\ \gamma \end{pmatrix} \tag{63}$$

we get with (54)

$$A = \gamma^3 \frac{dv}{dt} \tag{64}$$

Those to relation can now be written as

$$\frac{d(v\gamma)}{dt} = A \tag{65}$$

and

$$\frac{dv}{dt} = (1-v^2)^{3/2} A \tag{66}$$

Now we have to have a closer look at the structure of A. The integral is always positive, hence if A gets positive once, it grows monotonously afterwards. Besides that, it is bounded from below for finite times. This also holds for $\frac{dv}{dt}$. Lets first discuss what happens if A ever gets positive. Due to its monotonous grows $\frac{dv}{dt}$ would stay positive for all times afterwards. Additionally v cant be bounded away from the speed of light, since $\frac{dv}{dt}$ vanishes only if v approaches 1. In other words, if A gets positive the solution will be a runaway solution. Now lets assume we start with a negative A. As long A doesn't get positive, a collision of the to particles would happen in finite time, since we start with $v < 0$ and $\frac{dv}{dt}$ is also negative. That means for a negative A the particles would get arbitrary close, so the integral would diverge. It follows, that A will get positive unavoidably. Those two arguments show that all solution are runaways.
For completeness lets determine, if a collision can occur at all. If the particles get close enough, the potential energy would be unbounded and due to energy conservation the kinetic energy would also be unbounded. Hence v would approach one. In this case $v\gamma = \frac{v}{(1-v^2)^{1/2}}$ would diverge. But is derivative A is bounded form bellow for finite times, so this cant happen. In summary can be said that a collision can never occur and instead the particles turn around before and fly apart with velocities approaching the speed of light.

7.2 Inapplicability of the Abraham Lorentz Dirac equation for runaway solutions

In the derivation of the Abraham Lorentz Dirac equation for point particles a cut off was used. The way the cut off is often interpreted, is as mathematical procedure to tag diverging terms and that it makes only sense combined with taking the limit to zero of the cut off parameter. In part 6.3 we have seen, that a cut off actually has to be interpreted as exchanging the original charge distribution by another one. In Dirac's derivation the cut off is equivalent to drop all fields inside his tube. A charge distribution, which produces the same fields as a point particle outside the tube and no fields inside, has to distributed only on the surface of the tube. Since Dirac's tube consisted out of spheres in the comoving coordinate frame, he solved the problem for the Lorentz model. From those considerations follows, that it is at least questionable, if the resulting equations hold for a point particle. One could argue, that by taking the cut off parameter to zero, which means shrinking the radius of the charge distribution to zero, one arrives at a point particle, but some structure dependent terms don't vanish in this limes. If the charge distribution is not spherical symmetric, the mass increase would have tensor character, which would not vanish in the limit. With this interpretation of the cut off we see, that the cut off and the limit of the cut off parameter to zero don't always have to be applied in combination. If one tries to solve the problem for an actual point particle, a cut off can't be used. I don't know another satisfying way, to handle the diverging integrals, so I use a cut off nevertheless. This means, we work with some extended charge distribution. The cut off parameter is given by the size of the charge distribution and is no arbitrary small quantity any more. Dirac used in his derivation a Tailor series in the cut off parameter and neglected all higher order terms. We know now, that the cur off parameter has some fix finite value for every extended charge distribution. So it is not clear if the cut off parameter is small enough, that higher order terms can be neglected in all cases. Runaways are an example were the higher order terms can't be neglected. To see this, we start with the four velocity and calculate it's derivatives. From (54) follows

$$u^\alpha = \begin{pmatrix} \cosh(e^\tau) \\ 0 \\ 0 \\ \sinh(e^\tau) \end{pmatrix} \tag{67}$$

For simplicity we only use an estimate for the even derivatives for $\tau > 0$.

$$\frac{d^{2n}u^\alpha}{d\tau^{2n}} > (e^\tau)^{2n} u^\alpha \tag{68}$$

Those derivatives appear in the Tailor series with corresponding powers of ϵ, hence in the higher order terms are expressions like $\frac{(\epsilon e^\tau)^{2n}}{(2n)!} u^\alpha$. Those terms can be ignored as long ϵe^τ is sufficiently small, but it is obvious, that for every finite ϵ they will get arbitrary large after enough time. For finite ϵ the Abraham Lorentz Dirac equation only holds for a finite time in the case of runaways. If one believes, that the Abraham Lorentz Dirac equation holds exactly for a point particle, one has to admit at least, that an extended charge distribution, no matter how small, and a point particle behave differently at late times in the case of runaways.

To analyse the behaviour of extended charge distributions we need an equation of motion, which holds in every case. We have seen, that the Abraham Lorentz Dirac equation doesn't hold always, because in its derivation higher order terms have been neglected, who can give a finite contribution. It follows, that we can not use only the first terms of a Tailor series, but have to calculate the complete equation of motion, similar to Parrott.

8 My own research

8.1 A uniformly accelerated charge

Uniformly accelerated charge is interesting for several reasons, for us two are especially relevant. First the retarded time condition is solvable and so it is possible to give an analytic expression for the field strength tensor in coordinate time. Second the Lorentz-Dirac equation predicts no radiation-reaction force, but Lamor's formula predicts radiation, so one could expect a contradiction. Lets start with a derivation of the equation of motion. Uniform acceleration means uniform acceleration in the comoving reference frame and not in a fixed coordinate frame. The second wouldn't be possible, since the velocity is bounded be the speed of light. We choose a coordinate frame where the motion is one dimensional in direction of the z-coordinate. Our ansatz is

$$u_\alpha(\tau) = \begin{pmatrix} u_0 \\ 0 \\ 0 \\ u_3 \end{pmatrix} = \begin{pmatrix} \sqrt{1+u_3^2} \\ 0 \\ 0 \\ u_3 \end{pmatrix} \tag{69}$$

and

$$\begin{pmatrix} \gamma & 0 & 0 & -v\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -v\gamma & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} a_0 \\ 0 \\ 0 \\ a_3 \end{pmatrix} = \begin{pmatrix} u_0 & 0 & 0 & -u_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -u_3 & 0 & 0 & u_0 \end{pmatrix} \begin{pmatrix} \dot{u}_0 \\ 0 \\ 0 \\ \dot{u}_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -g \end{pmatrix} \tag{70}$$

where g is the constant acceleration in the comoving frame. The last line results in

$$\sqrt{1+u_3^2}\,\dot{u}_3 - \frac{u_3^2 \dot{u}_3}{\sqrt{1+u_3^2}} = -g \tag{71}$$

A separation of the variable leads to

$$\frac{du_3}{\sqrt{1+u_3^2}} = -g d\tau \tag{72}$$

After integration and the substitution $u_3 = \sinh(k)$ we finally arrive at

$$z_\alpha = \begin{pmatrix} \frac{1}{g}\sinh(g\tau) \\ 0 \\ 0 \\ -\frac{1}{g}\cosh(g\tau) \end{pmatrix} \tag{73}$$

Next lets find the dependence of the trajectory on the coordinate time. We start with $t = z_0$

$$\tau = \frac{1}{g}\sinh^{-1}(gt) \tag{74}$$

$$z_\alpha = \begin{pmatrix} t \\ 0 \\ 0 \\ -\frac{1}{g}\sqrt{1+g^2t^2} \end{pmatrix} \tag{75}$$

Before we are able to derive the field strength tensor we have to express the retarded time with the coordinate time. To do that we solve $(x^\alpha - z^\alpha)(x_\alpha - z_\alpha) = 0$ for t_{ret}.

$$(t - t_{ret})^2 - x^2 - y^2 - (z - \frac{1}{g}\sqrt{1 + g^2t_{ret}^2})^2 = 0 \tag{76}$$

$$t_{ret} = \frac{t(t^2 - x^2 - y^2 - z^2 - \frac{1}{g^2}) + \sqrt{z^2(t^2 - x^2 - y^2 - z^2 + \frac{1}{g^2})^2 + 4\frac{x^2+y^2}{g^2})}}{2(t^2 - z^2)} \tag{77}$$

There is second solution with a minus sign in front of the square root with corresponds to t_{adv}. This lengthy expression justifies some abbreviations.

$$R^2 = x^2 + y^2 + z^2 \tag{78}$$

$$m = \sqrt{z^2(t^2 - x^2 - y^2 - z^2 + \frac{1}{g^2})^2 + 4\frac{x^2+y^2}{g^2})} \tag{79}$$

Now we can state z^α, u^α and a^α at t_{ret}

$$z^\alpha(t_{ret}) = \begin{pmatrix} t_{ret} \\ 0 \\ 0 \\ \frac{1}{g}\sqrt{1+g^2t_{ret}^2} \end{pmatrix} = \begin{pmatrix} \frac{t(t^2-R^2-\frac{1}{g^2})+m}{2(t^2-z^2)} \\ 0 \\ 0 \\ \frac{z(t^2-R^2-\frac{1}{g^2})+\frac{tm}{z}}{2(t^2-z^2)} \end{pmatrix} \tag{80}$$

$$u^\alpha(t_{ret}) = \begin{pmatrix} \sqrt{1+g^2t_{ret}^2} \\ 0 \\ 0 \\ gt_{ret} \end{pmatrix} = \begin{pmatrix} g\frac{z(t^2-R^2-\frac{1}{g^2})+\frac{tm}{z}}{2(t^2-z^2)} \\ 0 \\ 0 \\ g\frac{t(t^2-R^2-\frac{1}{g^2})+m}{2(t^2-z^2)} \end{pmatrix} \tag{81}$$

$$a^\alpha(t_{ret}) = \begin{pmatrix} g^2t_{ret} \\ 0 \\ 0 \\ g\sqrt{1+g^2t_{ret}^2} \end{pmatrix} = \begin{pmatrix} g^2\frac{t(t^2-R^2-\frac{1}{g^2})+m}{2(t^2-z^2)} \\ 0 \\ 0 \\ g^2\frac{z(t^2-R^2-\frac{1}{g^2})+\frac{tm}{z}}{2(t^2-z^2)} \end{pmatrix} \tag{82}$$

To use (24) we still need r, w^α and $a^\alpha w_\alpha$

$$r = (x^\alpha - z^\alpha(t_{ret}))u_\alpha(t_{ret}) = \begin{pmatrix} t - t_{ret} \\ x \\ y \\ z - z(t_{ret}) \end{pmatrix} \circ \begin{pmatrix} gz(t_{ret}) \\ 0 \\ 0 \\ -gt_{ret} \end{pmatrix} = g(tz(t_{ret}) - t_{ret}z) = \frac{gm}{2z} \quad (83)$$

$$w^\alpha = (x^\alpha - z^\alpha(t_{ret})/r - u^\alpha(t_{ret}) = \frac{2z}{gm}\begin{pmatrix} t - t_{ret} \\ x \\ y \\ z - z(t_{ret}) \end{pmatrix} - \begin{pmatrix} gz(t_{ret}) \\ 0 \\ 0 \\ gt_{ret} \end{pmatrix} \quad (84)$$

$$a^\alpha w_\alpha = \begin{pmatrix} g^2 t_{ret} \\ 0 \\ 0 \\ g^2 z(t_{ret}) \end{pmatrix} \circ \begin{pmatrix} \frac{2z}{gm}(t - t_{ret}) - gz(t_{ret}) \\ -\frac{2z}{gm}x \\ -\frac{2z}{gm}y \\ -\frac{2z}{gm}(z - z(t_{ret})) + gt_{ret} \end{pmatrix} =$$
$$\frac{2zg}{m}(tt_{ret} - zz(t_{ret}) + z(t_{ret})^2 - t_{ret}^2) = \frac{zg}{m}(t^2 - R^2 + \frac{1}{g^2}) \quad (85)$$

We are able now to calculate the fields. Lets start with the Coulomb fields

$$F_c^{10} = E_{xc} = \frac{e}{r^2}(u^1 w^0 - u^0 w^1) = \frac{8z^3 e}{g^3 m^3}(-gz(t_{ret})x) = \frac{4exz^3}{g^2 m^3}\frac{z(t^2 - R^2 - \frac{1}{g^2}) + \frac{tm}{z}}{t^2 - z^2} \quad (86)$$

$$F_c^{20} = E_{yc} = \frac{e}{r^2}(u^2 w^0 - u^0 w^2) = \frac{8z^3 e}{g^3 m^3}(-gz(t_{ret})y) = \frac{4eyz^3}{g^2 m^3}\frac{z(t^2 - R^2 - \frac{1}{g^2}) + \frac{tm}{z}}{t^2 - z^2} \quad (87)$$

$$F_c^{30} = E_{zc} = \frac{e}{r^2}(u^3 w^0 - u^0 w^3) = \frac{4ez^2}{g^2 m^2}(gt_{ret}(\frac{2z}{gm}(t - t_{ret}) - gz(t_{ret}))$$
$$- gz(t_{ret})(\frac{2z}{gm}(z - z(t_{ret})) - gt_{ret})) = \frac{8ez^3}{g^2 m^3}(tt_{ret} - zz(t_{ret}) + z(t_{ret})^2 - t_{ret}^2) =$$
$$- \frac{4ez^3}{g^2 m^3}(t^2 - R^2 + \frac{1}{g^2}) \quad (88)$$

$$F_c^{32} = B_{xc} = \frac{e}{r^2}(u^3 w^2 - u^2 w^3) = \frac{8z^3 e}{g^3 m^3}(-gt_{ret}y) = -\frac{4eyz^3}{g^2 m^3}\frac{t(t^2 - R^2 - \frac{1}{g^2}) + m}{t^2 - z^2} \quad (89)$$

$$F_c^{13} = B_{yc} = \frac{e}{r^2}(u^1 w^3 - u^3 w^1) = \frac{8z^3 e}{g^3 m^3}(-gt_{ret}x) = \frac{4exz^3}{g^2 m^3}\frac{t(t^2 - R^2 - \frac{1}{g^2}) + m}{t^2 - z^2} \quad (90)$$

$$F_c^{21} = B_{zc} = 0 \quad (91)$$

We can now continue with the radiation fields, since the calculations are very similar I just state them.

$$F_r^{10} = E_{xr} = \frac{4exz^3}{g^2m^3} \frac{z(t^2 - z^2 + x^2 + y^2 + \frac{1}{g^2}) - \frac{tm}{z}}{t^2 - z^2} \tag{92}$$

$$F_r^{20} = E_{yr} = \frac{4eyz^3}{g^2m^3} \frac{z(t^2 - z^2 + x^2 + y^2 + \frac{1}{g^2}) - \frac{tm}{z}}{t^2 - z^2} \tag{93}$$

$$F_r^{30} = E_{zr} = \frac{-8e(y^2 + x^2)z^3}{g^2m^3} \tag{94}$$

$$F_r^{32} = B_{xr} = -\frac{4eyz^3}{g^2m^3} \frac{t(t^2 - z^2 + x^2 + y^2 + \frac{1}{g^2}) - m}{t^2 - z^2} \tag{95}$$

$$F_r^{13} = B_{xr} = \frac{4exz^3}{g^2m^3} \frac{t(t^2 - z^2 + x^2 + y^2 + \frac{1}{g^2}) - m}{t^2 - z^2} \tag{96}$$

$$F_r^{21} = B_{zr} = 0 \tag{97}$$

After adding radiation and Coulomb fields we arrive at

$$E_x = \frac{8exz^4}{g^2m^3} \tag{98}$$

$$E_y = \frac{8eyz^4}{g^2m^3} \tag{99}$$

$$E_z = -4ez^3 \frac{t^2 + x^2 + y^2 - z^2 + \frac{1}{g^2}}{g^2m^3} \tag{100}$$

$$B_x = -\frac{8eytz^3}{g^2m^3} \tag{101}$$

$$B_y = \frac{8extz^3}{g^2m^3} \tag{102}$$

Surprisingly the Coulomb and the radiation fields alone are more complicated then their sum. They were first derived by Max Born and can be found in [8] and of course in [5]. We are no able to check, what I claimed in the introduction of this chapter. Lets start with the Lorentz-Dirac force.

$$\dot{a^\alpha} + a^\beta a_\beta u^\alpha = \begin{pmatrix} g^2\cosh(g\tau) \\ 0 \\ 0 \\ g^2\sinh(g\tau) \end{pmatrix} + (\begin{pmatrix} g\sinh(g\tau) \\ 0 \\ 0 \\ g\cosh(g\tau) \end{pmatrix} \circ \begin{pmatrix} g\sinh(g\tau) \\ 0 \\ 0 \\ -g\cosh(g\tau) \end{pmatrix}) \begin{pmatrix} \cosh(g\tau) \\ 0 \\ 0 \\ \sinh(g\tau) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{103}$$

Lamor's formula on the other hand predicts an energy momentum flow of

$$a^\beta a_\beta u^\alpha = -g^2 \begin{pmatrix} \cosh(g\tau) \\ 0 \\ 0 \\ \sinh(g\tau) \end{pmatrix} \tag{104}$$

The difference is of cause $\dot{a^\alpha}$. How can energy momentum conservation hold here? The problem here is how to interpret those two quantities. One way out is to claim, that $a^\alpha(\tau_1) = a^\alpha(\tau_2)$ is a necessary condition for energy momentum conservation to hold, like Parrott did. The idea behind this is, that the integral $\int_{\tau_1}^{\tau_2} \dot{a^\alpha}$ vanishes then. But if this is true, than there must be a mistake in Dirac's derivation, because it is essentially based on energy momentum conservation. To build up some intuition lets perform the derivation of the general results in this special case again. This time we start with Lamor's formula. The calculation is much easier, if we move the trajectory in such a way that it will go through the coordinate origin and than introduce spherical coordinates. This is done by $z_{old} \to z_{new} + \frac{1}{g}$. We now have to integrate over sphere with radius r centred around the origin at time $t = r$. The most interesting component is the 0-component, so we only calculate this one. The normal vector of this surface is $\begin{pmatrix} 0 \\ \vec{e_r} \end{pmatrix}$. Contracting it with $4\pi T^{0\alpha}$ is just $(\vec{E} \times \vec{B}) \circ \vec{e_r}$. Hence we have to calculate

$$(\vec{E} \times \vec{B}) \circ \vec{e_r} = \frac{32te^2\left(x^2+y^2\right)\left(t^2+x^2+y^2+z^2\right)}{\sqrt{x^2+y^2+z^2}\left(t^4+(x^2+y^2+z^2)\left(\frac{4}{g^2}+x^2+y^2+\frac{4z}{g}+z^2\right)-2t^2\left(x^2+y^2+z(\frac{2}{g}+z)\right)\right)^3 g^4} \tag{105}$$

and with

$$\begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} r \\ r\sin(\theta)\cos(\phi) \\ r\sin(\theta)\sin(\phi) \\ r\cos(\theta) \end{pmatrix} \tag{106}$$

this simplifies to

$$\frac{g^2e^2\sin(\theta)^2}{r^2} \tag{107}$$

After integration we finally get just

$$\int_0^{2\pi} d\phi \int_0^{\pi} d\theta r^2 \sin(\theta) \frac{g^2e^2\sin(\theta)^2}{4\pi r^2} = \frac{2}{3}g^2e^2 \tag{108}$$

This matches perfectly with the general result for $\tau = 0$. The only difference is that in the general case we had to take the limit $r \to \infty$, what was not necessary here. The reason therefor is, that $\vec{B_c} = 0$ for $t = r$. The particle is at rest at $\tau = 0$, so it's Coulomb

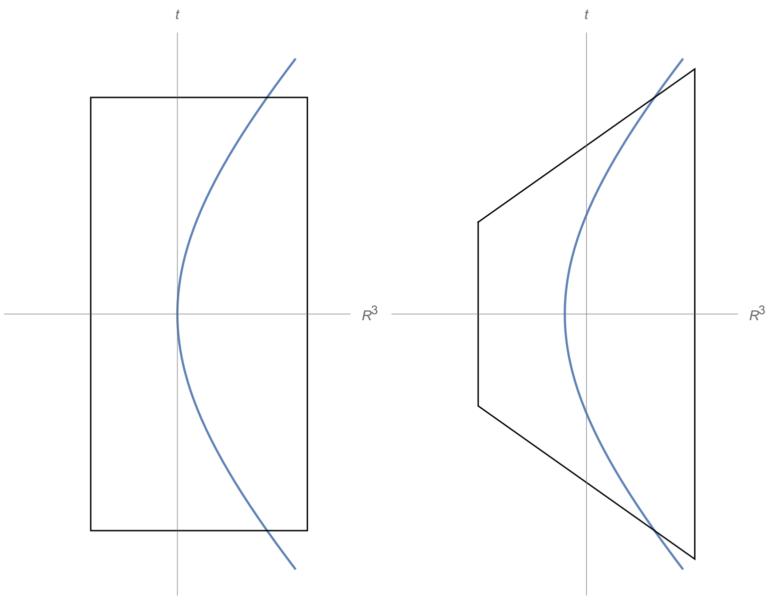

Figure 4: The tube used by Parrott

fields are just electric fields and no magnetic fields.

Now we can go on to Dirac's calculation. Instead of his quite complex tube, we use a very simple one. The caps are just planes of simultaneity in coordinate time $t = \tau_1$ and $t = -\tau_1$, and the tube itself are just spheres around the origin at times form $-\tau_1$ to τ_1, as shown in figure 4. The radius r of the spheres should be big enough, that the trajectory enters and leaves the tube trough the caps, but this isn't even necessary for the calculation, it's just for aesthetics. This tube has the advantage, that we can relay completely on symmetry arguments. After this we will see why this tube give the same results as Dirac's tube. First we have to find the normal vectors on the different surfaces. Given the simple geometry they can just be stated $\begin{pmatrix} 0 \\ \vec{e_r} \end{pmatrix}$ for the tube and $\begin{pmatrix} \pm 1 \\ 0 \end{pmatrix}$ for the upper and lower cap. We restrict our calculation again to the 0-component, so we don't even need mass renormalisation. Instead of a real calculation it is enough to realise, that $\vec{E}(t) = \vec{E}(-t)$ and $\vec{B}(-t) = -\vec{B}(t)$. For the cap integrals we need only $4\pi T^{00} = \frac{1}{2}(E^2 + B^2)$, which is the same for both caps. Therefore the caps together with the divergences just chancel each other. The tube integration needs the Pointing vector $\vec{E} \times \vec{B}$, which is asymmetric with respect to the time. Since the integration runs over symmetric region in time, this integral vanishes too. This matches with the general result. Now we have to convince ourself, that this result also holds for a tube formed like a Dirac tube. Gauss's theorem allows us to deform the tube as long the same amount of charge is contained in it. We will see soon, that the angel between trajectory and caps actually matters. Given this we have to use the caps similar to Dirac's caps, planes of simultaneity in the comoving frame, but are still able to use the same tube. The arguments from above for the tube and the 0-component of the caps still hold, but we get another contribution form the cap integral. The normal vector form the caps now has some additional part showing in z-direction with the same sign for the upper and lower cap. This additional part brings in one component of the Pointing vector, which is antisymmetric. Hence the caps cancel each other again, and the total result stays zero. Since we finished the calculations we can now try to make sense out of our results. We know there is no total energy flow trough a tube surrounding the trajectory and we know there is energy flow away from our particle. A natural guess is, that the energy comes form the fields of particle. To motivate this guess lets assume we would try to build machine, which produces energy out of nothing, by using this case of radiation and no radiation reaction force. For a machine it would of cause be impossible to constantly accelerate a charge infinitely long. But it would be enough to turn around the sign of the accelerating force in the middle. So the trajectory would be connected hyperboles with alternating direction. The radiation energy could be collected by big solar cell surrounding the machine. The energy for accelerating the charge would go completely in kinetic energy, which is refundable. The only exception is the moment, when the force changes direction. To calculate the efficiency of the machine we have to compare the amount of radiated energy with the extra energy need in the moment of changing direction of the force. There is no reason for an exact calculation, so we only perform a broad non relativistic estimate. This means the acceleration is just constant. With Lamor's formula we get for the radiated energy for on cycle

$$\frac{2g^2e^2}{3}t \tag{109}$$

with the time t, which is needed for the cycle. Additionally lets assume the time, which is needed to turn around the force is $\triangle t$, which is very small. The non relativistic version of the Lorentz-Dirac force is just $2/3e^2\dot{a}$, whose average value is $\frac{4e^2g}{3\triangle t}$ during the direction change of the force. So we get for the total energy with $v = 1/2gt$

$$E = F \triangle s = \frac{4e^2g}{3\triangle t}v \triangle t = \frac{2g^2e^2}{3}t \tag{110}$$

It shouldn't be surprising, that those two are equal, so our machine is useless. But we learn, that the total energy, which is radiated away during the whole time, flows from the charge into the fields just at the begin and the end of the phases of constant acceleration. In other words, the radiation energy is already contained in the fields during the phases of constant acceleration and just wanders along with the particle.

The next step is to actually test the guess from above, that energy is already contained in the fields and doesn't come from the particle. To do this we take spheres center at the origin at several times, from $r - t_1$ to $r + t_1$, and connect the first and the last sphere with a light cone with the trajectory, as shown in figure 5. So we constructed a tube similar to the one Parrott used. If the guess is correct the difference in the energy flow through the two light cones is the same as the flow though the spheres. Lets start with calculating the flow through a light cone. We only calculate the 0-component again and we move our trajectory in such a way, that the particle is at the origin at time t_1.

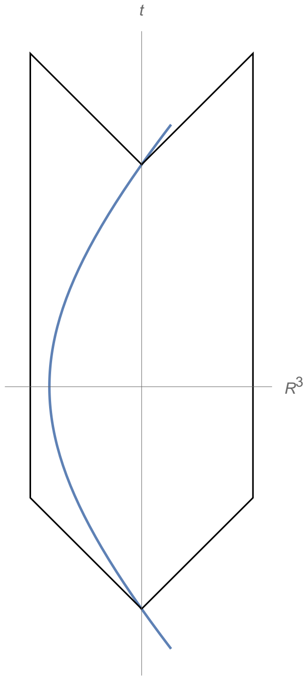

Figure 5: The tube used by Parrott

This is done by $z_{old} \rightarrow z_{new} + \frac{1}{g}\sqrt{1+g^2t_1^2}$. The normal vector on a light cone is $\begin{pmatrix} 1 \\ \vec{e_r} \end{pmatrix}$. Furthermore $t = t_1 + \sqrt{x^2+y^2+z^2}$ holds. So we have to integrate over

$$
\begin{aligned}
&\frac{1}{2}(E^2+B^2) - \vec{e_r} \circ (\vec{E} \times \vec{B}) = \\
&\frac{-\left(x^2+y^2+z^2\right)^{3/2}}{2\sqrt{x^2+y^2+z^2}\left(\left(1+g^2t_1^2\right)x^2+\left(1+g^2t_1^2\right)y^2+z\left(z+2g^2t_1^2z-2g^2t_1\sqrt{\frac{1}{g^2}+t_1^2}\sqrt{x^2+y^2+z^2}\right)\right)^3} \\
&-\frac{g^2t_1\left(x^2\left(-2\sqrt{\frac{1}{g^2}+t_1^2}z+t_1\sqrt{x^2+y^2+z^2}\right)+y^2\left(-2\sqrt{\frac{1}{g^2}+t_1^2}z+t_1\sqrt{x^2+y^2+z^2}\right)\right)}{2\sqrt{x^2+y^2+z^2}\left(\left(1+g^2t_1^2\right)x^2+\left(1+g^2t_1^2\right)y^2+z\left(z+2g^2t_1^2z-2g^2t_1\sqrt{\frac{1}{g^2}+t_1^2}\sqrt{x^2+y^2+z^2}\right)\right)^3} \\
&-\frac{g^2t_1\left(2z^2\left(-\sqrt{\frac{1}{g^2}+t_1^2}z+t_1\sqrt{x^2+y^2+z^2}\right)\right)}{2\sqrt{x^2+y^2+z^2}\left(\left(1+g^2t_1^2\right)x^2+\left(1+g^2t_1^2\right)y^2+z\left(z+2g^2t_1^2z-2g^2t_1\sqrt{\frac{1}{g^2}+t_1^2}\sqrt{x^2+y^2+z^2}\right)\right)^3}
\end{aligned}
\tag{111}
$$

In spherical coordinates this simplifies to

$$
\frac{2}{r^4\left(2+3g^2t_1^2-4g^2t_1\sqrt{\frac{1}{g^2}+t_1^2}\cos(\theta)+g^2t_1^2\cos(2\theta)\right)^2}
\tag{112}
$$

Integration over the angels leads to

$$\frac{2\pi\left(3+4g^2t_1^2\right)}{3r^2} \tag{113}$$

An integration over r is not possible, because this integral would diverge. So the total flow through the light cone at t_1 is a diverging integral. Luckily we only interested in the difference in flow through the light cones centred at the trajectory at $-t_1$ and t_1. Since t_1 only appears squared in the derived expression, the flow is exactly the same. This means the guess is wrong. The same amount of energy flows through both light cones, so the energy flowing through the spheres must come from the particle. When I encountered this first, I though this would contradict the result form above with the tube with straight caps. The contradiction resolve itself, if one considers not only the forward light cones but also the backward light cones. If one repeats the calculation for the flow through the backward light cones, one finds the same result as for the forward light cones. The flow through the spheres on the other hand just changes sign, because the calculation involves the Pointing vector, which just changes sign under $t \to -t$. Hence the particle has exactly the same absorption rate as radiation rate. This explains why the result for the total flow through the tube with straight capes is zero, and there is still an outflow of energy. A vanishing total flow just means out and in flow of energy has to be the same. This also matches with the lesson from the fictional machine from above. The energy is contained in the fields the hole time, but it get absorbed and re-emitted. This was missing in the guess.
Now lets discuss, how this results can be connected with Parrott's work and if we are able to explain now, why the result of Parrott's and Dirac's calculation differ. A change in form of the tube doesn't change the result of the calculation, according to Gauss's theorem, but this doesn't hold directly by the charge. To see this, imaging a point charge resting a the origin for the hole time. In this case we have only an electric Coulomb field. A surface integral over a sphere around the origin gives exactly the amount of charge times 4π. Next lets take an surface integral over a spherical sector. The side walls don't contribute, because their normal vector is orthogonal to the field. The result is the total charge times the space angel of the spherical sector, which is not the same. In case of a point charge one could try to argue that the hole sphere and the spherical sector both contain the particle, but is then forced to admit that the form of the surface matters, if the charge lies on the surface. In the case of an extended charge, the spherical sector and the hole sphere just contain a different amount of charge, so there is no surprise at all that the results differ. Inspired by this, we conclude that only the form of the caps directly by the trajectory matters and the rest of the tube can be deformed in any way we want, without changing the result.
From this follows, that the tube with forward light cones and the tube Parrott used give equivalent results. The same is true for Dirac's tube and the tube with straight light cones. Even the tube with backward light cones fits in this framework. To see this, repeat Parrott's calculation with advanced fields and a tube with backward light cones. Only the sign in front of the radiation reaction term changes. In the case of constant acceleration the retarded and the advanced fields are identical. So the change in sign

Figure 6: A modified cap connected to Parrott's tube

by changing from forward to backward light cones happens in the general case too. By now it is obvious, why Dirac's and Parrott's result differ. They use different caps. This implies, almost in a natural way, the question, which choice for the caps is physical meaningful. Is it one of the two above or do we have to use completely different caps?
At this point in my research, I didn't had any straight forward method at hand to provide an answer to this question. My first inspiration came from a mathematical constrain for Gauss's theorem. It is only valid for smooth surfaces, and light cone caps aren't smooth. Normally this can be circumvented easily, by rounding up the edges in a very small region. This changes the total result in case of continuous fields by something proportional to the size of the arbitrary small region, which can safely be ignored. But the fields at the tip of the light cones diverge and so the continuity argument doesn't work here. Hence Parrott's caps have to be replaced by some caps, how are smooth at the trajectory. Additional I had the hope, that any smooth space like surface as cap would lead to the same end result. Motivated by this, I tried to find a cap, which is smooth and still simple enough, that an exact solution is possible. Additionally the new cap should still fit tougher with Parrott's tube.
The energy momentum tensor is given in dependence of the four coordinates, I call them light cone coordinates, τ, r, ϕ and θ. The simplest way to define a three dimensional hyper surface is by setting one of then constant. This is exactly the way, how Parrott's tube is constructed. The caps are given by setting τ constant and the tube is given by setting r constant. In general every hyper surface can be described by replacing one of those coordinates with a function of the three other coordinates, the replaced coordinate isn't necessarily the same on the hole hyper surface. The only reasonably choice to get a closed tube, is to replace r. We have no interest to break the spherical symmetry, therefore r should only be a function of τ and not of the angels. After this consideration it is easy to find cap which full files all constrains. It is defined by $r = \tau_1 - \tau$. So the complete cap is just

$$c^\alpha = z^\alpha + (\tau_1 - \tau)(u^\alpha + w^\alpha) \tag{114}$$

This cap is a smooth space like surface and it can connected to Parrott's tube. To do it choose one $\tau_0 < \tau_1$ and choose as domain for the cap $[\tau_0, \tau_1]$. A fitting tube is then given by $r = \tau_1 - \tau_0$.

8.2 Calculation of the flow through the modified caps

Other than before, it isn't possible any more to just state the normal vector on the caps from above. To derive it we use some standard methods of differential geometry. The tangent vectors on the surface at one point are just given by derivatives with respect to the coordinates at this point. So we get the three tangent vectors by deriving c^α in direction τ, θ and ϕ. The normal vector is orthogonal to all tangent vectors. Up to the sign there exists only one unit vector, which is orthogonal to all tangent vectors. So if we are able to construct a vector orthogonal to the tangent vectors, we only need to fix the length to find the normal vector. The standard way of constructing something orthogonal is done with the total antisymmetric epsilon tensor $\epsilon^{\alpha\beta\gamma\delta}$. Contracting it with the three tangent vectors give us something orthogonal to all tangent vectors, which is exactly what we were looking for

$$n_\delta = \epsilon_{\alpha\beta\gamma\delta}\partial_\tau c^\alpha \partial_\theta c^\beta \partial_\phi c^\gamma \tag{115}$$

The length of n^δ is the volume spanned by the tangent vectors and an unit normal vector. This is nothing else than Jacobian determinate, so we don't even need to adjust it. The sign is a matter of the orientation, hence we can pick the one we want in the end, but we have to choose in a consistent way before. In the language of differential geometry the normal vector is just the hodge dual of the wedge product of all tangent vectors. The calculation is much simpler, if we use a basis of orthogonal unit vectors, which are adapted to the problem. Our choice is $u^\alpha, w^\alpha, \partial_\theta w^\alpha = \theta^\alpha$ and $\partial_\phi w^\alpha = \sin(\theta)\phi^\alpha$. All relevant quantities have to be written as a projection on those unit vectors. So instead of contracting the normal vector with the hole energy momentum tensor, we calculate the contraction with each basis vector first. Than we split up the energy momentum tensor and integrate each part separately. First we develop the derivatives

$$\partial_\tau c^\alpha = u^\alpha - (u^\alpha + w^\alpha) + (\tau_1 - \tau)(a^\alpha + \partial_\tau w^\alpha) \tag{116}$$

$$\partial_\theta c^\alpha = (\tau_1 - \tau)\theta^\alpha \tag{117}$$

$$\partial_\phi c^\alpha = (\tau_1 - \tau)\sin(\theta)\phi^\alpha \tag{118}$$

To make further progress we develop $\partial_\tau w^\alpha$

$$\partial_\tau w^\alpha = -w^\beta a_\beta u^\alpha + c_1\theta^\alpha + c_2\phi^\alpha \tag{119}$$

We used here that $\partial_\tau w^\alpha w_\alpha = 0$, because w^α is a unit vector and that $\partial_\tau w^\alpha u_\alpha = -w^\alpha a_\alpha$, which follows form $\partial_\tau(w^\alpha u_\alpha) = 0$. Moreover c_1 and c_2 are just some constants, which will drop out in the end. After developing $a^\alpha = -a^\beta w_\beta w^\alpha + ...$ too, we get

$$\partial_\tau c^\alpha = -(\tau_1 - \tau)w^\beta a_\beta u^\alpha - ((\tau_1 - \tau)a^\beta w_\beta + 1)w^\alpha + c_3\theta^\alpha + c_4\phi^\alpha \tag{120}$$

Finally we are ready to contract the normal vector. We start with u^δ

$$u^\delta n_\delta = -\sin(\theta)(\tau_1 - \tau)^2((\tau_1 - \tau)a^\beta w_\beta + 1)\epsilon_{\alpha\beta\gamma\delta}w^\alpha\theta^\beta\phi^\gamma u^\delta \tag{121}$$

All other terms have one vector twice contracted with the epsilon tensor and drop out therefore. Since all vectors are orthonormal $\epsilon_{\alpha\beta\gamma\delta}w^\alpha\theta^\beta\phi^\gamma u^\delta$ is just plus or minus one, so we choose as our convention $\epsilon_{\alpha\beta\gamma\delta}u^\alpha w^\beta\theta^\gamma\phi^\delta = 1$, so (121) gets positive. The next contraction is with w^δ

$$w^\delta n_\delta = -w^\beta a_\beta \sin(\theta)(\tau_1 - \tau)^3 \tag{122}$$

The contractions with θ^δ and ϕ^δ vanish. Before we can go through the integration term by term, we need $\eta^{\alpha\beta} = u^\alpha u^\beta - w^\alpha w^\beta - \theta^\alpha\theta^\beta - \phi^\alpha\phi^\beta$. After plugging this in energy momentum tensor, we are ready

$$\int \frac{1}{2r^4}u^\alpha u^\beta n_\beta d\tau d\theta d\phi = \int \frac{\sin(\theta)}{2(\tau_1-\tau)^2}((\tau_1-\tau)a^\beta w_\beta + 1)u^\alpha d\tau d\theta d\phi \tag{123}$$

Before we continue it is worth to go through all integrals over the angels, which appear, in a systematic way. Angels are only contained in w^α, θ^α and ϕ^α. In the comoving frame at the retarded time w^α is just

$$w_0^\alpha = \begin{pmatrix} 0 \\ \sin(\theta)\cos(\phi) \\ \sin(\theta)\sin(\phi) \\ \cos(\theta) \end{pmatrix} \tag{124}$$

In an arbitrary inertial frame w^α only differs from w_0^α by a Lorentz transformation. A Lorentz matrix is only a function of u^α, which has no dependence on the angles, and can be pulled out of the integration. The same applies for θ^α and ϕ^α. So we only need to integrate over the quantities in the rest frame. θ^α and ϕ^α will only appear in this combination $\theta^\alpha\theta^\beta + \phi^\alpha\phi^\beta$. After pulling out the Lorentz matrices, this can be written as

$$\theta_0^\alpha\theta_0^\beta + \phi_0^\alpha\phi_0^\beta = m^{\alpha\beta} - w_0^\alpha w_0^\beta \tag{125}$$

with

$$m^{\alpha\beta} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{126}$$

So integrations have only to be performed over different numbers of w^α. So lets go through them

$$\int \sin(\theta)d\theta d\phi = 4\pi \tag{127}$$

$$\int w_0^\alpha \sin(\theta)d\theta d\phi = 0 \tag{128}$$

$$\int w_0^\alpha w_0^\beta \sin(\theta)d\theta d\phi = \frac{4\pi}{3}m^{\alpha\beta} \tag{129}$$

$$\int w_0^\alpha w_0^\beta w_0^\gamma \sin(\theta)d\theta d\phi = 0 \tag{130}$$

$$\int w_0^\alpha w_0^\beta w_0^\gamma w_0^\delta \sin(\theta) d\theta d\phi = \frac{4\pi}{15}(m^{\alpha\beta}m^{\gamma\delta} + m^{\alpha\gamma}m^{\beta\delta} + m^{\alpha\delta}m^{\gamma\beta}) \tag{131}$$

The only missing thing is the effect of the Lorentz transformations on the $m^{\alpha\beta}$. To determinate it, first we use that

$$m^{\alpha\beta} = \delta_0^\alpha \delta_0^\beta - \eta^{\alpha\beta} \tag{132}$$

and second $\Lambda_0^\alpha = \Lambda_\alpha^0 = u^\alpha$. So we find

$$\Lambda_\gamma^\alpha m^{\gamma\delta} \Lambda_\delta^\beta = \Lambda_\gamma^\alpha (\delta_0^\gamma \delta_0^\delta - \eta^{\gamma\delta}) \Lambda_\delta^\beta = u^\alpha u^\beta - \eta^{\alpha\beta} \tag{133}$$

With this formulas at hand, we just have to plug them in and simplify. The first integral is just

$$\int \frac{2\pi u^\alpha}{(\tau_1 - \tau)^2} d\tau \tag{134}$$

The second is

$$\int \frac{-1}{2r^4} w^\alpha w^\beta n_\beta d\tau d\theta d\phi = \int \frac{\sin(\theta)}{2(\tau_1 - \tau)} w^\beta a_\beta w^\alpha d\tau d\theta d\phi = \int \frac{2\pi}{3(\tau_1 - \tau)} a_\beta (u^\alpha u^\beta - \eta^{\alpha\beta}) d\tau = \int \frac{-2\pi a^\beta}{3(\tau_1 - \tau)} d\tau \tag{135}$$

The three integral with contractions with $\phi^\alpha, \theta^\alpha$ and $a_\perp^\alpha$ vanish, since they are orthogonal to n^α. So the missing two terms contain both $(u^\alpha + w^\alpha)n_\alpha$, which is just $\sin(\theta)(\tau_1 - \tau)^2$. The next integral is

$$\int \frac{1}{r^3} a_\perp^\alpha (u^\beta + w^\beta) n_\beta d\tau d\theta d\phi = \int \frac{a^\alpha + a^\gamma w_\gamma w^\gamma}{\tau_1 - \tau} \sin(\theta) d\tau d\theta d\phi = \int \frac{4\pi a^\alpha + \frac{4\pi}{3} a_\gamma (u^\gamma u^\alpha - \eta^{\gamma\alpha})}{\tau_1 - \tau} d\tau = \int \frac{8\pi a^\alpha}{3(\tau_1 - \tau)} d\tau \tag{136}$$

For the last integral we use $a_\perp^\alpha a_{\alpha\perp} = a^\alpha a_\alpha + (a_\alpha w^\alpha)^2$

$$\int \frac{-1}{r^2} (a^\gamma a_\gamma + (a_\gamma w^\gamma)^2)(u^\alpha + w^\alpha)(u^\beta + w^\beta) n_\beta d\tau d\theta d\phi = \int - \sin(\theta)(a^\gamma a_\gamma + (a_\gamma w^\gamma)^2)(u^\alpha + w^\alpha) d\tau d\theta d\phi = \int -\frac{8\pi}{3} a^\gamma a_\gamma u^\alpha d\tau \tag{137}$$

After collecting all terms and introducing a cut off ϵ, we get

$$\int \frac{2\pi u^\alpha}{(\tau_1 - \tau)^2} + \frac{2\pi a^\alpha}{\tau_1 - \tau} d\tau - \int \frac{8\pi}{3} a^\gamma a_\gamma u^\alpha d\tau = \frac{2\pi u^\alpha(\tau_0)}{\tau_1 - \tau_0} - \frac{2\pi u^\alpha(\tau_1 - \epsilon)}{\epsilon} - \int_{\tau_1}^{\tau_0} \frac{8\pi}{3} a^\gamma a_\gamma u^\alpha d\tau \tag{138}$$

We must add this cap now for τ_1 and subtracted for τ_2 to Parrott's tube. The tube runs from τ_0 to $\tau_2 - \tau_1 + \tau_0$ instead from τ_1 to τ_2. After putting the missing $\frac{e^2}{4\pi}$ in, we get a very similar result as Parrott did.

$$\frac{e^2}{2\epsilon}(u^\alpha(\tau_2 - \epsilon) - u^\alpha(\tau_1 - \epsilon)) - \frac{2e^2}{3}\int a^\gamma a_\gamma u^\alpha d\tau \tag{139}$$

Obviously the cap modification failed, to achieve the intended goal to get a force orthogonal to the four velocity. At least the concerns relating the applicability of Gauss's theorem, could be resolved. Additionally the hope, that any smooth space like cap gives the same result is disappeared. What is also surprising, that the caps look close to the trajectory quite similar to Dirac's caps. One would therefore expect the same result, since only the cap form close to the trajectory matters for the end result. So lets examine how close those caps are. To do this we look again at the case of constant accelerated charge, since we have already derived all necessary quantities. Furthermore we choose $\tau_1 = 0$, so Dirac's cap, the hyperplane of simultaneity in the comoving frame, is just given by $t = 0$. That means we only have to investigate the maximal deviation of the 0-component of the modified cap close to the trajectory. This deviation is maximal in the direction of the z-coordinate, $\theta = 0$.

$$c^0 = \frac{1}{g}\sinh(g\tau) - \tau(\cosh(g\tau) + \sinh(g\tau)) = -g\tau^2 + \mathcal{O}(\tau^3) \tag{140}$$

Normally it should be unproblematic, that the deviation is of the order τ^2, because getting close to the trajectory means taking the limit $\lim_{\tau\to 0}$ and for any smooth function the deviation would only give a vanishing contribution. But we are dealing with a function, how diverges like $\frac{1}{r^2}$ close to the trajectory, so there still can be a finite contribution form the deviation. At least we now have understood why this modified caps give a different result than Dirac's caps. To make further progress we need to modify the caps in some other way. This is done in the next chapter

8.3 Flow to orthogonal caps

The first part of this chapter is to motivate the form of the caps, which I have chosen. Sadly there is no argument, which completely fixes their form or I am at least not aware of it. But it is possible to give some motivation for my choice. First let us recall our aim of finding an expression for the force , which is orthogonal to the four velocity. This worked for Dirac's caps in the general case and we were able to reproduce this result even without mass renormalisation or using a tailor series in the case of constant acceleration. So those caps should full file our aim. Second the only caps, how are physically distinguished by the trajectory, are the caps orthogonal to it. So our choices for the caps are the one, how are orthogonal to the trajectory, or some with a deviation, which is at order of $\frac{1}{r^3}$ or smaller. Our first step is to parametrise such an orthogonal plane with light cone coordinates. Luckily this is possible. The normal vector of the plane is the tangent vector of the trajectory at the interception point. This is just $u^\alpha(\tau_1)$.

If we now connect an arbitrary point with some point in the plane, the connection vector and the four velocity are orthogonal, if and only if the point is in the plane. Our point in the plane is $z^\alpha(\tau_1)$. It makes sense to choose r as function of the other coordinates, for the same reasons as above. So we get the equation for every point x^α in the plane

$$(x^\alpha - z^\alpha(\tau_1))u_\alpha(\tau_1) = 0 \tag{141}$$

Now we write x^α in light cone coordinates and solve for r

$$(z^\alpha + r(u^\alpha + w^\alpha) - z^\alpha(\tau_1))u_\alpha(\tau_1) = 0 \tag{142}$$

$$r = \frac{(z^\alpha(\tau_1) - z^\alpha)u_\alpha(\tau_1)}{(u^\alpha + w^\alpha)u_\alpha(\tau_1)} \tag{143}$$

A cap orthogonal to the trajectory at τ_1 is given by

$$c^\alpha = z^\alpha + \frac{(z^\alpha(\tau_1) - z^\alpha)u_\alpha(\tau_1)}{(u^\alpha + w^\alpha)u_\alpha(\tau_1)}(u^\alpha + w^\alpha) \tag{144}$$

But before we are able to dive in the calculation we have to clear out several other problems we haven't faced in chapter 8.2. First there is no simple way to connect this cap with the tube of Parrott. My first try was to modify the cap and to use that it is enough be at least $\frac{1}{r^3}$ close to (144) near the trajectory. I did come up with

$$r = \frac{(z^\alpha(\tau_1) - z^\alpha)u_\alpha(\tau_1)}{(u^\alpha + w^\alpha)u_\alpha(\tau_1)} \frac{(\tau_1 - \rho - \tau)^5}{(\tau_1 - \rho - \tau)^5 + (\tau_1 - \tau)^5} + \rho \frac{(\tau_1 - \tau)^5}{(\tau_1 - \rho - \tau)^5 + (\tau_1 - \tau)^5} \tag{145}$$

For the power anything else then five could be chosen, as long as it is big enough to ensure there is no deviation around $\tau = \tau_1$ bigger then $\frac{1}{\rho^3}$. This cap would now fit perfectly with Parrott's tube with $r = \rho$ starting by $\tau_1 - \rho$. Sadly this new cap turned out to be to complicated for me to perform the integration. So I decided to find my own tube. This was quite tempting, since it is done easily in this case. Just exchange τ_1 with $\rho + \tau$. Additionally I realised shortly after, that the integration over the caps can be circumvented. This is only the case, because we have to use a cut off around the trajectory. Hence we have to integrate the caps form some ϵ to ρ and the tube with radius ρ. But if we set $\rho = \epsilon$ and just do the tube, we would have gotten exactly the same result. The reason therefor is that the caps and both tubes together enclose some volume, which doesn't contain any charge. This is great news, because we just got ride of half of the work, but it should be crystal clear that the form of the caps still matters. Second there appeared another problem. Due to $\tau_1 \to \rho + \tau$ the integrand turns into a delay equation. This isn't surprising, since Dirac originally had to deal with a delay equation, and we use similar caps. The situation we are facing is still much easier, because the delay is just constant in our case, while for Dirac's tube it depended on the trajectory in a complex way. The exact difference of our and Dirac's caps and the consequences are discussed in chapter *. My main concern was that an integration over τ is hard for a delay equation and maybe not analytically possible. But this integration can also be circumvented, because we are interested on the force and not it's integral. If one absorbs the cap integrals in the tube integral, in the way described above, * reduces to

$$\int T^{\alpha\beta} d^3\sigma_\beta = mu^\alpha(\tau_1) - mu^\alpha(\tau_2) \tag{146}$$

And now we just differentiate both sides with respect to τ_2. On the left side we get just rid of the unwanted integral, while on the right side we get the usual expression for the force.
Now were are ready to start the calculation. Sadly it is much longer than the one in the last chapter, because r is now a function of τ, θ and ϕ and not just τ. The structure on the other hand is exactly the same. So we can reuse all results from above, we just have to exchange r. Lets start with the derivatives of r

$$\partial_\tau r = \frac{((u^\alpha(\tau+\epsilon) - u^\alpha)u_\alpha(\tau+\epsilon) + (z^\alpha(\tau+\epsilon) - z^\alpha)a_\alpha(\tau+\epsilon))(u^\beta + w^\beta)u_\beta(\tau+\epsilon)}{((u^\alpha + w^\alpha)u_\alpha(\tau+\epsilon))^2} - \frac{(z^\beta(\tau+\epsilon) - z^\beta)u_\beta(\tau+\epsilon)((a^\alpha + \partial_\tau w^\alpha)u_\alpha(\tau+\epsilon) + (u^\alpha + w^\alpha)a_\alpha(\tau+\epsilon))}{((u^\alpha + w^\alpha)u_\alpha(\tau+\epsilon))^2} \tag{147}$$

$$\partial_\theta r = -\frac{(z^\alpha(\tau+\epsilon) - z^\alpha)u_\alpha(\tau+\epsilon)(\partial_\theta w^\beta u_\beta(\tau+\epsilon))}{((u^\alpha + w^\alpha)u_\alpha(\tau+\epsilon))^2} \tag{148}$$

$$\partial_\phi r = -\frac{(z^\alpha(\tau+\epsilon) - z^\alpha)u_\alpha(\tau+\epsilon)(\partial_\phi w^\beta u_\beta(\tau+\epsilon))}{((u^\alpha + w^\alpha)u_\alpha(\tau+\epsilon))^2} \tag{149}$$

Next we calculate the tangent vectors

$$\partial_\tau c^\alpha = u^\alpha + \partial_\tau r(u^\alpha + w^\alpha) + r(a^\alpha + \partial_\tau w^\alpha) \tag{150}$$

$$\partial_\theta c^\alpha = \partial_\theta r(u^\alpha + w^\alpha) + r(\partial_\theta w^\alpha) \tag{151}$$

$$\partial_\phi c^\alpha = \partial_\phi r(u^\alpha + w^\alpha) + r(\partial_\phi w^\alpha) \tag{152}$$

To evaluate the normal vector, we develop the tangent vectors in the unit vectors. We put those components together with the unit vectors in a determinant and develop after the last column. For the the unit vector u^α we get

$$n_\delta u^\delta = \epsilon_{\alpha\beta\gamma\delta}\partial_\tau c^\alpha \partial_\theta c^\beta \partial_\phi c^\gamma u^\delta = \begin{vmatrix} 1 + \partial_\tau r - ra^\beta w_\beta & \partial_\theta r & \partial_\phi r & 1 \\ \partial_\tau r - ra^\beta w_\beta & \partial_\theta r & \partial_\phi r & 0 \\ -ra^\beta\theta_\beta - r\partial_\tau w^\beta\theta_\beta & r & 0 & 0 \\ -ra^\beta\phi_\beta - r\partial_\tau w^\beta\phi_\beta & 0 & \sin(\theta)r & 0 \end{vmatrix} \tag{153}$$

It follows for the component of n^α in u^α direction

$$n_\alpha u^\alpha = -(r^2\sin(\theta)(\partial_\tau r - ra^\beta w_\beta) + r^2\partial_\phi r(a^\beta\phi_\beta + \partial_\tau w^\beta\phi_\beta) + r^2\sin(\theta)\partial_\theta r(a^\beta\theta_\beta + \partial_\tau w^\beta\theta_\beta)) \tag{154}$$

To get the contraction of n^α with w^α, we just need to move the one in the last column down by one row.

$$n_\alpha w^\alpha = r^2\sin(\theta)(1+\partial_\tau r - r a^\beta w_\beta) + r^2\partial_\phi r(a^\beta\phi_\beta + \partial_\tau w^\beta\phi_\beta) + r^2\sin(\theta)\partial_\theta r(a^\beta\theta_\beta + \partial_\tau w^\beta\theta_\beta) \tag{155}$$

The contractions with θ^α and ϕ^α are obtained in the same way

$$n_\alpha\theta^\alpha = -r\sin(\theta)\partial_\theta r \tag{156}$$

$$n_\alpha\phi^\alpha = -r\partial_\phi r \tag{157}$$

We want to calculate

$$\begin{aligned}\int \frac{4\pi}{e^2}T^{\alpha\beta}n_\beta d\theta d\phi &= \int (\frac{1}{2r^4}(u^\alpha u^\beta - w^\alpha w^\beta + \theta^\alpha\theta^\beta + \phi^\alpha\phi^\beta) + \frac{1}{r^3}(a_\perp^\beta(u^\alpha + w^\alpha) + a_\perp^\alpha(u^\beta + w^\beta)) \\ &- \frac{1}{r^2}a_\perp^\gamma a_{\perp\gamma}(u^\alpha + w^\alpha)(u^\beta + w^\beta))n_\beta d\theta d\phi = \int \overbrace{\frac{-u^\alpha\sin(\theta)}{2r^2}\partial_\tau r}^{I} - \overbrace{\frac{w^\alpha\sin(\theta)}{2r^2}\partial_\tau r}^{II} \\ &+ \overbrace{\frac{u^\alpha\sin(\theta)}{2r}a^\beta w_\beta}^{III} + \overbrace{\frac{w^\alpha\sin(\theta)}{2r}a^\beta w_\beta}^{IV} - \overbrace{\frac{w^\alpha\sin(\theta)}{2r^2}}^{V} - \overbrace{\frac{u^\alpha}{2r^2}a^\beta\phi_\beta\partial_\phi r}^{VIa} - \overbrace{\frac{u^\alpha\sin(\theta)}{2r^2}a^\beta\theta_\beta\partial_\theta r}^{VIb} \\ &- \overbrace{\frac{w^\alpha}{2r^2}a^\beta\phi_\beta\partial_\phi r}^{VIIa} - \overbrace{\frac{w^\alpha\sin(\theta)}{2r^2}a^\beta\theta_\beta\partial_\theta r}^{VIIb} - \overbrace{\frac{u^\alpha}{2r^2}\partial_\tau w^\beta\phi_\beta\partial_\phi r}^{VIIIa} - \overbrace{\frac{u^\alpha\sin(\theta)}{2r^2}\partial_\tau w^\beta\theta_\beta\partial_\theta r}^{VIIIb} \\ &- \overbrace{\frac{w^\alpha}{2r^2}\partial_\tau w^\beta\phi_\beta\partial_\phi r}^{IXa} - \overbrace{\frac{w^\alpha\sin(\theta)}{2r^2}\partial_\tau w^\beta\theta_\beta\partial_\theta r}^{IXb} - \overbrace{\frac{\sin(\theta)\theta^\alpha}{2r^3}\partial_\theta r}^{Xa} - \overbrace{\frac{\phi^\alpha}{2r^3}\partial_\phi r}^{Xb} + \overbrace{\frac{\sin(\theta)a_\perp^\alpha}{r}}^{XI} \\ &+ \overbrace{\frac{\sin(\theta)(u^\alpha + w^\alpha)}{r^2}a^\beta\theta_\beta\partial_\theta r}^{XIIa} + \overbrace{\frac{u^\alpha + w^\alpha}{r^2}a^\beta\phi_\beta\partial_\phi r}^{XIIb} - \overbrace{a_\perp^\gamma a_{\perp\gamma}(u^\alpha + w^\alpha)\sin(\theta)}^{XIII} d\theta d\phi\end{aligned} \tag{158}$$

Now we have go through allot of terms, but we are still lucky, because all angel dependent terms in the denominator chancel out. With the formulas on page 35 the integrations are straight forward.

$$\begin{aligned}I &= \int \frac{-\sin(\theta)u^\alpha}{2r^2}\partial_\tau r d\theta d\phi = \\ &\int \frac{-\sin(\theta)u^\alpha}{2((z^\gamma(\tau+\epsilon) - z^\gamma)u_\gamma(\tau+\epsilon))^2}\big((1 - u^\gamma u_\gamma(\tau+\epsilon) + (z^\gamma(\tau+\epsilon) - z^\gamma)a_\gamma(\tau+\epsilon))(u^\beta + w^\beta)u_\beta(\tau+\epsilon) \\ &- (z^\beta(\tau+\epsilon) - z^\beta)u_\beta(\tau+\epsilon)((a^\gamma + \partial_\tau w^\gamma)u_\gamma(\tau+\epsilon) + (u^\gamma + w^\gamma)a_\gamma(\tau+\epsilon)\big)d\theta d\phi = \\ &\frac{-2\pi u^\alpha}{((z^\gamma(\tau+\epsilon) - z^\gamma)u_\gamma(\tau+\epsilon))^2}\big((1 - u^\gamma u_\gamma(\tau+\epsilon) + (z^\gamma(\tau+\epsilon) - z^\gamma)a_\gamma(\tau+\epsilon))u^\beta u_\beta(\tau+\epsilon) \\ &- (z^\beta(\tau+\epsilon) - z^\beta)u_\beta(\tau+\epsilon)(a^\gamma u_\gamma(\tau+\epsilon) + u^\gamma a_\gamma(\tau+\epsilon)\big)\end{aligned} \tag{159}$$

$$II = \int \frac{-\sin(\theta)w^\alpha}{2r^2}\partial_\tau r d\theta d\phi =$$

$$\int \frac{-\sin(\theta)w^\alpha}{2((z^\gamma(\tau+\epsilon)-z^\gamma)u_\gamma(\tau+\epsilon))^2}\big((1-u^\gamma u_\gamma(\tau+\epsilon)+(z^\gamma(\tau+\epsilon)-z^\gamma)a_\gamma(\tau+\epsilon))(u^\beta+w^\beta)u_\beta(\tau+\epsilon)-$$
$$(z^\beta(\tau+\epsilon)-z^\beta)u_\beta(\tau+\epsilon)((a^\gamma+\partial_\tau w^\gamma)u_\gamma(\tau+\epsilon)+(u^\gamma+w^\gamma)a_\gamma(\tau+\epsilon)\big)d\theta d\phi =$$
$$\frac{-2\pi}{3((z^\gamma(\tau+\epsilon)-z^\gamma)u_\gamma(\tau+\epsilon))^2}\big((1-u^\gamma u_\gamma(\tau+\epsilon)+(z^\gamma(\tau+\epsilon)-z^\gamma)a_\gamma(\tau+\epsilon))u_\beta(\tau+\epsilon)(u^\alpha u^\beta-\eta^{\alpha\beta})-$$
$$(z^\beta(\tau+\epsilon)-z^\beta)u_\beta(\tau+\epsilon)(a_\gamma(\tau+\epsilon)(u^\alpha u^\gamma-\eta^{\alpha\gamma})+u_\gamma(\tau+\epsilon)\partial_\tau\Lambda^\gamma_\delta\Lambda^\alpha_\mu m^{\delta\mu})\big) \quad (160)$$

To simplify this further we need an expression for $\partial_\tau\Lambda^\gamma_\delta\Lambda^\alpha_\mu m^{\delta\mu}$. We express $m^{\delta\mu}$ as $\delta^\delta_0\delta^\mu_0 - \eta^{\delta\mu}$. The two deltas give $a^\gamma u^\alpha$. To evaluate the second part we go into the comoving frame. The Lorentz matrix is just a unit matrix, while its derivative contains just accelerations in the time-space part. In this coordinate system the Thomas precession and its time derivative in the space components vanish. So we get

$$\partial_\tau\Lambda^\gamma_\delta\Lambda^\alpha_\mu\eta^{\delta\mu} = \begin{pmatrix} 0 & -a^1 & -a^2 & -a^3 \\ a^1 & 0 & 0 & 0 \\ a^2 & 0 & 0 & 0 \\ a^3 & 0 & 0 & 0 \end{pmatrix} = a^\gamma u^\alpha - a^\alpha u^\gamma \quad (161)$$

With both terms combined we get $\partial_\tau\Lambda^\gamma_\delta\Lambda^\alpha_\mu m^{\delta\mu} = u^\gamma a^\alpha$.

$$III = \int \frac{\sin(\theta)u^\alpha}{2r}a^\beta w_\beta d\theta d\phi = \int \frac{\sin(\theta)u^\alpha}{2(z^\gamma(\tau+\epsilon)-z^\gamma)u_\gamma(\tau+\epsilon)}(u^\gamma+w^\gamma)u_\gamma(\tau+\epsilon)a^\beta w_\beta d\theta d\phi =$$
$$\frac{2\pi u^\alpha}{3(z^\gamma(\tau+\epsilon)-z^\gamma)u_\gamma(\tau+\epsilon)}u_\gamma(\tau+\epsilon)a_\beta(u^\beta u^\gamma-\eta^{\beta\gamma}) = \frac{-2\pi u^\alpha}{3(z^\gamma(\tau+\epsilon)-z^\gamma)u_\gamma(\tau+\epsilon)}u_\gamma(\tau+\epsilon)a^\gamma \quad (162)$$

$$IV = \int \frac{\sin(\theta)w^\alpha}{2r}a^\beta w_\beta d\theta d\phi = \int \frac{\sin(\theta)w^\alpha}{2(z^\gamma(\tau+\epsilon)-z^\gamma)u_\gamma(\tau+\epsilon)}(u^\gamma+w^\gamma)u_\gamma(\tau+\epsilon)a^\beta w_\beta d\theta d\phi =$$
$$\frac{2\pi}{3(z^\gamma(\tau+\epsilon)-z^\gamma)u_\gamma(\tau+\epsilon)}u^\gamma u_\gamma(\tau+\epsilon)a_\beta(u^\alpha u^\beta-\eta^{\alpha\beta}) = \frac{-2\pi a^\alpha}{3(z^\gamma(\tau+\epsilon)-z^\gamma)u_\gamma(\tau+\epsilon)}u^\gamma u_\gamma(\tau+\epsilon) \quad (163)$$

$$V = \int \frac{-\sin(\theta)w^\alpha}{2r^2}d\theta d\phi = \int \frac{-\sin(\theta)w^\alpha}{2((z^\gamma(\tau+\epsilon)-z^\gamma)u_\gamma(\tau+\epsilon))^2}((u^\beta+w^\beta)u_\beta(\tau+\epsilon))^2 d\theta d\phi =$$
$$\frac{-4\pi u^\beta u_\beta(\tau+\epsilon)}{3((z^\gamma(\tau+\epsilon)-z^\gamma)u_\gamma(\tau+\epsilon))^2}u_\gamma(\tau+\epsilon)(u^\alpha u^\gamma-\eta^{\alpha\gamma}) \quad (164)$$

$$VIa = \int \frac{-u^\alpha}{2r^2}\partial_\phi r a^\beta \phi_\beta d\theta d\phi = \int \frac{u^\alpha}{2(z^\gamma(\tau+\epsilon)-z^\gamma)u_\gamma(\tau+\epsilon)}\sin(\theta)\phi^\gamma u_\gamma(\tau+\epsilon)a^\beta\phi_\beta d\theta d\phi \tag{165}$$

VI b contributes almost exactly the same term, only ϕ and θ are exchanged. So in sum in the rest system they contribute $\phi_0^\gamma\phi_0^\beta + \theta_0^\gamma\theta_0^\beta$, which is just $m^{\alpha\beta} - w_0^\alpha w_0^\beta$.

$$VIa+VIb = \frac{u^\alpha}{2(z^\gamma(\tau+\epsilon)-z^\gamma)u_\gamma(\tau+\epsilon)}u_\gamma(\tau+\epsilon)a_\beta(4\pi((u^\beta u^\gamma-\eta^{\beta\gamma})-\frac{4\pi}{3}(u^\beta u^\gamma-\eta^{\beta\gamma}) = \frac{-4\pi u^\alpha}{3(z^\gamma(\tau+\epsilon)-z^\gamma)u_\gamma(\tau+\epsilon)}u_\gamma(\tau+\epsilon)a^\gamma \tag{166}$$

The same thing will happen for all terms with a and b, so we start with the sum form now on. If we go through VII a and b we get zero because then there are only odd numbers of w^α.

$$VIIIa+VIIIb = \int \frac{u^\alpha\sin(\theta)}{2(z^\gamma(\tau+\epsilon)-z^\gamma)u_\gamma(\tau+\epsilon)}u_\gamma(\tau+\epsilon)\partial_\tau w_\beta(\phi^\beta\phi^\gamma+\theta^\gamma\theta^\beta)d\theta d\phi = 0 \tag{167}$$

$$IXa+IXb = \int \frac{w^\alpha\sin(\theta)}{2(z^\gamma(\tau+\epsilon)-z^\gamma)u_\gamma(\tau+\epsilon)}u_\gamma(\tau+\epsilon)\partial_\tau w_\beta(\phi^\beta\phi^\gamma+\theta^\gamma\theta^\beta)d\theta d\phi = \frac{1}{2(z^\gamma(\tau+\epsilon)-z^\gamma)u_\gamma(\tau+\epsilon)}u_\gamma(\tau+\epsilon)\Lambda^\alpha_\sigma\Lambda^\beta_\nu\Lambda^\gamma_\xi\eta_{\beta\delta}\partial_\tau\Lambda^\delta_\rho(\frac{4\pi}{3}m^{\sigma\rho}m^{\nu\xi} - \frac{4\pi}{15}(m^{\sigma\rho}m^{\nu\xi}+m^{\sigma\nu}m^{\rho\xi}+m^{\sigma\xi}m^{\nu\rho})) = \frac{1}{2(z^\gamma(\tau+\epsilon)-z^\gamma)u_\gamma(\tau+\epsilon)}u_\gamma(\tau+\epsilon)\eta_{\beta\delta}(\frac{4\pi}{3}a^\alpha u^\delta(u^\beta u^\gamma-\eta^{\beta\gamma}) - \frac{4\pi}{15}(a^\alpha u^\delta(u^\beta u^\gamma-\eta^{\beta\gamma})+(u^\alpha u^\beta-\eta^{\alpha\beta})a^\gamma u^\delta+(u^\alpha u^\gamma-\eta^{\alpha\gamma})a^\beta u^\delta) = 0 \tag{168}$$

$$Xa+Xb = \int \frac{-\theta^\alpha\sin(\theta)\partial_\theta r - \phi^\alpha\partial_\phi r}{2r^3}d\theta d\phi = \int \frac{\sin(\theta)(\theta^\alpha\theta^\beta+\phi^\alpha\phi^\beta)}{2((z^\gamma(\tau+\epsilon)-z^\gamma)u_\gamma(\tau+\epsilon))^2}u_\beta(\tau+\epsilon)(w^\gamma+u^\gamma)u_\gamma(\tau+\epsilon)d\theta d\phi = \frac{4\pi}{3((z^\gamma(\tau+\epsilon)-z^\gamma)u_\gamma(\tau+\epsilon))^2}u_\beta(\tau+\epsilon)u^\gamma u_\gamma(\tau+\epsilon)(u^\alpha u^\beta-\eta^{\alpha\beta}) \tag{169}$$

$$XI = \int \frac{\sin(\theta)(a^\alpha+a^\gamma w_\gamma w^\alpha)}{r}d\theta d\phi = \int \frac{\sin(\theta)(a^\alpha+a^\gamma w_\gamma w^\alpha)}{(z^\gamma(\tau+\epsilon)-z^\gamma)u_\gamma(\tau+\epsilon)}(u^\beta+w^\beta)u_\beta(\tau+\epsilon)d\theta d\phi = \frac{4\pi a^\alpha}{(z^\gamma(\tau+\epsilon)-z^\gamma)u_\gamma(\tau+\epsilon)}u^\beta u_\beta(\tau+\epsilon)+\frac{4\pi a_\gamma}{3(z^\gamma(\tau+\epsilon)-z^\gamma)u_\gamma(\tau+\epsilon)}u^\beta u_\beta(\tau+\epsilon)(u^\alpha u^\gamma-\eta^{\alpha\gamma}) = \frac{8\pi a^\alpha}{3(z^\gamma(\tau+\epsilon)-z^\gamma)u_\gamma(\tau+\epsilon)}u^\beta u_\beta(\tau+\epsilon) \tag{170}$$

$$XIIa + XIIb = \int \frac{u^\alpha + w^\alpha}{r^2}(a^\beta \theta_\beta \sin(\theta)\partial_\theta r + a^\beta \phi_\beta \partial_\phi r) d\theta d\phi =$$
$$\int \frac{-\sin(\theta)(u^\alpha + w^\alpha)}{(z^\gamma(\tau+\epsilon) - z^\gamma)u_\gamma(\tau+\epsilon)} a_\beta u_\gamma(\tau+\epsilon)(\theta^\gamma\theta^\beta + \phi^\gamma\phi^\beta) d\theta d\phi =$$
$$\frac{-8\pi u^\alpha}{3(z^\gamma(\tau+\epsilon) - z^\gamma)u_\gamma(\tau+\epsilon)} a_\beta u_\gamma(\tau+\epsilon)(u^\gamma u^\beta - \eta^{\gamma\beta}) = \frac{8\pi u^\alpha}{3(z^\gamma(\tau+\epsilon) - z^\gamma)u_\gamma(\tau+\epsilon)} a^\gamma u_\gamma(\tau+\epsilon) \tag{171}$$

$$XIII = \int -(a_\gamma a^\gamma + (a_\gamma w^\gamma)^2)\sin(\theta)(u^\alpha + w^\alpha) d\theta d\phi = -\frac{8\pi}{3} a_\gamma a^\gamma u^\alpha \tag{172}$$

The last one is Lamor's formula again, what one already could have expected. Now lets combine all terms.

$$\frac{2\pi}{3}\Bigg(\frac{1}{((z^\gamma(\tau+\epsilon) - z^\gamma)u_\gamma(\tau+\epsilon))^2}\Big(-3u^\alpha\big(1 - u^\gamma u_\gamma(\tau+\epsilon) + (z^\gamma(\tau+\epsilon) - z^\gamma)a_\gamma(\tau+\epsilon)\big)u^\beta u_\beta(\tau+\epsilon) +$$
$$3u^\alpha(z^\beta(\tau+\epsilon) - z^\beta)u_\beta(\tau+\epsilon)(a^\gamma u_\gamma(\tau+\epsilon) + u^\gamma a_\gamma(\tau+\epsilon))$$
$$-\big(1 - u^\gamma u_\gamma(\tau+\epsilon) + (z^\gamma(\tau+\epsilon) - z^\gamma)a_\gamma(\tau+\epsilon)\big)u_\beta(\tau+\epsilon)(u^\alpha u^\beta - \eta^{\alpha\beta})$$
$$+(z^\beta(\tau+\epsilon) - z^\beta)u_\beta(\tau+\epsilon)\big(a_\gamma(\tau+\epsilon)(u^\alpha u^\gamma - \eta^{\alpha\gamma}) + u_\gamma(\tau+\epsilon)u^\gamma a^\alpha\big) - 2u^\beta u_\beta(\tau+\epsilon)u_\gamma(\tau+\epsilon)(u^\alpha u^\gamma - \eta^{\alpha\gamma})$$
$$+ 2u_\beta(\tau+\epsilon)u^\gamma u_\gamma(\tau+\epsilon)(u^\alpha u^\beta - \eta^{\alpha\beta})\Big) +$$
$$\frac{1}{(z^\gamma(\tau+\epsilon) - z^\gamma)u_\gamma(\tau+\epsilon)}\big(-u^\alpha u_\gamma(\tau+\epsilon)a^\gamma - a^\alpha u^\gamma u_\gamma(\tau+\epsilon) - 2u^\alpha u_\gamma(\tau+\epsilon)a^\gamma + 4u^\alpha a^\gamma u_\gamma(\tau+\epsilon)$$
$$+ 4a^\alpha u^\beta u_\beta(\tau+\epsilon)\big) - 4a^\gamma a_\gamma u^\alpha\Bigg) \tag{173}$$

Much simplification of this horrible expression is not possible, but at least some.

$$\frac{2\pi}{3}\Bigg(\frac{1}{((z^\gamma(\tau+\epsilon) - z^\gamma)u_\gamma(\tau+\epsilon))^2}\big((u^\alpha(\tau+\epsilon) - 4u^\alpha u^\beta u_\beta(\tau+\epsilon))(1 - u^\gamma u_\gamma(\tau+\epsilon) + (z^\gamma(\tau+\epsilon) - z^\gamma)a_\gamma(\tau+\epsilon)\big)$$
$$+\frac{1}{(z^\gamma(\tau+\epsilon) - z^\gamma)u_\gamma(\tau+\epsilon)}\big(3u^\alpha(a^\gamma u_\gamma(\tau+\epsilon) + u^\gamma a_\gamma(\tau+\epsilon)) + a_\gamma(\tau+\epsilon)(u^\alpha u^\gamma - \eta^{\alpha\gamma}) + u_\gamma(\tau+\epsilon)u^\gamma a^\alpha)$$
$$+ u^\alpha u_\gamma(\tau+\epsilon)a^\gamma + 3a^\alpha u^\gamma u_\gamma(\tau+\epsilon)\big) - 4a^\gamma a_\gamma u^\alpha\Bigg) =$$
$$\frac{2\pi}{3}\Bigg(\frac{1}{((z^\gamma(\tau+\epsilon) - z^\gamma)u_\gamma(\tau+\epsilon))^2}\big((u^\alpha(\tau+\epsilon) - 4u^\alpha u^\beta u_\beta(\tau+\epsilon))(1 - u^\gamma u_\gamma(\tau+\epsilon) + (z^\gamma(\tau+\epsilon) - z^\gamma)a_\gamma(\tau+\epsilon)\big) +$$
$$\frac{1}{(z^\gamma(\tau+\epsilon) - z^\gamma)u_\gamma(\tau+\epsilon)}\big(4u^\alpha(a^\gamma u_\gamma(\tau+\epsilon) + u^\gamma a_\gamma(\tau+\epsilon)) + 4a^\alpha u^\gamma u_\gamma(\tau+\epsilon) - a^\alpha(\tau+\epsilon)\big) - 4a^\gamma a_\gamma u^\alpha\Big) \tag{174}$$

This equation is still not the final result, one step is missing. The tube integration over τ did run from $\tau_1 - \epsilon$ to $\tau_2 - \epsilon$, so deriving with respect to τ_2 is not enough to get ride of the integral. Additionally the shift $\tau \to \tau - \epsilon$ is necessary. After reintroducing the $\frac{e^2}{4\pi}$ the full equation of motion is

$$ma^\alpha = eF^{\alpha\beta}_{ext}u_\beta - \frac{e^2}{6}\Bigg(\frac{1}{((z^\gamma - z^\gamma(\tau-\epsilon))u_\gamma)^2}\Big(\big(u^\alpha - 4u^\alpha(\tau-\epsilon)u^\beta u_\beta(\tau-\epsilon)\big)\big(1 - u^\gamma u_\gamma(\tau-\epsilon) + (z^\gamma - z^\gamma(\tau-\epsilon))a_\gamma\Big) + \frac{1}{(z^\gamma - z^\gamma(\tau-\epsilon))u_\gamma}\Big(4u^\alpha(\tau-\epsilon)(a^\gamma(\tau-\epsilon)u_\gamma + u^\gamma(\tau-\epsilon)a_\gamma) + 4a^\alpha(\tau-\epsilon)u^\gamma(\tau-\epsilon)u_\gamma - a^\alpha\Big) - 4a^\gamma(\tau-\epsilon)a_\gamma(\tau-\epsilon)u^\alpha(\tau-\epsilon)\Bigg) \quad (175)$$

A Tailor series in ϵ results in the Abraham-Lorentz-Dirac equation and the usual mass renormalisation term up to order $\mathcal{O}(\epsilon)$ as expected. This equation of motion can hardly be called the equation of motion for the charged particle, because it holds only for some very special charge distribution. So its better call it an equation of motion for some model for a charged particle. The details of the charge distribution, which a determined by the cut off, are discussed in the next chapter.

8.4 The particle model

As discussed in chapter 6.3, a cut off is equivalent with replacing the original charge distribution with another one. This means equation (175) does not hold for a point particle as first intended, but for an extended charge distribution. The cut off works only, if there are no fields inside the tube and the fields outside the tube are identical with the fields of a point particle. It follows, that the charge must be spread out over the surface of the tube. To find the form of the charge distribution, we have to understand the form of the tube. The tube consist out of cuts between the light cone at $z^\alpha(\tau - \epsilon)$ and the hyperplane of simultaneity of the comoving coordinate frame at time τ. The cut of an light cone and a space like plane is always a sphere, so the whole tube consist out of a connection of spheres in the comoving coordinate frame. This is similar to the Lorentz model, but there are two important differences. First the spheres are not centred at the trajectory in general, but at the point where the particle would have been, if it continued its movement with constant velocity after emitting the fields. The difference to the Lorentz model is only very small, it is of order ϵ^2. The second difference is, that the charge is not distributed uniformly over the spheres. The Coulomb fields alone are spherical symmetric, but the radiation fields are angel depended. They vanish into the direction of the acceleration. This happens also for the tube Dirac used. An exact expression for the charge distribution can be obtained with the help of the Maxwell equations. One has to calculate the divergence of the field strength tensor with cut off $\partial_\alpha F^{\alpha\beta}_{cut} = 4\pi j^\beta$.

It is possible to give some physical justification for the model obtained here. In the Lorentz model a change in form of the charge distribution, corresponding to a change in the trajectory, happens instantaneously and simultaneously. If one has a force in mind, which is responsible for the form of the charge distribution and whose origin is at the trajectory, such an instantaneous change in form would require the force to be transmitted between space like separated points. Such a force is unphysical, because it could lead to paradoxes. The easiest thing to avoid an instantaneous form change is to build in a delay, which describes the time the charge distribution needs to react to a change in the trajectory. If one assumes, that the force, who determines the form, propagates on light cones centred at the trajectory, one arrives at the model used be Parrott. The model we are using here is very similar, we only assumed that the charge is a sphere in the comoving frame and not in the comoving frame at the time, where the fields or the force were emitted. This is still somehow unphysical, because then at the emission of the force the velocity later must already have been known, but it is still better than the Lorentz model. Additionally one has to keep in mind, that the aim of this thesis is to derive an equation of motion and not to find the most realistic extended charge model. To get a reasonable equation of motion is much harder for the charge model used by Parrott, because there is no easy way to use caps orthogonal to the trajectory.

8.5 Mass renormalisation

In the Abraham-Lorentz-Dirac equation mass renormalisation seems to appear in a quite natural form. For equation (175) this is not the case. If there are terms corresponding to a mass increase, they depend on the exact form of the trajectory. This can be seen best in the case of constant acceleration. We already know, that in this case their is no radiation reaction force, so their is only mass increase. Plugging equation (73) into (175) leads to

$$ma^\alpha = -\frac{ge^2}{2\sinh(g\epsilon)}a^\alpha + eF_{ext}^{\alpha\beta}u_\beta \tag{176}$$

This is of course equivalent to the usual term in lowest order of ϵ, but the important difference is that the complete term is a function of the trajectory and not just constant. The mass renormalisation suggest by Dirac is choosing a value for m, maybe even a negative one, such that $m + \frac{e^2}{2\epsilon}$ is equivalent to the value obtained for the mass from experiments m_{exp} for all ϵ. If the aim of this idea is to incorporate the increase of inertia in the mass, it does fail, because the mechanical mass shouldn't be a function of the trajectory. If the aim is just to get ride of the diverging term for small ϵ, this still works, but seems much more artificial. Another reason why one has to be very careful, with the mass renormalisation is, that the term in equation (175), from which the usual term for mass increase arises, contains partly a delay. A delayed mass increase is perfectly natural for extended charge distribution, but in Tailor series the delay vanishes and there appears an additional term. This is the origin of the $\frac{da^\alpha}{d\tau}$ term in the usual radiation reaction force. If mass renormalisation can't be justified in a physical way and boils down to do nothing more then dropping the divergent term, one could argue, that the complete term, the one with delay, should be dropped and not just the divergent part after using a Tailor series. But than the radiation reaction force would change.

In the framework of an extended charge distribution mass renormalisation is not necessary, because the mass increase does not propose any difficulties. For a point particle a cut off can't be used and some other method is needed to handle the diverging terms. Such a method should exist, since classical electrodynamics must be contained in quantum electrodynamics in some limit and quantum electrodynamics is renormalisable, but I don't know, what exactly are those methods.

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Mir ist bekannt, dass Zuwiderhandlung auch nachträglich zur Aberkennung des Abschlusses führen kann.

München, April 15, 2016	
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