On the Dynamics of Multi-Time Systems

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ABSTRACT

Carrying over the idea of a wave function to relativistic quantum mechanics naturally leads to a multi-time wave function of the form $\psi(t_1, \mathbf{x}_1, ..., t_N, \mathbf{x}_N)$. It depends on one time coordinate t_k and one space coordinate $\mathbf{x}_k$ for each particle $k = 1, ..., N$. As already proposed by Dirac in 1932, the dynamics of a multi-time wave function can be described by a system of N equations of motion, which determine the evolution along the N time coordinates. The existence of a common solution to such a multi-time system is associated with an integrability condition, the so-called consistency condition, that substantially restricts the possibilities to introduce an interaction between the particles. The aim of this doctoral thesis is to treat several physically relevant possibilities for interacting dynamics in a mathematically rigorous fashion.

After a physical introduction to multi-time wave functions and their meaning as probability amplitudes similarly as in non-relativistic quantum mechanics, the mathematical solution theory of systems of equations for a multi-time wave function is discussed. The consistency conditions in Dirac equations with added interaction potentials are the topic of the first main part of the thesis. A generalization of the no-go result by Petrat and Tumulka (2014) about non-existence of solutions in multi-time systems with interaction potentials is proven. It entails that no Poincaré invariant interacting potentials are compatible with the consistency condition. This possibility for the formulation of relativistic interacting quantum dynamics is hence ruled out.

Dirac's suggestion from 1932 to circumvent the problem with interaction potentials is to employ second quantized potentials, which is taken on in the subsequent chapter. Together with Fock and Podolsky, he devised a model of quantum electrodynamics that describes the interaction of N Dirac fermions through an electromagnetic field. In the second main part of this thesis, that model is formulated mathematically, using a scalar field for simplicity and a cut-off to bypass the well-known ultraviolet divergences. First, the single-time equations are treated by proving essential self-adjointness of the Hamiltonians and also invariance of certain regular functions under the respective time-evolutions. Then, a consistent multi-time evolution is constructed by glueing together the single-time evolutions. Besides the results on existence and uniqueness of solutions, it is also demonstrated that the model is indeed interacting.

In the subsequent part, a well-defined interaction is constructed in another way, namely by particle creation and annihilation via interior-boundary conditions (IBCs). This proposed solution to the ultraviolet problem of quantum field theory is treated in a relativistic setting for the first time. In a simplified model in $1+1$ dimensions where Dirac fermions create and annihilate each other when they meet, the class of IBCs that lead to probability conservation is identified. The theorem on existence and uniqueness in this model is the first rigorous result in a multi-time system with a variable number of particles.

As final results, general theorems on linear systems of arbitrary order in the time derivatives are presented. Those show how the solution theory of multi-time equations changes if time derivatives of higher order are present.

ZUSAMMENFASSUNG

Die Übertragung des Konzepts einer Wellenfunktion auf die relativistische Quantenmechanik führt auf natürliche Weise zu einer Vielzeiten-Wellenfunktion, welche die Form $\psi(t_1, \mathbf{x}_1, ..., t_N, \mathbf{x}_N)$ besitzt. Sie hängt von einer Zeitkoordinate t_k und einer Ortskoordinate $\mathbf{x}_k$ für jedes Teilchen $k = 1, ..., N$ ab. Wie schon von Dirac 1932 vorgeschlagen, kann die Dynamik einer Vielzeiten-Wellenfunktion durch ein System aus N Bewegungsgleichungen beschrieben werden, welche die Entwicklung entlang der N Zeitkoordinaten bestimmen. Die Existenz einer gemeinsamen Lösung zu solch einem Vielzeiten-System ist hierbei mit einer Integrabilitätsbedingung verbunden, der so genannten Konsistenzbedingung, die die Möglichkeiten, eine Wechselwirkung zwischen den Teilchen einzuführen, erheblich einschränkt. Das Ziel dieser Doktorarbeit ist es, verschiedene physikalisch relevante Möglichkeiten für wechselwirkende Dynamiken auf mathematisch rigorose Weise zu behandeln.

Nach einer physikalischen Einleitung über Vielzeiten-Wellenfunktionen und deren Bedeutung als Wahrscheinlichkeitsamplitude, ähnlich wie in der nicht-relativistischen Quantenmechanik, wird die mathematische Lösungstheorie von Gleichungssystemen für eine Vielzeiten-Wellenfunktion besprochen. Die Konsistenzbedingung in Dirac-Gleichungen mit addierten Wechselwirkungspotentialen ist Gegenstand des ersten Hauptteils der Arbeit. Es wird eine Verallgemeinerung des Unmöglichkeits-Resultats von Petrat und Tumulka (2014) über die Nichtexistenz von Lösungen in Vielzeiten-Systemen mit Wechselwirkungspotentialen bewiesen. Es beinhaltet, dass keine Poincaré-invarianten wechselwirkenden Potentiale mit der Konsistenzbedingung konform gehen. Diese Möglichkeit für die Formulierung von relativistischer, wechselwirkender Quantendynamik ist damit ausgeschlossen.

Diracs Vorschlag aus dem Jahr 1932, das Problem mit Wechselwirkungspotentialen zu umgehen, ist durch Benutzung zweitquantisierter Potentiale, was im folgenden Kapitel angegangen wird. Zusammen mit Fock und Podolsky ersann er ein Modell der Quantenelektrodynamik, das die Wechselwirkung von N Dirac-Fermionen durch ein elektromagnetisches Feld beschreibt. Im zweiten Hauptteil der Arbeit wird das Modell mathematisch formuliert, wobei der Einfachheit halber ein skalares Feld und zur Verhinderung der wohlbekannten Ultraviolett-Divergenzen ein Cut-Off benutzt wird. Zuerst werden die Ein-Zeit-Gleichungen behandelt, indem die wesentliche Selbstadjungiertheit der Hamiltonians und auch die Invarianz gewisser regulärer Funktionen unter den entsprechenden Zeitentwicklungen gezeigt wird. Sodann wird eine konsistente Vielzeiten-Entwicklung konstruiert, indem man die Ein-Zeit-Entwicklungen zusammenfügt. Neben den Resultaten über Existenz und Eindeutigkeit von Lösungen wird auch aufgezeigt, dass das Modell tatsächlich wechselwirkend ist.

Im darauf folgenden Abschnitt wird eine wohldefinierte Wechselwirkung mittels einer anderen Methode konstruiert, und zwar durch Teilchenerzeugung und -vernichtung via *interior-boundary conditions* (IBCs). Diese vorgeschlagene Lösung des Ultraviolettproblems in der Quantenfeldtheorie wird zum ersten Mal in einem relativistischen Rahmen behandelt. In einem vereinfachten Modell in $1 + 1$ Dimensionen, in welchem Dirac-Fermionen sich bei Kontakt gegenseitig erzeugen und vernichten können, werden jene IBCs klassifiziert, die

zu Wahrscheinlichkeitserhaltung führen. Das Existenz- und Eindeutigkeitstheorem in diesem Modell ist das erste rigorose Resultat in einem Vielzeiten-System mit einer variablen Teilchenzahl.
Als letzte Resultate werden allgemeine Theoreme über lineare Systeme beliebiger Ordnung in den Zeitableitungen präsentiert. Diese zeigen auf, wie sich die Lösungstheorie von Vielzeiten-Gleichungen verändert, wenn höhere Zeitableitungen vorkommen.

CONTENTS

CHAPTER

ONE

INTRODUCTION TO MULTI-TIME WAVE FUNCTIONS

Progress in fundamental physics is commonly connected with finding a new theory, one that extends a known theory or that even unifies several established ones. Quantum mechanics successfully describes the processes in the microcosm, while Albert Einstein's theory of relativity works excellently for objects with high velocity. It may therefore be surprising that until today, as both theories are known for about a century, no mathematically closed unification of relativity and quantum mechanics has been found on the level of the equations of motion.

The quantum field theories of the standard model are indeed very successful for the empirical prediction of scattering cross-sections and for typical experiments such as particle accelerators, e.g. the large hadron collider (LHC). But the relativistic invariance of quantum field theory is only shown at the level of scattering theory, for initial and final times approaching $\pm\infty$. If a new fundamental theory is supposed to entail non-relativistic quantum mechanics, a well-defined dynamics for all times has to exist. This is one of the main motivations to investigate the mathematical existence theory for models of relativistic quantum mechanics in this thesis.

The models we consider have in common that they utilize one basic object for the formulation of relativistic quantum mechanics: the multi-time wave function. This straightforwardly generalizes the wave function of non-relativistic quantum mechanics, which we write $\varphi(t, \mathbf{x}_1, ..., \mathbf{x}_N)$, a function of one time variable t and, for N particles, N space variables $\mathbf{x}_k \in \mathbb{R}^d, k = 1, ..., N$. The relativistic counterpart of this wave function is the **multi-time wave function** $\psi(t_1, \mathbf{x}_1, ..., t_N, \mathbf{x}_N)$. In a system with N particles, it depends on N time coordinates and N space coordinates, or put in other words, it is a function on configuration space-time with N space-time points $x_k = (t_k, \mathbf{x}_k)$ as arguments.

The use of a multi-time wave function is everything but a radical break with the usual ideas of relativistic quantum mechanics. Actually, many quantum field theory models can be reformulated with the help of a multi-time wave function, in the same sense as the Heisenberg and Schrödinger picture are called equivalent in non-relativistic quantum mechanics (see also (1.5)). However, there are also new possibilities for dynamics of multi-time wave functions, as explained in 1.3. In any case, if one intends to utilize an analogue of the wave function to construct Lorentz invariant quantum dynamics, one is directly lead to the multi-time wave function, which also provides a rather direct picture of physical processes and has the well-known interpretation that $|\psi|^2$ is a probability density (see Sec. 1.4).

Paul Dirac proposed multi-time wave functions already in 1932 [1]. They arise by a simple

train of thought: If we look at a single-time wave function $\varphi(t, \mathbf{x}_1, ..., \mathbf{x}_N)$ and intend to perform a Lorentz transformation, it is unclear how to do so since time and space arguments are mixed in such a transformation. But the argument of φ can be read as a collection of N simultaneous space-time points $(t, \mathbf{x}_1), ..., (t, \mathbf{x}_N)$, which under a Lorentz transformation is mapped to $(t'_1, \mathbf{x}'_1), ..., (t'_N, \mathbf{x}'_N)$ with, in general, different times $t'_1 \neq t'_2 \neq ... \neq t'_N$. Thus, it is natural to consider a multi-time wave function $\psi(x_1, ..., x_N)$ and it is viable that a manifestly Lorentz covariant formulation of quantum mechanics can be achieved by using it.
We demonstrate how manifestly Lorentz covariant quantum dynamics can be formulated with the help of multi-time wave functions at several examples in this thesis. We are mainly concerned with systems of partial differential equations of the form

$$\begin{aligned} i\partial_{t_1}\psi(x_1, ..., x_N) &= \mathcal{H}_1(x_1, ..., x_N)\psi(x_1, ..., x_N) \\ i\partial_{t_2}\psi(x_1, ..., x_N) &= \mathcal{H}_2(x_1, ..., x_N)\psi(x_1, ..., x_N) \\ &\dots \\ i\partial_{t_N}\psi(x_1, ..., x_N) &= \mathcal{H}_N(x_1, ..., x_N)\psi(x_1, ..., x_N), \end{aligned} \tag{1.1}$$

which we call **multi-time systems**. One can think of $\mathcal{H}_k$ as a self-adjoint Hamiltonian for particle k, but more general operators may appear, see e.g. [2]. The free dynamics of relativistic fermions, e.g. electrons, is described by $\mathcal{H}_k = \mathcal{H}_k^0$ for every k, where $\mathcal{H}_k^0$ is the free Dirac Hamiltonian (see also around Eq. (2.4)) acting on particle number k. The question how to introduce a relativistic interaction in quantum physics is highly non-trivial. In multi-time systems, one mathematical difficulty is the following: A common solution ψ only exists if the equations of motion can be integrated in any order with the same result $\psi(t_1, ..., t_N)$. This leads to an integrability condition, which reads

$$[\mathcal{H}_j, \mathcal{H}_k] - i\frac{\partial \mathcal{H}_k}{\partial t_j} + i\frac{\partial \mathcal{H}_j}{\partial t_k} = 0, \qquad \forall k \neq j. \tag{1.2}$$

This so-called **consistency condition** is examined in more mathematical detail in Section 1.5. It is also a crucial ingredient in many proofs of our central results on interacting relativistic quantum dynamics. The main results of this thesis, which span the Chapters 2, 3, and 4, are the following:

(a) Multi-time systems of the form (1.1) with Hamiltonians featuring **interaction potentials** $V_k(x_1, ..., x_N)$ are usually inconsistent since the condition (1.2) fails there. For potentials that do not couple the spins of the particles, this was proven by Petrat and Tumulka [3]. We extend their results to spin-coupling potentials, show that although some potentials satisfying condition (1.2) exist ($\rightarrow$ **Lemma 2.4**), there are none that are Poincaré invariant and for which (1.1) has a smooth solution ($\rightarrow$ **Thm. 2.1**).

(b) A model that satisfies the consistency condition and describes the interaction of relativistic fermions through a second-quantized field was already constructed in 1932 by **Dirac, Fock, and Podolsky** [4]. We present a rigorous formulation of this model, using a cut-off on the field, and prove existence ($\rightarrow$ **Thm. 3.1**) and uniqueness ($\rightarrow$ **Thm. 3.2**) of solutions in a suitable sense.

(c) A novel idea to circumvent the use of cut-offs and renormalization in quantum field theories is the method of **interior-boundary conditions (IBCs)** [5, 6]. IBCs are boundary conditions between different sectors of Fock space that implement particle

creation and annihilation in such a way that the total probability is conserved. We analyze this method in the context of relativistic quantum mechanics in the multi-time formalism. We present a model in $1+1$ dimensions for which the existence and uniqueness of solutions ($\rightarrow$ **Thm. 4.4**) can be proven by fixed point methods and also probability conservation is shown ($\rightarrow$ **Thm. 4.5**) rigorously.

In the subsequent sections of this introductory chapter, we elucidate the history of the multi-time formalism, meaning and dynamics of multi-time wave functions. Many of the points made in this chapter were also formulated in one or the other way in the Phd thesis of Lienert [7], and in the overview paper [8]. A basic introduction to the importance of the multi-time formalism was also given in the author's Master thesis [9].
Now, we first explain the structure of this thesis.

1.1 Structure of this thesis

Chapter 1 The first chapter is of an introductory nature and contains important physics aspects of multi-time wave functions. In the following Section 1.2, a brief overview of the history of the multi-time formalism as well as of recent literature about this topic is given and its relation to other formulations of relativistic quantum mechanics is explained. We then present different possibilities for the dynamics in Sec. 1.3 and go on with the meaning of multi-time wave functions for detection probabilities in Sec. 1.4. In Sec. 1.5, we discuss the consistency condition, a necessary condition for the existence of solutions, first on a heuristic level (Sec. 1.5.1) then collect known rigorous results on solution theory in Sec. 1.5.2 and close with some remarks on the geometric meaning of the consistency condition (Sec. 1.5.3).

Chapter 2 We now switch to the formal mathematical treatment. We show a no-go-theorem on the compatibility of multi-time systems with interaction potentials that generalizes the result of Petrat and Tumulka from [3]. We consider free Dirac Hamiltonians with added interaction potentials $V_k(x_1, ..., x_N)$ and prove that no sufficiently regular solution to the multi-time system can exist if the potentials are Poincaré invariant. To this end, we first give a proof of a useful version of the consistency condition for our case (Sec. 2.3) and then classify the consistent potentials (Sec. 2.4). This chapter is based on the paper [10] by Dirk-A. Deckert and the present author.

Chapter 3 In this chapter, we give a rigorous formulation of the QED model by Dirac, Fock, Podolsky [4]. In our version of the model, the interaction of N Dirac fermions is mediated by a second-quantized scalar field φ, with a smooth cut-off function ρ (see Sec. 3.2). To show existence and uniqueness of solutions to the multi-time system, we first examine the single-time equations (Sec. 3.3.2) and in particular make sure that certain smooth functions are invariant under the single-time evolutions. We then build a multi-time evolution out of the single-time ones, using new ideas to prove existence of solutions from the consistency condition in Sec. 3.3.3. The publication of the results of this chapter is in preparation and will be done jointly by Dirk-A. Deckert and the present author.

Chapter 4 We extend the concept of interior-boundary conditions (IBCs) to the relativistic domain. IBCs have been put forward as a possible way to bypass the problem of ultraviolet divergences in quantum field theory [5, 6], but have so far only been formulated

for non-relativistic models. In order to provide a rigorous result, we restrict to one spatial dimension and treat massless Dirac particles that can create and annihilate each other when they meet. We show existence and uniqueness of solutions to that model and identify the class of IBCs that ensure probability conservation on every Cauchy surface. This chapter is largely based on the article [11] by Matthias Lienert and the present author.

Chapter 5 In this last mathematical chapter, we discuss systems of differential equations of higher order $n \in \mathbb{N}$ for a multi-time wave function, where the solution theory from the previous chapters is generalized. We discuss the necessary initial values (Sec. 5.1) and prove theorems on the uniqueness (Sec. 5.2) and existence (Sec. 5.3) of solutions to these systems of arbitrary order. Lastly, we derive an analogue of the consistency condition for this more general setting in Sec. 5.4.

Chapter 6 We close the thesis with an outlook and conclusion.
The appendix contains an overview of the notations used in this work on page 101.

1.2 Formulations of relativistic quantum mechanics

> *Die Quantentheorie der Wellenfelder ist in ihrer bisherigen relativistischen Fassung noch schwerwiegenden Einwänden ausgesetzt. Insbesondere scheint die Wechselwirkung der Elektronen mit sich selbst einstweilen die Anwendung der Theorie in manchen Fällen unmöglich zu machen. Von einer endgültigen Formulierung der Theorie sind wir also noch weit entfernt.* – Heisenberg and Pauli, 1930 [12][1].

After Paul Dirac had found his important Dirac equation for a single particle with spin $\frac{1}{2}$,

$$(i\gamma^\mu \partial_\mu - m)\,\psi(t, \mathbf{x}) = 0, \tag{1.3}$$

several routes were followed to build an interacting theory of several particles from there[2]. Heisenberg and Pauli, among others, pursued the idea to utilize the standard quantization rules of imposing (anti)commutation relations [13, 12], which is similar to the general way that quantum field theory (QFT) is presented today in many textbooks. Dirac proposed a different way to generalize (1.3) to several particles, which is by the introduction of a multi-time wave function. The idea was first presented in [1] for two particles. In that case, the natural non-interacting generalization of (1.3) is

$$\begin{aligned}(i\gamma_1^\mu \partial_{1,\mu} - m_1)\,\psi(t_1, \mathbf{x}_1, t_2, \mathbf{x}_2) &= 0\\ (i\gamma_2^\mu \partial_{2,\mu} - m_2)\,\psi(t_1, \mathbf{x}_1, t_2, \mathbf{x}_2) &= 0,\end{aligned} \tag{1.4}$$

where $\partial_{k,\mu} = \frac{\partial}{\partial x_k^\mu}$, and γ_k^μ denotes the matrix γ^μ acting on the spin indices of particle k (cf. Eq. (2.6)) for $k = 1, 2$.
Dirac also made the first steps towards the introduction of a consistent interaction in [1] and further in the publication [4] together with Fock and Podolsky, which is treated rigorously

[1] Translation to English: *The quantum theory of wave fields is still subject to severe objections in its present relativistic version. Especially the interaction of the electrons with themselves interim seems to make the application of the theory impossible in some cases. We are thus far away from a definitive formulation of the theory.*

[2] While most unexplained notations in this thesis are standard, we provide a list of notations on page 101.

in Chapter 3. In both articles, multi-time wave functions of the form $\psi(t_1, \mathbf{x}_1, ..., t_N, \mathbf{x}_N)$ are introduced without further comment, which underlines how natural this step appeared to the authors. The idea was taken on by Felix Bloch [14], who first recognized the necessity of the consistency condition (1.2) (see Sec. 1.5 for a mathematical consideration) and was also pivotal for the wave function on space-like hypersurfaces of Tomonaga [15] and Schwinger [16].
During the further development of quantum electrodynamics (QED), however, the emphasis on the manifest Lorentz invariance that was provided by the multi-time formalism shrank. The problem of ultraviolet divergences was pressing since it lead to an ill-defined theory. Physicists concentrated on finding recipes to extract experimental predictions from the ill-defined theory and finally succeeded in perturbatively calculating S-matrix elements with the help of renormalization. Although these predictions were correct with astonishing precision, the question of how to obtain a mathematically closed formulation of relativistic quantum mechanics – that is also interacting and describes processes like the interaction of light and matter – remains unsolved (see also [17, ch. 11]).
To approach this question, we go back to the old idea of a multi-time wave function and investigate the solution theory of some models of relativistic quantum mechanics below. Mathematically, the multi-time wave function is a map $\psi : \Omega \subset \mathbb{R}^{(d+1)N} \to \mathbb{C}^K$ with some domain Ω, discussed in detail in the following section, and some $K \in \mathbb{N}$ depending on spin, so it is a natural object to treat and simpler than operator-valued distributions as in the Heisenberg picture. The usual formulation of QFTs with field operators $\hat{\Phi}(x)$ in the Heisenberg picture can be easily translated to multi-time wave functions with the following formula (eq. (15) in [18], see also assertion 3 in [2])

$$\psi(x_1, ..., x_N) = \frac{1}{\sqrt{N!}} \left\langle 0 \left| \hat{\Phi}(x_1)...\hat{\Phi}(x_N) \right| \Psi \right\rangle, \tag{1.5}$$

where $|0\rangle$ denotes the vacuum state in Fock space and $|\Psi\rangle$ the (initial) Heisenberg state. In many models, the number of particles is allowed to vary, which can be implemented in the multi-time formalism without further ado, see also Chapter 4.
In recent years, there has been renewed interest in multi-time wave functions and the construction of interacting dynamics for them in mathematical physics, see [8] for an overview. Petrat and Tumulka showed in [3] that interaction potentials are not a viable option to formulate interactions in the multi-time formalism, which we generalize in Chapter 2. They furthermore demonstrated how QFT-like models can be written down with the help of multi-time wave functions [2, 19] and that these models are consistent in principle – leaving aside ultraviolet divergences. Lienert constructed a one-dimensional model where interaction between Dirac particles is formulated by boundary conditions [20, 21], a model that our considerations on IBCs in the multi-time formalism (Chapter 4) are based on.

1.3 Dynamics for multi-time wave functions

A multi-time system such as (1.1) with one differential equation for each time coordinate is the standard way to write down evolution equation for multi-time equations and was already put forward by Dirac [1]. The mathematical challenges connected with these systems, in particular the integrability condition (1.2) needed for a common solution of the N equations, are discussed in Section 1.5.
We now investigate the correct domain for the multi-time wave function. For N particles and K spin components, the wave function is a map $\psi : \Omega \subset \mathbb{R}^{(d+1)N} \to \mathbb{C}^K$ with

domain Ω. Although the domain is sometimes chosen to be the full configuration space-time $\Omega = \mathbb{R}^{(d+1)N}$, the physically sensible and natural domain of ψ is the set of space-like configurations

$$\mathscr{S}^{(N)} := \left\{ (t_1, \mathbf{x}_1, ..., t_N, \mathbf{x}_N) \in \mathbb{R}^{(d+1)N} \middle| \forall j \neq k : |t_j - t_k| < \|\mathbf{x}_j - \mathbf{x}_k\| \right\}. \tag{1.6}$$

The two main reasons for this are (adopted from [10, p. 4] with small changes):

- *Sufficiency:* In order to interpret Born's rule on any space-like hypersurface, it is sufficient for ψ to have domain $\mathscr{S}^{(N)}$ (see Sec. 1.4). A Lorentz transformation of a simultaneous configuration as presented in the introduction always yields a space-like configuration. It is unclear how to interpret ψ on time-like configurations. Indeed, the mere concept of "N-particle configuration" implies the use of $\mathscr{S}^{(N)}$ because the presence of N particles is always understood with respect to a frame, e.g. a laboratory frame, which is represented by a space-like hypersurface.

- *Necessity:* The consistency condition (1.20) that is necessary for the existence of solutions to a multi-time system needs to be satisfied in the domain of ψ. In quantum field theory, this condition usually amounts to the commutation of field operators such as $[\hat{\Phi}(x_1), \hat{\Phi}(x_2)] = 0$, which is known to be true only on space-like configurations. This is the reason why multi-time formulations of quantum field theory, such as [2, 19] and the model by Dirac, Fock, Podolsky presented in Chapter 3 are well-posed on $\mathscr{S}^{(N)}$, but not on $\mathbb{R}^{(d+1)N}$.

Because of these reasons, we primarily consider multi-time wave functions defined on $\mathscr{S}^{(N)}$ in this work.
The single-time wave function φ can easily be obtained from the multi-time wave function by setting all times equal, i.e.

$$\varphi(t, \mathbf{x}_1, ..., \mathbf{x}_N) = \psi(t, \mathbf{x}_1, ..., t, \mathbf{x}_N). \tag{1.7}$$

If ψ solves the multi-time system (1.1), it follows by the chain rule that

$$i\partial_t \varphi(t, \mathbf{x}_1, ..., \mathbf{x}_N) = \sum_{k=1}^{N} \mathcal{H}_k(t, \mathbf{x}_1, ..., t, \mathbf{x}_N) \varphi(t, \mathbf{x}_1, ..., \mathbf{x}_N), \tag{1.8}$$

so we retrieve the usual Schrödinger equation with the Hamiltonian $\mathcal{H}$ being the sum over all the partial Hamiltonians $\mathcal{H}_k$. Thus, models like (1.1) are *single-time reducible* in the terminology of Lienert (see [7, sec. 1.2.3]). In this thesis, we only consider models of this type which most strongly resemble the known quantum theories.
We mention, however, that *single-time irreducible* dynamics for multi-time wave functions are also possible, in case that the resource of different time variables is actually used in the equations of motion. One class of examples for those are integral equations or integro-differential equations. The most prominent example comes from the attempt to describe bound states in QED: the Bethe-Salpeter equation (see [22, ch. 6] and [9, ch. 3]). Lienert [23] proposed a number of integral equations similar to this one to implement direct interactions in a relativistic quantum setting. His equations for two particles are of the form

$$\psi(x_1, x_2) = \psi^{\text{free}}(x_1, x_2) + \int d^4x_1' d^4x_2' \; G_1(x_1 - x_1') G_2(x_2 - x_2') K(x_1' - x_2') \psi(x_1', x_2'), \tag{1.9}$$

where $G_j, j = 1, 2$, is a Green's function of the free equation for particle j, ψ^{free} is the solution of the free multi-time system and K is an interaction kernel. There has been some progress on the solution theory of equations of the shape (1.9) in recent times [24, 25], but only for slightly idealized situations.
The hope is that the direct interaction on the quantum level helps to solve the problem of ultraviolet divergences in an analogous way as the formulation of classical electrodynamics by Wheeler and Feynman does (compare [26, 27]). While this is a very promising idea, it is beyond the scope of this thesis.

1.4 The meaning of multi-time wave functions

Since the multi-time wave function ψ is not merely an abstract mathematical object, but is supposed to describe physics, we have to clarify its meaning. The statistical meaning of the non-relativistic wave function φ is given by Born's rule, which says that detection probabilities of particle positions at time t are given by the $|\varphi(t)|^2$-distribution. Conservation of probability derives from the continuity equation

$$\frac{\partial|\varphi(t,\mathbf{x}_1,...,\mathbf{x}_N)|^2}{\partial t} = -\text{div } \mathbf{j}(\text{t},\mathbf{x}_1,...,\mathbf{x}_\text{N}), \tag{1.10}$$

with the probability current $\mathbf{j} = \frac{1}{m}\Im(\varphi^*\nabla\varphi)$. Note that the divergence and Nabla operators act in dN dimensions. We can build one single object with $dN+1$ components from this;

$$j := \left(|\varphi|^2, \tfrac{1}{m}\Im(\varphi^*\partial_{\mathbf{x}_1^1}\varphi), ..., \tfrac{1}{m}\Im(\varphi^*\partial_{\mathbf{x}_N^d}\varphi)\right), \tag{1.11}$$

which is then divergence-free in $dN+1$-dimensional Euclidean space, i.e.

$$\text{div } j = 0. \tag{1.12}$$

For a relativistic generalization, we require a description of detection probabilities on any space-like hypersurface. The relativistic generalization of the current (1.11) suggests to look for a tensor $j^{\mu_1\cdots\mu_N}$ with $(d+1)N$ components in Minkowski space-time that is built from the multi-time wave function and divergence-free in the sense of

$$\frac{\partial}{\partial x_k^{\mu_k}} j^{\mu_1\cdots\mu_N}(x_1,...,x_N) = 0, \quad k = 1,...,N. \tag{1.13}$$

The most important example is the Dirac tensor current

$$j^{\mu_1\cdots\mu_N} = \overline{\psi}\,\gamma_1^{\mu_1}\ldots\gamma_N^{\mu_N}\psi, \tag{1.14}$$

but one should note that the form of the tensor current depends on the evolution equations, it is e.g. different for the Klein-Gordon equation.
A divergence-free tensor current provides a simple way of generalizing the Born rule to relativistic space-time. Following [28], we may call this the *curved Born rule*: For (ideal) detectors on a space-like hypersurface Σ, the detection probabilities of particles are distributed according to the $|\psi|^2_\Sigma$-distribution, where

$$|\psi|^2_\Sigma(x_1,...,x_N) = j^{\mu_1\cdots\mu_N}(x_1,...,x_N)n_{\mu_1}(x_1)...n_{\mu_N}(x_N), \tag{1.15}$$

with $n_\mu(x)$ the future-pointing normal vector field on Σ. Probability conservation then amounts to

$$\int_\Sigma |\psi|^2_\Sigma = 1 \quad \text{for every space-like } \Sigma. \tag{1.16}$$

It was conjectured by several authors and has now been proven as a theorem in [28] that the Born rule on equal-time hypersurfaces actually implies the curved Born rule if the dynamics satisfies reasonable assumptions of interaction locality and finite speed of propagation. Hence, multi-time wave functions usually have the clear statistical meaning that is familiar from non-relativistic quantum physics.

Remark: Although statistical predictions may be correct, a theory that only provides a linear evolution for the multi-time wave function usually produces conceptual issues as they were described by Schrödinger in his famous cat example. There are several formulations of relativistic quantum mechanics that solve the so-called measurement problem [29]. Remarkably, the tensor current $j^{\mu_1 \cdots \mu_N}$ constructed from the multi-time wave function is an important building block in two of the most promising examples: Hypersurface-Bohm-Dirac models [30] and relativistic collapse models (see e.g. [31]). We present the law of motion of the former models as an example. In addition to the multi-time wave function that gives the tensor current, a Hypersurface-Bohm-Dirac model uses a foliation of space-time into space-like hypersurfaces Σ, and n denotes the unit normal vector field to the foliation. Then, the world lines of the particles are given by parametrizations $X_k(s), k = 1, ..., N$, that solve [18],

$$\frac{X_k^{\mu_k}(s)}{ds}\Big|_{\Sigma_s} \propto j^{\mu_1 \cdots \mu_N}(X_1^{\Sigma_s}, ..., X_N^{\Sigma_s}) \prod_{j \neq k} n_{\mu_j}(X_j^{\Sigma_s}), \tag{1.17}$$

where $X_j^{\Sigma_s}$ denotes the position where the world line of particle j crosses the respective hypersurface Σ_s. For a detailed discussion of realistic formulations relativistic quantum mechanics, see i.a. [7, ch. 2], [18], [17, ch. 11, 12]. For our topic, we just note that the multi-time wave function serves as a tool both for formulating the dynamics of primitive objects in quantum theories solving the measurement problem and also for deriving statistical predictions in the usual way. Since we focus on the mathematical status of the multi-time evolution in this work, we do not discuss the measurement problem any further.

1.5 The consistency condition

After the physical discussion above, we turn to the more mathematical properties of systems of differential equations for a multi-time wave function. We investigate systems of the form (1.1) as an initial value problem where the wave function ψ can be given on any space-like hypersurface or for any combination of initial times. For simplicity, we often only consider $t_1 = ... = t_N = 0$ as the initial value surface.
For most multi-time systems of interest, the properties of *each equation separately* are well-studied. The decisive question is therefore what happens if the N equations have to be solved simultaneously. In various proofs in this thesis, there are two qualitative insights we obtain:

- Uniqueness of solutions to the system is often not extraordinarily complicated: If we know that the solution to each equation is unique, putting them together to a system will not generate new solutions. Only on a domain with boundary like $\mathscr{S}^{(N)}$, additional considerations about the boundary might be necessary.

- Existence of solutions is more difficult. In particular, it requires an integrability condition that makes sure that the time evolutions defined by the different equations are compatible with each other.

The condition needed for existence is usually called *consistency condition* and is the topic of this section.
We focus on multi-time systems of the shape

$$\begin{aligned} i\partial_{t_1}\psi(x_1,...,x_N) &= \mathcal{H}_1(t_1,...,t_N)\psi(x_1,...,x_N) \\ i\partial_{t_2}\psi(x_1,...,x_N) &= \mathcal{H}_2(t_1,...,t_N)\psi(x_1,...,x_N) \\ &\dots \\ i\partial_{t_N}\psi(x_1,...,x_N) &= \mathcal{H}_N(t_1,...,t_N)\psi(x_1,...,x_N), \end{aligned} \tag{1.18}$$

where for fixed $(t_1,...,t_N) \in \mathbb{R}^N$ and each k,

$$\mathcal{H}_k(t_1,...,t_N) : \mathrm{dom}(\mathcal{H}_k(t_1,...,t_N)) \subset L^2(\mathbb{R}^{3N},\mathbb{C}^K) \to L^2(\mathbb{R}^{3N},\mathbb{C}^K) \tag{1.19}$$

is a linear operator called *partial Hamiltonian*. Although it is not natural from the relativistic point of view to separate space and time by using a spatial Hilbert space such as $L^2(\mathbb{R}^{3N},\mathbb{C}^K)$, it does allow to use the strong methods of functional analysis and thus gain information on existence and uniqueness of solutions for a wide class of systems. The solution theory to each single equation in (1.18) is then reduced to the question of self-adjointness of $\mathcal{H}_k$. In this sense, we often consider $\psi(t_1,...,t_N)$ for some fixed $(t_1,...,t_N) \in \mathbb{R}^N$ as a function of N space coordinates only, which is an element of the Hilbert space $L^2(\mathbb{R}^{3N},\mathbb{C}^K)$.
It was first recognized by Bloch [14] and investigated more closely by Petrat and Tumulka [3] that a necessary (and in some cases also sufficient) condition for the existence of solution to (1.18) is the so-called *consistency condition*

$$\left([\mathcal{H}_j,\mathcal{H}_k] - i\frac{\partial\mathcal{H}_k}{\partial t_j} + i\frac{\partial\mathcal{H}_j}{\partial t_k}\right)\psi = 0, \qquad \forall k \neq j. \tag{1.20}$$

In the first step, it is a condition on the operators *applied* to a solution ψ. However, if we want to obtain a solution for all possible initial values, it becomes a condition on the operators themselves and one often sees eq. (1.20) written without the ψ.
We give a general idea why this condition is necessary before coming to rigorous results.

1.5.1 Idea behind the consistency condition

We first discuss why the consistency condition (1.20) is expected to be necessary for existence of solutions. Looking a the multi-time system (1.18), the condition is connected with the path independence in the space of time coordinates: If we are given initial values at $t_1 = ... = t_N = 0$, it should make no difference if we first evolve to some t_j while keeping the other times constant and then in t_k-direction, or the other way around.
This is particularly transparent for time-independent partial Hamiltonians. In this case, the integration of a single equation in (1.18) gives

$$\psi(s_1,...,s_{k-1},t_k,s_{k+1}...,s_N) = e^{-i\mathcal{H}_k(t_k-s_k)}\psi(s_1,...,s_{k-1},s_k,s_{k+1}...,s_N) \tag{1.21}$$

Therefore, a natural guess for the solution of (1.18), given initial values $\psi|_{t_1=...=t_N=0} = \psi^0 \in L^2(\mathbb{R}^3,\mathbb{C}^K)$, is

$$\psi(t_1,...,t_N) = e^{-i\mathcal{H}_1 t_1}e^{-i\mathcal{H}_2 t_2}\dots e^{-i\mathcal{H}_N t_N}\psi^0. \tag{1.22}$$

But the order of the exponentials was chosen haphazardly, and any other order is also a natural candidate for the solution. For the well-definedness of the solution, we therefore expect that the exponentials should commute, which amounts to

$$[\mathcal{H}_j,\mathcal{H}_k] = 0, \qquad \forall k \neq j, \tag{1.23}$$

in the spectral sense. That is, the actions of the respective differential equations in the system (1.18) have to commute. The analogue of this condition in the case of time-dependent Hamiltonians is (1.20). We proceed to its rigorous justification.

Remark: Equation (1.22) and other orders give $N!$ candidates for the solution of system (1.18) in the case of time-independent Hamiltonians. In fact, one could time-evolve along an arbitrary path from $(0, ..., 0)$ to $(t_1, ..., t_N)$ and thus even obtain infinitely many candidates. We explained that the consistency condition is needed for them to all be equal. By this explanation, one could erroneously think that a violation of the consistency condition (1.23) leads to a problem with *uniqueness* of solutions. This is not true since, as mentioned above, uniqueness of solutions can be proven with high generality without any further conditions. The actual problem is that formula (1.22) only gives a solution of the multi-time system if the exponentials commute, and if not, there is no solution at all. So the consistency condition is really connected to the *existence* of a solution.

1.5.2 Known results on existence and uniqueness

We begin with the simple case of time-independent partial Hamiltonians $\mathcal{H}_k$, for which the standard methods of functional analysis can directly be applied. Theorems on existence and uniqueness in this case are found in [3, thm. 1] and [7, cor. 1.2.2][3] We reformulate both results with the help of the following definition of different solution senses.

Definition: Consider a system of the form (1.18) with self-adjoint partial Hamiltonians on a Hilbert space $\mathscr{H}$. We call a function $\psi : \mathbb{R}^N \to \mathscr{H}, (t_1, ..., t_N) \mapsto \psi(t_1, ..., t_N)$,

- solution in the weak $\mathscr{H}$-sense iff the partial Hamiltonians are time-independent and for all $(t_1, ..., t_N) \in \mathbb{R}^N$ and all $k = 1, ..., N$,

$$\psi(t_1, ..., t_N) = e^{-i\mathcal{H}_k t_k} \psi(t_1, ..., t_{k-1}, 0, t_{k+1}, ..., t_N). \tag{1.24}$$

- solution in the strong $\mathscr{H}$-sense iff $\psi : \mathbb{R}^N \to \mathscr{H}$ is differentiable and satisfies for all $(t_1, ..., t_N) \in \mathbb{R}^N$ and all $k = 1, ..., N$,

$$i\partial_{t_k} \psi(t_1, ..., t_N) = \mathcal{H}_k(t_1, ..., t_N)\psi(t_1, ..., t_N). \tag{1.25}$$

These notions of solution are used in the literature with many different names, the first one for instance is called *strong solution* in [3]. Since we want to understand the notion of strong solution in the sense of a point-wise evaluable function $\psi(t_1, \mathbf{x}_1, ..., t_N, \mathbf{x}_N)$ and not a Hilbert space element $\psi(t_1, ..., t_N)$, we reserve the classical solution sense of PDE theory for the former and add "in the $\mathscr{H}$-sense" for the latter kind of solutions here.

Proposition 1.1. *For the multi-time system* (1.18) *with time-independent self-adjoint partial Hamiltonians on a Hilbert space $\mathscr{H}$ and with given initial values $\psi^0 \in \mathscr{H}$,*

- *there exists a unique solution ψ in the weak $\mathscr{H}$-sense with $\psi(0, ..., 0) = \psi^0$ if and only if the consistency condition $[\mathcal{H}_j, \mathcal{H}_k] = 0$ holds (in the spectral sense) for each $k \neq j$.*

- *there exists a unique solution ψ in the strong $\mathscr{H}$-sense with $\psi(0, ..., 0) = \psi^0$ if and only if the consistency condition $[\mathcal{H}_j, \mathcal{H}_k] = 0$ holds (in the spectral sense) for each $k \neq j$ and $\psi^0 \in \mathrm{dom}(\mathcal{H}_k)$ for each k.*

[3] Note that, in the latter source in the formulation of Lienert's corollary 1.2.2, the word self-adjoint should be replaced by essentially self-adjoint.

The additional condition in [7, cor. 1.2.2] that there is a common domain of essential self-adjointness of the Hamiltonians can be dropped since it follows from the commutativity, compare [32, cor. 5.28]. Since the statement is slightly stronger than in the literature, we provide a proof.

Proof. The first part was proven in [3, thm. 1] and follows directly from a theorem about unitary N-parameter groups in the book of Reed and Simon [33, thm. VIII.12].
We now show both directions of the second part.
"$\Longrightarrow$": If ψ is a solution in the strong $\mathscr{H}$-sense, it is by the definition above also a solution in the weak $\mathscr{H}$-sense. Using the first part of the proposition, this implies the consistency condition $[\mathcal{H}_j, \mathcal{H}_k] = 0$ for each $k \neq j$. For condition (1.25) to be true at $t_1 = ... = t_N = 0$, we also need $\psi^0 \in \mathrm{dom}(\mathcal{H}_k)$ for every $k = 1, ..., N$.
"$\Longleftarrow$": Uniqueness of the solution is clear by the first part of the proposition. We now prove the existence of a solution constructively. We may define

$$\psi(t_1, ..., t_N) = e^{-i\mathcal{H}_1 t_1} \dots e^{-i\mathcal{H}_N t_N} \psi^0, \tag{1.26}$$

and notice that, by the commutativity of the Hamiltonians, any order of the exponentials yields the same function. Let $k \in \{1, ..., N\}$. Then we have

$$\begin{aligned} i\partial_{t_k}\psi(t_1, ..., t_N) &= i\partial_{t_k}\left(e^{-i\mathcal{H}_{k+1}t_{k+1}} \dots e^{-i\mathcal{H}_N t_N} e^{-i\mathcal{H}_1 t_1} \dots e^{-i\mathcal{H}_k t_k}\psi^0\right) \\ &= e^{-i\mathcal{H}_{k+1}t_{k+1}} \dots e^{-i\mathcal{H}_N t_N} e^{-i\mathcal{H}_1 t_1} \dots e^{-i\mathcal{H}_k t_k}\mathcal{H}_k\psi^0 \\ &= \mathcal{H}_k\psi(t_1, ..., t_N). \end{aligned} \tag{1.27}$$

Since $\psi^0 \in \mathrm{dom}(\mathcal{H}_k)$, we may use the commutativity and we see that this expression is well-defined. □

For time-dependent Hamiltonians $\mathcal{H}_k(t_1, ..., t_N)$, the solution theory of a multi-time system becomes much more intricate. Even the existence theory of a single equation requires more conditions than just self-adjointness of the Hamiltonian for all times, compare the classical results by Kato [34] or Yosida [35, ch. XIV] (also found in [36, thm. X.70]). For bounded Hamiltonians, the time evolution is given by the Dyson expansion and Petrat and Tumulka could prove the following analogue of the above proposition [3, thm. 2], which we reformulate according to our definition.

Proposition 1.2. ***(Petrat and Tumulka).*** *Let $\mathcal{H}_1, ..., \mathcal{H}_N$ be smooth functions on $\mathbb{R}^N$ with values in the bounded operators on a Hilbert space $\mathscr{H}$. Then the multi-time system* (1.18) *possesses a solution $\psi : \mathbb{R}^N \to \mathscr{H}$ in the strong $\mathscr{H}$-sense for every initial condition $\psi(0, ..., 0) \in \mathscr{H}$ if and only if the consistency condition*

$$[\mathcal{H}_j, \mathcal{H}_k] - i\frac{\partial \mathcal{H}_k}{\partial t_j} + i\frac{\partial \mathcal{H}_j}{\partial t_k} = 0 \quad \forall j \neq k \tag{1.28}$$

holds.

We prove further results on the consistency condition, especially one for classical solutions of a multi-time system (Prop. 2.2), in the subsequent chapters. Chapter 2 deals with the restrictions the consistency condition puts on multi-time systems and shows that no Poincaré invariant interaction potentials are compatible with it.

Remark:

(a) It is conceivable that the proof of Petrat and Tumulka can be refined to yield a similar theorem for Hamiltonians of the form $\mathcal{H}_k(t_1, ..., t_N) = \mathcal{H}_k^{\text{free}} + V_k(t_1, ..., t_N)$, where $\mathcal{H}_k^{\text{free}}$ is a time-independent term and V_k is bounded. This works since one can move to the interaction picture and then build the Dyson expansion for the terms $V_k(t_1, ..., t_N)$ only. The model by Dirac, Fock, and Podolsky treated in chapter 3, however, has an unbounded interaction term and thus requires more work.

(b) The two results presented here use the full configuration space-time $\mathbb{R}^{4N}$ as domain of the multi-time wave function, albeit the reasoning in Sec. 1.3 about the natural domain $\mathscr{S}^{(N)}$. This is a simplification that allows for the formulation of general results that underline the meaning of the consistency condition (1.20). It is more complicated to treat models on a non-trivial domain like $\mathscr{S}^{(N)}$, so we tackle this only for concrete cases in the chapters 3 and 4, and not in great generality.

1.5.3 Geometric view of the consistency condition

This subsection is taken from [10] by D.-A. Deckert and the present author. We discuss on a non-rigorous level how the results on the consistency condition can be reformulated with the help of differential geometry (compare section 2.3 in [3]). For each multi-time argument $(t_1, ..., t_N)$, the multi-time wave function is an element of the Hilbert space $\mathscr{H} = L^2(\mathbb{R}^{3N}, \mathbb{C}^K)$. We can define a vector bundle E over the base manifold $\mathbb{R}^N$ with identical fibres $\mathscr{H}$ at every point. (This is therefore a trivial vector bundle $E = \mathbb{R}^N \times \mathscr{H}$). A multi-time wave function is then a section of E.
A natural notion of parallel transport on E can be given by the single-time evolution operators $U_k(t_k)$ (which would be $e^{-i\mathcal{H}_k t_k}$ for time-independent $\mathcal{H}_k$). This means that we define a connection ∇ on E with components $\nabla_k = \partial_{t_k} + i\mathcal{H}_k$, whereby the parallel transport in direction t_k is given by U_k. Solutions of (1.18) are then sections that are covariantly constant, i.e. satisfy $\nabla\psi = 0$.
The well-definedness of solutions requires that the parallel transport along a closed curve does not change the vector. So we need that for any loop γ, $U_\gamma = \mathbb{1}$. This is equivalent to saying that the vector bundle has a trivial holonomy group, $\text{Hol}(\nabla) = \{\mathbb{1}\}$. By the theorem of Ambrose and Singer [37], the holonomy group is in direct correspondence to the curvature form $F(\nabla)$; in particular:

$$\text{Hol}(\nabla) = \{\mathbb{1}\} \iff F(\nabla) = 0. \tag{1.29}$$

Therefore, the existence of a well-defined solution implies that ∇ is a flat curvature for E. By the formula for calculating the curvature from the connection, this means

$$0 = F_{jk} = \frac{\partial \mathcal{H}_j}{\partial t_k} - \frac{\partial \mathcal{H}_k}{\partial t_j} - i[\mathcal{H}_j, \mathcal{H}_k], \quad \forall j \neq k, \tag{1.30}$$

which is the consistency condition (1.20). However, this is only rigorously true in some particular cases since many of the results in differential geometry are only available for finite-dimensional fibres and/or continuous connection components. Thus, a generalization to unbounded Hamiltonians in L^2 is not in reach.

CHAPTER
TWO

CONSISTENCY OF MULTI-TIME DIRAC EQUATIONS WITH INTERACTION POTENTIALS

This chapter is based on the paper [10] by D.-A. Deckert and the present author. We consider first-order multi-time systems of the shape (1.1) and discuss the question how to introduce an interaction between the N particles. A ready idea would be an interaction potential $V(x_k, x_j)$ as in non-relativistic quantum mechanics. But the consistency condition explained in Sec. 1.5 is restrictive in the admission of solutions and actually rules out relativistic interacting potentials. To make this statement precise is the purpose of this chapter, achieved in Theorem 2.1.

2.1 Definition of the model

The model for our investigation is given by the system of evolution equations

$$i\frac{\partial}{\partial t_k}\psi(x_1,...,x_N) = \mathcal{H}_k(x_1,...,x_N)\psi(x_1,...,x_N), \quad k=1,...,N, \tag{2.1}$$

where the *partial Hamiltonians* $\mathcal{H}_k$ are given by

$$\mathcal{H}_k = \mathcal{H}_k^0 + V_k, \tag{2.2}$$

with $\mathcal{H}_k^0$ being the free Dirac Hamiltonian of the k-th particle. The interaction shall be described by the operator V_k which is given in terms of a (self-adjoint) spin-matrix valued multiplication operator $V_k(x_1,...,x_N)$ that depends on the space-time coordinates $x_1,\ldots,x_N$. For this model, as was first recognized by Bloch [14] and further investigated by Petrat and Tumulka [3], a necessary condition for existence of solutions to (2.1) is the aforementioned *consistency condition*

$$\left([\mathcal{H}_j,\mathcal{H}_k] - i\frac{\partial V_k}{\partial t_j} + i\frac{\partial V_j}{\partial t_k}\right)\psi = 0, \qquad \forall k \neq j. \tag{2.3}$$

In [3], Petrat und Tumulka conjectured that interacting systems of the form (2.1) with general non-vanishing potentials that lead to interaction between the particles are excluded as they would violate the consistency condition (2.3). They gave a proof of this claim under the assumption that the potentials V_k depend on the spin-index of the k-th particle only. This rules out a number of conceivable potentials, but not all of them: Potentials such as the one of the Breit equation [38, 39], which can be derived as an approximation to

the Bethe-Salpeter equation of QED (see [22]), contain a more complicated spin-coupling, which poses the question whether more general potentials may indeed comply with condition (2.3) and thereby to well-posedness of (2.1) in terms of an initial value problem.

As main results of this chapter, we present a concrete example of a spin-coupling interaction potential which satisfies the consistency condition. However, we also show that the class of potentials admitted by the consistency condition is rather small. In particular, under certain smoothness conditions on possible solutions ψ, we identify this class completely and show that it does not contain Poincaré invariant potentials. Therefore, combining the mathematical consistency condition with the physical requirement of Poincaré invariance, our results show that any type of potential acting as a multiplication operator must be excluded as possible candidate for modeling the interaction between the N particles.

After the following paragraph about the employed notation and conventions, we present our results in Section 2.2 and the proofs and more detailed derivations in Sections 2.3 and 2.4.

Notations and conventions. We consider 4-dimensional Minkowski space-time with metric $\eta = \mathrm{diag}(1, -1, -1, -1)$, with the usual notation that Greek indices run from 0 to 3 and Latin indices $a, b, ...$ only over the spatial components $1, 2, 3$. The Einstein summation convention is employed for Greek indices only. Particle labels are denoted also by Latin indices, $j, k, ...$ and run from 1 to the total particle number N. Throughout, the abbreviation $\partial_{k,\mu} := \frac{\partial}{\partial x_k^\mu}$ is used.
The gamma matrices are arbitrary 4×4-matrices that form a representation of the Clifford algebra, i.e. fulfill the anti-commutation relation

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\mathbb{1}, \quad \mu, \nu = 0, 1, 2, 3. \tag{2.4}$$

Moreover, the matrix γ^0 is hermitian, γ^a anti-hermitian, and a fifth gamma matrix is defined as

$$\gamma^5 := i\gamma^0\gamma^1\gamma^2\gamma^3. \tag{2.5}$$

The free Dirac Hamiltonian for the k-th particle is given by $\mathcal{H}_k^0 = -i\sum_{a=1}^3 \gamma_k^0\,\gamma_k^a\partial_{k,a} + \gamma_k^0 m_k$, where m_k is the mass of the k-th particle and we use the following convention for the matrices: Since we are always working in the N-fold tensor product of $\mathbb{C}^4$, we write for some 4×4-matrix M:

$$M_k := \mathbb{1} \otimes \cdots \otimes \mathbb{1} \underbrace{\otimes\, M \,\otimes}_{\text{k-th place}} \mathbb{1} \otimes \cdots \otimes \mathbb{1}. \tag{2.6}$$

It is well-known that the Dirac operator (4.5) is self-adjoint on $\mathrm{dom}(\mathcal{H}_k^0) = H^1(\mathbb{R}^3, \mathbb{C}^4)$; see [40]. Furthermore, it will be convenient to use the notation $\alpha_k^\mu := \gamma_k^0\gamma_k^\mu$ so that we may write the multi-time system (2.1) as

$$(i\alpha_k^\mu\partial_{k,\mu} - \gamma_k^0 m_k)\psi(x_1, ..., x_N) = V_k(x_1, ...x_N)\psi(x_1, ..., x_N), \quad k = 1, ..., N. \tag{2.7}$$

Hence, the wave function $\psi(x_1, ..., x_N)$ takes values in $(\mathbb{C}^4)^{\otimes N} \cong \mathbb{C}^K$, $K := 4^N$.

2.2 Results

In order to present the results two remarks are in order. First, we need to make precise what is meant by the notion *interaction potential*. External potentials of the form $V_k(x_k)$ that do not generate entanglement must be excluded, and also potentials that seemingly depend on different coordinates, but that actually only arise from external potentials by a change of coordinates in the spinor space $\mathbb{C}^K$. Therefore we define:

Definition: A collection of potentials V_k, $k = 1, ..., N$, given as spin-matrix valued multiplication operators $V_k(x_1, ..., x_N)$ is called non-interacting iff there is a unitary map $U(x_1, ..., x_N) : \mathbb{C}^K \to \mathbb{C}^K$ such that $\tilde{\psi} := U(x_1, ...x_N)(\psi(x_1, ...x_N))$ satisfies a system of the form (2.1) where for each k, the potential $V_k(x_k)$ is independent of all other coordinates $x_1, ...x_{k-1}, x_{k+1}, ...x_N$. In the other case, we call the collection of potentials interacting.

Petrat and Tumulka called potentials that are connected via a unitary map U *gauge-equivalent* [3], which means that interacting potentials in the sense of our definition are exactly those that are not gauge-equivalent to external potentials.
Second, as we emphasized in 1.3 that the natural domain of a multi-time wave function is not the whole configuration space-time $\mathbb{R}^{4N}$, but the subset $\mathscr{S}^{(N)}$, all results will be proven mainly on $\mathscr{S}^{(N)}$ and only besides on $\mathbb{R}^{4N}$. Lastly, we have to make precise what is meant by Poincaré invariance of potentials. For Λ in the proper Lorentz group and $a \in \mathbb{R}^4$, the Poincaré transformation maps $x \mapsto x' = \Lambda x + a$ and the multi-time wave function transforms as

$$\psi'(x_1, ...x_N) = S(\Lambda)^{\otimes N} \psi\left(\Lambda^{-1}(x_1 - a), ..., \Lambda^{-1}(x_N - a)\right), \tag{2.8}$$

with the spin transformation matrix $S(\Lambda)$ that fulfills $S(\Lambda)\gamma S^{-1}(\Lambda) = \Lambda\gamma$. We call a potential V_k Poincaré invariant if it satisfies

$$V_k(x_1, ..., x_N) = S(\Lambda)^{\otimes N} V_k\left(\Lambda^{-1}(x_1 - a), ..., \Lambda^{-1}(x_N - a)\right) S^{-1}(\Lambda)^{\otimes N}, \tag{2.9}$$

which is the condition for (2.1) to be Poincaré invariant. Our main result can then be stated as follows:

Theorem 2.1. *Let $N = 2$, $\Omega = \mathbb{R}^{4N}$ or $\Omega = \mathscr{S}^{(N)}$. If $V_k(x_1, ..., x_N)$ are interacting potentials in $C^1(\Omega, \mathbb{C}^{K\times K})$ and for all initial values $\varphi \in C_c^\infty(\mathbb{R}^{3N} \cap \Omega, \mathbb{C}^K)$, there is a solution $\psi \in C^2(\Omega, \mathbb{C}^K)$ to the multi-time system of Dirac equations* (2.1)*, then the potentials V_k are not Poincaré invariant.*

We only formulate the theorem for the case $N = 2$, although we expect it to hold for general N and we prove several intermediate results for any N. For larger numbers of particles, however, some parts in the proofs which are based on a direct computation in terms of gamma matrices quickly become very complex and hardly traceable. In several partial results, we will also not restrict to $\Omega = \mathbb{R}^{4N}$ or $\Omega = \mathscr{S}^{(N)}$, but consider any open set $\Omega \subset \mathbb{R}^{4N}$. The strategy of proof is illustrated as follows:

(a) **Existence $\Longrightarrow$ Consistency:** If a solution to (2.1) exists, then the consistency condition (2.3) has to hold.

(b) **Consistency $\Longrightarrow$ Restrictions on potentials:** If the consistency condition (2.3) holds, then the admissible potentials are restricted and no Poincaré invariant ones are possible.

2.2.1 Step (a): The consistency condition.

We already discussed in Sec. 1.5.1 why one expects the consistency condition (2.3) to be necessary for existence of solutions. Petrat and Tumulka have proven that the existence of a solution for every initial datum in the Hilbert space necessitates the consistency condition (2.3) in two different cases [3, thm. 1 and 2] we presented in Sec. 1.5.2:

- for time-independent, possibly unbounded partial Hamiltonians $\mathcal{H}_k$,
- for time-dependent, but smooth and bounded partial Hamiltonians $\mathcal{H}_k$.

Here, we generalize the results of Petrat and Tumulka to the relevant case of unbounded Hamiltonians that may include a time-dependence in the potentials. Our proposition is a rather direct consequence of the differentiability of solutions and makes the idea of Bloch [14, p. 304] mathematically precise.

Proposition 2.2. *Let $\Omega \subset \mathbb{R}^{4N}$ be open. Suppose the multi-time system (2.1), with V_k being a function in $C^1(\Omega, \mathbb{C}^{K\times K})$, possesses a solution $\psi \in C^2(\Omega, \mathbb{C}^K)$. Then the consistency condition (2.3) holds for all $(x_1, ..., x_N) = X \in \Omega$.*

The proof is given in Section 2.3.

2.2.2 Step (b): Consistent potentials.

The consistency condition puts strong restrictions on the spin-coupling induced by the potentials. The following example shows the inconsistency for one natural looking choice.

Example: We consider a two-particle system (2.1) with $V_1 = \alpha_2^\mu A_\mu(x_1, x_2)$ and $V_2 = \alpha_1^\mu B_\mu(x_1, x_2)$ for some smooth, compactly supported functions A_μ, B_μ. This is suggested by the usual way of adding a 4-vector potential to the single-time Dirac equation, which is by adding $\alpha^\mu A_\mu$ to the Hamiltonian. One could think that interaction is achieved by choosing the gamma matrices of the other particle, as done here. But then the consistency condition is

$$\begin{aligned} &\left[\alpha_2^\mu A_\mu, -i\alpha_2^\nu \partial_{2,\nu} + \gamma_2^0 m_2\right] = 0 \\ \iff &-2m_2\gamma_2^\mu A_\mu + i\alpha_2^\nu \alpha_2^\mu \left(\partial_{2,\nu} A_\mu\right) + iA_\mu[\alpha_2^\nu, \alpha_2^\mu]\partial_{2,\nu} = 0. \end{aligned} \tag{2.10}$$

There is no possibility that the respective terms will cancel each other, so any A_μ different from zero will make the equations inconsistent. In particular, the derivative term with $\partial_{2,\nu}$ has to vanish separately, which will be a crucial ingredient in the proof of Theorem 2.3. A similar calculation excludes potentials of the form $V_k \sim F_{\mu\nu}(x_1, x_2)\gamma_1^\mu\gamma_2^\nu$, too.

To have a chance of being consistent, the potentials may only depend on few matrices, which are the identity matrix and γ^5. To see this, we need to reformulate the consistency condition to a more useful version. That the bracket in (2.3) applied to any solution ψ ought to be zero implies that it must also be zero on every initial value $\varphi = \psi|_{t_1=...=t_N=0}$. The initial values will be defined on a $3N$-dimensional set U, an intersection of Ω with the time-zero hypersurface. The assumption that there are solutions for all initial values in a certain class, e.g. the smooth compactly supported functions, allows us to draw general conclusions.

Theorem 2.3. *We assume:*

*(**A**) $U \subseteq \mathbb{R}^{3N}$ is open and simply connected. For a multi-time Dirac system* (2.1) *with continuously differentiable V_k, we have for each $\varphi \in C_c^\infty(U, \mathbb{C}^K)$,*

$$\left([\mathcal{H}_j, \mathcal{H}_k] - i\frac{\partial V_k}{\partial t_j} + i\frac{\partial V_j}{\partial t_k} \right) \varphi = 0, \qquad \forall k \neq j. \tag{2.11}$$

Then, for each $k \neq j$, the k-th spin component of the potential V_j is spanned by $\mathbb{1}_k$ and γ_k^5.

The proof is given in Section 2.4.1. One can directly see that the above example is not in the class of admissible potentials.
Theorem 2.3 allows us to proceed by a basis decomposition. All possible matrix structures that might appear in V_1 and V_2 can be listed and the consistency condition can be explicitly evaluated, as will be done in Section 2.4.2. In Lemma 2.7, we show that the consistency condition is equivalent to the system of equations (2.33a) to (2.33p), and that only eight possibly interacting terms remain.
It turns out that these possibilities for interacting terms in the potentials cannot be excluded by general arguments. In fact, interacting potentials that fulfill the consistency condition exist, for example the ones in the following lemma.

Lemma 2.4. *Let C_ν and c_ν be constants for $\nu = 0, 1, 2, 3$ with at least one C_ν and c_ν different from zero, and define $x := x_2 - x_1$. Consider the multi-time Dirac system* (2.1) *for two particles with potentials*

$$\begin{aligned} V_1 &= \gamma_1^\mu C_\mu \exp\left(2i\gamma_1^5 c_\lambda x^\lambda\right) - m_1\gamma_1^0 \\ V_2 &= \gamma_1^5 \alpha_2^\nu c_\nu. \end{aligned} \tag{2.12}$$

(a) This system is consistent, i.e. (2.3) *holds.*

(b) This system is interacting.

This is proven in Section 2.4.3. With this example at hand, it becomes clear that we cannot prove inconsistency of arbitrary interacting potentials. But obviously, the potential V_1 in (2.12) is not Lorentz invariant. Since the use of multi-time equations aims at a relativistic formulation of quantum mechanics, it is natural to require Poincaré invariance, i.e. Lorentz invariance and translation invariance, of the potentials. We show that the latter excludes the former by finding that every translation invariant potential has to be of a certain shape.

Lemma 2.5. *Suppose the assumptions* (**A**) *of Theorem 2.3 hold. If, in addition, the potentials are both interacting and translation invariant, i.e. satisfy*

$$V_k(x_1, x_2) = V_k(x_1 + a, x_2 + a) \quad \forall a \in \mathbb{R}^4, \tag{2.13}$$

then they are necessarily of the form

$$V_k = M_1 e^{c_{k,\nu} x^\nu} + M_2 e^{-c_{k,\nu} x^\nu} + const. \tag{2.14}$$

for some $M_1, M_2 \in \mathbb{C}^{K\times K}$ and $c_k \in \mathbb{C}^4$, where $x = x_1 - x_2$.

A slightly stronger version of this lemma will be formulated and proven in Section 2.4.4. Our main Theorem 2.1 can then be proven by a simple collection of facts:

Proof of Theorem 2.1:
First case: $\Omega = \mathbb{R}^{4N}$. Suppose a system (2.1) with potentials $V_k \in C^1(\mathbb{R}^{4N}, \mathbb{C}^{K\times K})$ that are interacting has a solution $\psi \in C^2(\mathbb{R}^{4N}, \mathbb{C}^K)$ for all initial values $\varphi \in C_c^\infty(\mathbb{R}^{3N}, \mathbb{C}^K)$. Consequently, by Proposition 2.2, the consistency condition (2.11) has to be true for all $\varphi \in C_c^\infty(\mathbb{R}^{3N}, \mathbb{C}^K)$. Then, by Lemma 2.5, if the potentials are translation invariant, they are of the form (2.14), which is not Lorentz invariant. Therefore, the potentials cannot be Poincaré invariant.
Second case: $\Omega = \mathscr{S}^{(N)}$. The proof for the domain $\mathscr{S}^{(N)}$ goes through as above because the necessary lemmas were all proven for general domains that are open and simply connected, which is true for $\mathscr{S}^{(N)}$. □

Under the assumptions on higher regularity of solutions, we have thus generalized the results of Petrat and Tumulka [3] in the sense that our Theorem 2.1 covers arbitrary multiplication operators with spin-coupling. The class of potentials that are consistent and translation invariant (equation (2.14)) does not contain any physically interesting potentials, but only potentials that oscillate with the distance of the particles. That these are not Lorentz invariant further motivates to disregard them because multi-time equations are intended for a fully and manifest Lorentz invariant formulation of quantum mechanics.

2.3 Proof of the consistency condition

Proof of Proposition 2.2: Suppose $\psi \in C^2(\Omega, \mathbb{C}^K)$ solves the equations (2.1). Let $j \neq k$. By the theorem of Schwarz, the time-derivatives on ψ commute, which at some point $X \in \Omega$ gives:

$$\begin{aligned}
\left(i\partial_{t_k} i\partial_{t_j} - i\partial_{t_j} i\partial_{t_k}\right)\psi = 0 &\Rightarrow i\partial_{t_k}\left(\mathcal{H}_j\psi\right) - i\partial_{t_j}\left(\mathcal{H}_k\psi\right) = 0 && (2.15)\\
&\Rightarrow \mathcal{H}_j i\partial_{t_k}\psi + \left(i\partial_{t_k}V_j\right)\psi - \left(i\partial_{t_j}V_k\right)\psi - \mathcal{H}_k i\partial_{t_j}\psi = 0 && (2.16)\\
&\Rightarrow \left(\mathcal{H}_j\mathcal{H}_k + \left(i\partial_{t_k}V_j\right) - \left(i\partial_{t_j}V_k\right) - \mathcal{H}_k\mathcal{H}_j\right)\psi = 0. && (2.17)
\end{aligned}$$

In (2.15) and (2.17), we used that ψ solves the multi-time equations (2.1), and (2.16) follows by the product rule. As $X \in \Omega$ was arbitrary, equation (2.3) holds on Ω, as claimed. □

Remark:

(a) The assumption that the solution ψ is at least twice differentiable in the time direction seems unproblematic because the spatial smoothness of initial data is usually inherited in the time direction due to the nature of physically relevant evolution equations. E.g. for the one-particle Dirac equation with smooth external electromagnetic potential A_μ, it was proven in [41] that solutions that are smooth on one (space-like) Cauchy surface are indeed smooth on all of $\mathbb{R}^4$.

(b) For the domain $\Omega = \mathscr{S}^{(N)}$, this theorem even covers potentials of the form

$$V \sim \frac{1}{(t_k - t_j)^2 - |\mathbf{x}_k - \mathbf{x}_j|^2}, \tag{2.18}$$

since $V \in C^\infty(\mathscr{S}^{(N)}, \mathbb{C}^{K\times K})$ is ensured by the singularity being outside of $\mathscr{S}^{(N)}$. One might call V relativistic Coulomb potential because it is a Lorentz invariant function that, on equal-time hypersurfaces, coincides with the well-known Coulomb potential.

2.4 Spin-coupling potentials

2.4.1 Proof of the theorem on the allowed matrix structures

Proof of Theorem 2.3: We start with a system (2.7) and evaluate the consistency condition (2.11). Let $k \neq j$, then:

$$\begin{aligned}
&\left[i\alpha_k^\mu \partial_{k,\mu} - \gamma_k^0 m_k - V_k, i\alpha_j^\nu \partial_{j,\nu} - \gamma_j^0 m_j - V_j\right] && (2.19)\\
&= \left[i\alpha_k^\mu \partial_{k,\mu} - \gamma_k^0 m_k, -V_j\right] + \left[-V_k, i\alpha_j^\nu \partial_{j,\nu} - \gamma_j^0 m_j\right] + \left[V_k, V_j\right] && (2.20)\\
&= \left[V_k, V_j\right] + m_k\left[\gamma_k^0, V_j\right] - m_j\left[\gamma_j^0, V_k\right] - i\left[\alpha_k^\mu \partial_{k,\mu}, V_j\right] + i\left[\alpha_j^\nu \partial_{j,\nu}, V_k\right]. && (2.21)
\end{aligned}$$

In (2.20), we used that the derivatives w.r.t. different coordinates commute by Schwarz. We consider the last term in more detail:

$$\begin{aligned}
i\left[\alpha_j^\nu \partial_{j,\nu}, V_k\right] &= i\alpha_j^\nu \partial_{j,\nu} V_k - iV_k \alpha_j^\nu \partial_{j,\nu}\\
&= i\alpha_j^\nu \left(\partial_{j,\nu} V_k\right) + i\alpha_j^\nu V_k \partial_{j,\nu} - iV_k \alpha_j^\nu \partial_{j,\nu}\\
&= i\alpha_j^\nu \left(\partial_{j,\nu} V_k\right) + i\left[\alpha_j^\nu, V_k\right]\partial_{j,\nu}\\
&= i\alpha_j^\nu \left(\partial_{j,\nu} V_k\right) + i\sum_{a=1}^{3}\left[\alpha_j^a, V_k\right]\partial_{j,a},
\end{aligned} \tag{2.22}$$

where in the last line, the summand with $\nu = 0$ was dropped because $\alpha^0 = \mathbb{1}$ commutes with everything. Doing the same for the second last term yields that the consistency condition is equivalent to

$$\begin{aligned}
0 = &\left[V_k, V_j\right] + m_k\left[\gamma_k^0, V_j\right] - m_j\left[\gamma_j^0, V_k\right]\\
&-i\alpha_k^\mu \left(\partial_{k,\mu} V_j\right) + i\alpha_j^\nu \left(\partial_{j,\nu} V_k\right)\\
&-i\sum_{a=1}^{3}\left[\alpha_k^a, V_j\right]\partial_{k,a} + i\sum_{a=1}^{3}\left[\alpha_j^a, V_k\right]\partial_{j,a}.
\end{aligned} \tag{2.23}$$

The derivatives in (2.23) are in some sense linearly independent, which is made clear in the following auxiliary claim.

Lemma 2.6. *Let $U \subseteq \mathbb{R}^{3N}$ be open. Let $f : U \to \mathbb{C}^K$ be a function and suppose there are complex $K \times K$-matrices $\Lambda_{k,j}(\mathbf{x}_1, ..., \mathbf{x}_N)$ such that*

$$\left(f(\mathbf{x}_1, ..., \mathbf{x}_N) + \sum_{k=1}^{N}\sum_{j=1}^{3}\Lambda_{k,j}\frac{\partial}{\partial x_k^j}\right)\varphi(\mathbf{x}_1, .., .\mathbf{x}_N) = 0, \quad \forall(\mathbf{x}_1, ..., \mathbf{x}_N) \in U, \tag{2.24}$$

holds for all $\varphi \in C_c^\infty(U, \mathbb{C}^K)$. Then, for all j and k, $\Lambda_{k,j}(\mathbf{x}_1, ..., \mathbf{x}_N) = 0$, and f must be the zero function.

Proof of the Lemma: We choose some fixed k and j and show that $\Lambda_{k,j} = 0$ first. Pick some point $(\mathbf{x}_1, ..., \mathbf{x}_N) = \mathbf{X} \in U$. There exists $\varphi \in C_c^\infty(U, \mathbb{C}^K)$ with the property that $\varphi(\mathbf{X}) = 0$ and $\partial_{l,m}\varphi(\mathbf{X}) = \delta_{lk}\delta_{mj}$. Thus, evaluating (2.24) at the point $\mathbf{X}$, we have

$$0 = f(\mathbf{X})\varphi(\mathbf{X}) + \sum_{l=1}^{N}\sum_{m=1}^{3}\Lambda_{l,m}(\mathbf{X})\delta_{lk}\delta_{mj} = \Lambda_{k,j}(\mathbf{X}). \tag{2.25}$$

Because all factors $\Lambda_{k,j}$ are equal to zero, Eq. (2.24) directly implies that f is the zero function. □

Applying this lemma to the consistency condition (2.23), we obtain that the prefactors of the derivative terms have to vanish separately, which means

$$\left[\alpha_j^a, V_k\right] = 0, \quad \forall k \neq j,\ \forall a \in \{1,2,3\}. \tag{2.26}$$

This will give us the desired constraint on the matrix structures that may appear in each V_k. We note that the following matrices form a basis of the complex 4×4 matrices (for a proof see e.g. [42, p. 53ff.]) :

$$\alpha^\mu, \quad \gamma^5\alpha^\mu, \quad \gamma^\mu, \quad \gamma^5\gamma^\mu, \qquad \mu = 0,1,2,3. \tag{2.27}$$

Although the matrix V_k is a tensor product of N 4×4-matrices, we can disregard all factors of the tensor product apart from the j-th to check when the condition (2.26) can be satisfied. We can express V_k in the above basis and just compute all commutators of α^a with basis elements. The following list, where we omit the index j, results:

$$\begin{aligned}
\left[\alpha^a, \alpha^0\right] &= 0 \\
\left[\alpha^a, \alpha^b\right] &= 2\gamma^a\gamma^b = -2i\varepsilon_{abc}\gamma^5\alpha^c \\
\left[\alpha^a, \gamma^5\alpha^0\right] &= 0 \\
\left[\alpha^a, \gamma^5\alpha^b\right] &= (2-2\delta^{ab})\gamma^5\gamma^b\gamma^a = 2i\varepsilon_{abc}\alpha^c \\
\left[\alpha^a, \gamma^0\right] &= -2\gamma^a \\
\left[\alpha^a, \gamma^b\right] &= -2\delta^{ab}\gamma^0 \\
\left[\alpha^a, \gamma^5\gamma^0\right] &= -2\gamma^5\gamma^a \\
\left[\alpha^a, \gamma^5\gamma^b\right] &= -2\delta^{ab}\gamma^5\gamma^0.
\end{aligned} \tag{2.28}$$

If V_k contains combinations of $\alpha_j^0 = \mathbb{1}_j$ and γ_j^5, the commutators in (2.23) vanish. But the commutators with all other elements of the basis give non-zero and linearly independent matrices, which implies that other matrices cannot be present in V_k in order for condition (2.26) to be fulfilled. □

2.4.2 Basis decomposition

By Theorem 2.3, the consistency condition implies that V_k only depends on the spin of the j-th particle via the identity matrix or γ_j^5. Therefore, we can expand the potentials as

$$\begin{aligned}
V_1 &= \mathbb{1}_2 V_{11} + \gamma_2^5 V_{15}, \\
V_2 &= \mathbb{1}_1 V_{21} + \gamma_1^5 V_{25}.
\end{aligned} \tag{2.29}$$

In the terms V_{i1} and V_{i5}, all matrices depending on the i-th spin index may appear in principle, so we have

$$\begin{aligned}
V_{11} &= \alpha_1^\mu W_{1,\mu} + \gamma_1^5\alpha_1^\mu Y_{1,\mu} + \gamma_1^\mu A_\mu + \gamma_1^5\gamma_1^\mu B_\mu \\
V_{15} &= \alpha_1^\mu X_{1,\mu} + \gamma_1^5\alpha_1^\mu Z_{1,\mu} + \gamma_1^\mu C_\mu + \gamma_1^5\gamma_1^\mu D_\mu \\
V_{21} &= \alpha_2^\nu W_{2,\nu} + \gamma_2^5\alpha_2^\nu X_{2,\nu} + \gamma_2^\nu E_\nu + \gamma_2^5\gamma_2^\nu F_\nu \\
V_{25} &= \alpha_2^\nu Y_{2,\nu} + \gamma_2^5\alpha_2^\nu Z_{2,\nu} + \gamma_2^\nu G_\nu + \gamma_2^5\gamma_2^\nu H_\nu,
\end{aligned} \tag{2.30}$$

where $A_0, B_k, C_0, D_k, E_0, F_k, G_0, H_k, W_{i,\mu}, X_{i,\mu}, Y_{i,\mu}, Z_{i,\mu}$ are arbitrary real scalar functions and $A_k, B_0, C_k, D_0, E_k, F_0, G_k, H_0$ are arbitrary functions with purely imaginary values,

such that the potentials are self-adjoint. It will soon become understandable why this nomenclature makes sense, especially what W_1, X_1, Y_1, Z_1 have to do with W_2, X_2, Y_2, Z_2.

Lemma 2.7. *Consider a multi-time system* (2.1) *for two particles for which the assumption* ***(A)*** *of Theorem 2.3 holds. Then the potentials can be expanded as*

$$V_1 = \gamma_1^\mu A_\mu + \gamma_1^5\gamma_1^\mu B_\mu + \gamma_2^5\left(\gamma_1^\mu C_\mu + \gamma_1^5\gamma_1^\mu D_\mu\right) + V_{1,ext} \tag{2.31}$$
$$V_2 = \gamma_2^\nu E_\nu + \gamma_2^5\gamma_2^\nu F_\nu + \gamma_1^5\left(\gamma_2^\nu G_\nu + \gamma_2^5\gamma_2^\nu H_\nu\right) + V_{2,ext} \tag{2.32}$$

where $V_{i,ext}$ is not interacting and the functions A_μ to H_μ, $\mu = 0,1,2,3$, are scalars. Furthermore, the consistency condition is equivalent to the following system of equations:

$$\partial_{1,\mu}W_{2,\nu} = \partial_{2,\nu}W_{1,\mu} \tag{2.33a}$$
$$\partial_{1,\mu}X_{2,\nu} = \partial_{2,\nu}X_{1,\mu} \tag{2.33b}$$
$$\partial_{1,\mu}Y_{2,\nu} = \partial_{2,\nu}Y_{1,\mu} \tag{2.33c}$$
$$\partial_{1,\mu}Z_{2,\nu} = \partial_{2,\nu}Z_{1,\mu} \tag{2.33d}$$
$$B_\mu Y_{2,\nu} + D_\mu Z_{2,\nu} = \tfrac{i}{2}\partial_{2,\nu}A_\mu \tag{2.33e}$$
$$(m_1\delta_{0\mu} + A_\mu)Y_{2,\nu} + C_\mu Z_{2,\nu} = \tfrac{i}{2}\partial_{2,\nu}B_\mu \tag{2.33f}$$
$$-B_\mu Z_{2,\nu} - D_\mu Y_{2,\nu} = \tfrac{i}{2}\partial_{2,\nu}C_\mu \tag{2.33g}$$
$$-(m_1\delta_{0\mu} + A_\mu)Z_{2,\nu} - C_\mu Y_{2,\nu} = \tfrac{i}{2}\partial_{2,\nu}D_\mu \tag{2.33h}$$
$$F_\nu X_{1,\mu} + H_\nu Z_{1,\mu} = \tfrac{i}{2}\partial_{1,\mu}E_\nu \tag{2.33i}$$
$$(m_2\delta_{0\nu} + E_\nu)X_{1,\mu} + G_\nu Z_{1,\mu} = \tfrac{i}{2}\partial_{1,\mu}F_\nu \tag{2.33j}$$
$$-F_\nu Z_{1,\mu} - H_\nu X_{1,\mu} = \tfrac{i}{2}\partial_{1,\mu}G_\nu \tag{2.33k}$$
$$-(m_2\delta_{0\nu} + E_\nu)Z_{1,\mu} - G_\nu X_{1,\mu} = \tfrac{i}{2}\partial_{1,\mu}H_\nu \tag{2.33l}$$
$$B_\mu G_\nu = C_\mu F_\nu \tag{2.33m}$$
$$B_\mu H_\nu = C_\mu(m_2\delta_{0\nu} + E_\nu) \tag{2.33n}$$
$$(m_1\delta_{0\mu} + A_\mu)G_\nu = D_\mu F_\nu \tag{2.33o}$$
$$(m_1\delta_{0\mu} + A_\mu)H_\nu = D_\mu(m_2\delta_{0\nu} + E_\nu) \tag{2.33p}$$

Proof of Lemma 2.7: Having used Theorem 2.3 already and expanded the potentials as in (2.30), we now evaluate the missing part of the consistency condition:

$$0 \stackrel{!}{=} [V_k, V_j] + m_k\left[\gamma_k^0, V_j\right] - m_j\left[\gamma_j^0, V_k\right] - i\alpha_k^\mu\partial_{k,\mu}V_j + i\alpha_j^\mu\partial_{j,\mu}V_k \tag{2.34}$$

We have

$$\begin{aligned} m_1\left[\gamma_1^0, V_2\right] &= 2m_1\gamma_1^0\gamma_1^5V_{25}, \\ m_2\left[\gamma_2^0, V_1\right] &= 2m_2\gamma_2^0\gamma_2^5V_{15}, \end{aligned} \tag{2.35}$$

and

$$\begin{aligned} [V_1, V_2] &= \left[V_{11}, \gamma_1^5\right]V_{25} + \left[\gamma_2^5, V_{21}\right]V_{15} + \left[V_{15}\gamma_2^5, \gamma_1^5V_{25}\right] \\ &= \left(-2\gamma_1^5\gamma_1^\mu A_\mu - 2\gamma_1^\mu B_\mu\right)V_{25} \\ &\quad + \left(2\gamma_2^5\gamma_2^\nu E_\nu + 2\gamma_2^\nu F_\nu\right)V_{15} \\ &\quad + 2\alpha_1^\mu\gamma_1^5X_{1,\mu}\left(\gamma_2^5\gamma_2^\nu G_\nu + \gamma_2^\nu H_\nu\right) + 2\alpha_1^\mu Z_{1,\mu}\left(\gamma_2^5\gamma_2^\nu G_\nu + \gamma_2^\nu H_\nu\right) \\ &\quad + 2\gamma_1^\mu\gamma_1^5C_\mu\left(\gamma_2^5\alpha_2^\nu Y_{\nu,2} + \alpha_2^\nu Z_{2,\nu}\right) - 2\gamma_1^\mu D_\mu\left(\gamma_2^5\alpha_2^\nu Y_{\nu,2} + \alpha_2^\nu Z_{2,\nu}\right). \end{aligned} \tag{2.36}$$

The derivative terms are

$$-i\alpha_1^\mu \partial_{1,\mu} V_{21} - i\alpha_1^\mu \gamma_1^5 \partial_{1,\mu} V_{25} + i\alpha_2^\nu \partial_{2,\nu} V_{11} + i\alpha_2^\nu \gamma_2^5 \partial_{2,\nu} V_{15}. \tag{2.37}$$

As the 16 matrices in (2.27) are linearly independent, their tensor products give us $16^2 = 256$ linearly independent matrices that appear in the consistency condition. Their respective prefactors have to vanish separately. This gives the following table, in which every of the 16 cells stands for 16 terms (for $\mu, \nu = 0, 1, 2, 3$) that have to vanish.

$\otimes$	α_2^ν	$\gamma_2^5\alpha_2^\nu$	γ_2^ν	$\gamma_2^5\gamma_2^\nu$
α_1^μ	$-i\partial_{1,\mu}W_{2,\nu} + i\partial_{2,\nu}W_{1,\mu}$	$-i\partial_{1,\mu}X_{2,\nu} + i\partial_{2,\nu}X_{1,\mu}$	$2F_\nu X_{1,\mu} + 2H_\nu Z_{1,\mu} - i\partial_{1,\mu}E_\nu$	$(2m_2\delta_{0\nu} + 2E_\nu)X_{1,\mu} + 2G_\nu Z_{1,\mu} - i\partial_{1,\mu}F_\nu$
$\gamma_1^5\alpha_1^\mu$	$-i\partial_{1,\mu}Y_{2,\nu} + i\partial_{2,\nu}Y_{1,\mu}$	$-i\partial_{1,\mu}Z_{2,\nu} + i\partial_{2,\nu}Z_{1,\mu}$	$2F_\nu Z_{1,\mu} + 2H_\nu X_{1,\mu} + i\partial_{1,\mu}G_\nu$	$(2m_2\delta_{0\nu} + 2E_\nu)Z_{1,\mu} + 2G_\nu X_{1,\mu} + i\partial_{1,\mu}H_\nu$
γ_1^μ	$-2B_\mu Y_{2,\nu} - 2D_\mu Z_{2,\nu} + i\partial_{2,\nu}A_\mu$	$-2B_\mu Z_{2,\nu} - 2D_\mu Y_{2,\nu} - i\partial_{2,\nu}C_\mu$	$-2B_\mu G_\nu + 2C_\mu F_\nu$	$-2B_\mu H_\nu + 2E_\nu C_\mu + 2m_2 C_\mu \delta_{0\nu}$
$\gamma_1^5\gamma_1^\mu$	$-(2m_1\delta_{0\mu} + 2A_\mu)Y_{2,\nu} - 2C_\mu Z_{2,\nu} + i\partial_{2,\nu}B_\mu$	$-(2m_1\delta_{0\mu} + 2A_\mu)Z_{2,\nu} - 2C_\mu Y_{2,\nu} - i\partial_{2,\nu}D_\mu$	$-(2m_1\delta_{\mu 0} + 2A_\mu)G_\nu + 2D_\mu F_\nu$	$-(2A_\mu + 2m_1\delta_{0\mu})H_\nu + (2m_2\delta_{0\nu} + 2E_\nu)D_\mu$

Setting every entry of this table equal to zero gives the required system of equations (2.33a)–(2.33p).

It remains to show that the potentials can be expanded as in (2.31), (2.32). Let us add up equations (2.33a) to (2.33d) with the respective matrices, factorizing $\alpha_1^\mu \alpha_2^\nu$, which leads to

$$\begin{aligned} &- \partial_{1,\mu}W_{2,\nu} + \partial_{2,\nu}W_{1,\mu} + \gamma_2^5\left(-\partial_{1,\mu}X_{2,\nu} + \partial_{2,\nu}X_{1,\mu}\right) \\ &+ \gamma_1^5\left(-\partial_{1,\mu}Y_{2,\nu} + \partial_{2,\nu}Y_{1,\mu}\right) + \gamma_1^5\gamma_2^5\left(-\partial_{1,\mu}Z_{2,\nu} + \partial_{2,\nu}Z_{1,\mu}\right) = 0. \end{aligned} \tag{2.38}$$

The names we gave to the terms in the potential are suited to make the symmetry of this equation visible. Defining

$$f_{j,\mu} := W_{j,\mu} + \gamma_2^5 X_{j,\mu} + \gamma_1^5 Y_{j,\mu} + \gamma_1^5\gamma_2^5 Z_{j,\mu}, \tag{2.39}$$

equation (2.38) becomes

$$\partial_{1,\mu} f_{2,\nu} = \partial_{2,\nu} f_{1,\mu}. \tag{2.40}$$

Then, we adapt the argument of Petrat and Tumulka [3, p. 34]: Define

$$g_{j,\mu\nu} = \partial_{j,\mu} f_{j,\nu} - \partial_{j,\nu} f_{j,\mu}. \tag{2.41}$$

For $i \neq j$, we have

$$\partial_{i,\lambda} g_{j,\mu\nu} = \partial_{j,\mu}\partial_{i,\lambda} f_{j,\nu} - \partial_{j,\nu}\partial_{i,\lambda} f_{j,\mu} = \partial_{j,\mu}\partial_{j,\nu} f_{i,\lambda} - \partial_{j,\nu}\partial_{j,\mu} f_{i,\lambda} = 0. \tag{2.42}$$

This implies that $g_{j,\mu\nu}$ is a function of x_j only. Define for arbitrary fixed $\tilde{x}_1, \tilde{x}_2$ the function $\tilde{f}_{j,\mu}(x_j) := f_{j,\mu}(x_j, \tilde{x}_i)$ and $h_{j,\mu}(x_1, x_2) := f_{j,\mu}(x_1, x_2) - \tilde{f}_{j,\mu}(x_j)$. Since (2.42) implies

$$g_{j,\mu\nu} = \partial_{j,\mu} f_{j,\nu} - \partial_{j,\nu} f_{j,\mu} = \partial_{j,\mu}\tilde{f}_{j,\nu} - \partial_{j,\nu}\tilde{f}_{j,\mu}, \tag{2.43}$$

we have

$$\partial_{j,\mu} h_{j,\nu} - \partial_{j,\nu} h_{j,\mu} = 0, \quad j = 1, 2. \tag{2.44}$$

Moreover, eq. (2.40) gives us

$$\partial_{1,\mu} h_{2,\nu} - \partial_{2,\nu} h_{1,\mu} = 0. \tag{2.45}$$

These two equations together form the integrability condition, from which it follows that a self-adjoint matrix-valued function $M(x_1, x_2)$ exists such that $h_{j,\mu} = \partial_{j,\mu} M(x_1, x_2)$, i.e.

$$f_{j,\mu}(x_1, x_2) = \partial_{j,\mu} M(x_1, x_2) + \tilde{f}_{j,\mu}(x_j). \tag{2.46}$$

Therefore, the unitary map $e^{iM(x_1,x_2)}$ maps the potential f_j to the purely external potential $\tilde{f}_j$, which shows that f_j is not interacting according to our definition.
The generalization to the case where the consistency condition only holds on $\mathscr{S}^{(N)}$ works exactly like in [3, p. 35]. □

2.4.3 A consistent example

As a side remark before we prove Lemma 2.4, note that the connection of the consistent potential with the above basis decomposition is more transparent if the potential is rewritten as $V_1 = -i\gamma_1^\mu C_\mu \sin(2c_\nu x^\nu) + \gamma_1^5 \gamma_1^\mu C_\mu \cos(2c_\nu x^\nu) - m_1 \gamma_1^0$.

Proof of Lemma 2.4:
Part a): We have to evaluate the consistency condition

$$\begin{aligned} &\left[i\alpha_1^\mu \partial_{1,\mu} - m_1\gamma_1^0 - \gamma_1^\mu C_\mu \exp\left(2i\gamma_1^5 c_\lambda x^\lambda\right) + m_1\gamma_1^0, i\alpha_2^\nu \partial_{2,\nu} - m_2\gamma_2^0 - \gamma_1^5 \alpha_2^\nu c_\nu \right] = 0 \\ \iff &- \left[\gamma_1^\mu C_\mu \exp\left(2i\gamma_1^5 c_\lambda x^\lambda\right), i\alpha_2^\nu \partial_{2,\nu} \right] + \left[\gamma_1^\mu C_\mu \exp\left(2i\gamma_1^5 c_\lambda x^\lambda\right), \gamma_1^5 \alpha_2^\nu c_\nu \right] = 0 \\ \iff &\gamma_1^\mu C_\mu \alpha_2^\nu \left(i\partial_{2,\nu} \exp\left(2i\gamma_1^5 c_\lambda x^\lambda\right) + 2\gamma_1^5 c_\nu \exp\left(2i\gamma_1^5 c_\lambda x^\lambda\right)\right) = 0, \end{aligned} \tag{2.47}$$

which is indeed true. Note that in the case at hand the consistency condition is satisfied identically, not only applied to certain functions.
Part b): Now we assume (for a contradiction) that there is a gauge transformation $U(x_1, x_2) : \mathbb{C}^K \to \mathbb{C}^K$ that yields non-interacting potentials. Such a map can be written as $U(x_1, x_2) = e^{iM(x_1,x_2)}$ with a self-adjoint $K \times K$-matrix M. We define the transformed quantities

$$\tilde{\psi} := U\psi, \quad \tilde{\gamma^\mu} := U\gamma^\mu U^\dagger. \tag{2.48}$$

If ψ is a solution of the system (2.1), it follows that $\tilde{\psi}$ satisfies

$$(i\tilde{\alpha}_k^\mu \partial_{k,\mu} - \tilde{\gamma_k^0} m_k)\tilde{\psi} = \tilde{V}_k \tilde{\psi} - \tilde{\alpha}_k^\mu (\partial_{k,\mu} \tilde{M})\tilde{\psi}, \tag{2.49}$$

where $\tilde{V}$ and $\tilde{M}$ stand for the same expressions as V and M, but with all appearing matrices replaced by the ones with a tilde[1]. Therefore, the condition that the transformed potential only depends on x_k amounts to the requirement that

$$V_k(x_1, x_2) - \alpha_k^\mu \partial_{k,\mu} M(x_1, x_2) \tag{2.50}$$

[1]Since the gamma matrices are always only defined up to a similarity transformation, the tildes do not really matter and can basically be omitted. Note that a gauge transformation just refers to a (local) change of coordinates in the spinor space.

is in fact only a matrix-valued function of x_k, so its derivative with respect to another coordinate has to vanish. Using that V_2 is constant, this implies the following two equations:

$$\partial_{1,\lambda}\alpha_2^\mu\partial_{2,\mu}M(x_1,x_2)=0 \tag{2.51}$$

$$\partial_{2,\delta}\alpha_1^\nu\partial_{1,\nu}M(x_1,x_2)=c_\delta 2i\gamma_1^5\gamma_1^\mu C_\mu\exp\left(2i\gamma_1^5c_\nu x^\nu\right) \tag{2.52}$$

Now consider the contraction

$$\begin{aligned}&\alpha_1^\lambda\alpha_2^\delta\partial_{1,\lambda}\partial_{2,\delta}M(x_1,x_2)\\ =\;&\alpha_1^\lambda\left(\alpha_2^\delta\partial_{1,\lambda}\partial_{2,\delta}M(x_1,x_2)\right)=0\\ =\;&\alpha_2^\delta\left(\alpha_1^\lambda\partial_{1,\lambda}\partial_{2,\delta}M(x_1,x_2)\right)=\alpha_2^\delta c_\delta 2i\gamma_1^5\gamma_1^\mu C_\mu\exp\left(2i\gamma_1^5c_\nu x^\nu\right)\end{aligned} \tag{2.53}$$

where we have used, after different regrouping of the summands, equation (2.51) in the second line and (2.52) in the third line. This is a contradiction because the C_μ, c_μ are not all zero. Hence, a matrix M with the required properties does not exist. We have therefore proven that the potential is not gauge-equivalent to a non-interacting one, so it is interacting. □

2.4.4 Classification of consistent potentials

Instead of proving Lemma 2.5 directly, we give a slightly stronger reformulation that implies it, but uses the basis decomposition discussed in Section 2.4.2.

Lemma 2.8. *Suppose the consistency condition is fulfilled (in the sense of **(A)** in Theorem 2.3) for a two-particle Dirac system* (2.1) *for which the gauge transformation which makes W_i, X_i, Y_i, Z_i purely external has already taken place. If the potentials are translation invariant, i.e. satisfy*

$$V_i(x_1,x_2)=V_i(x_1+a,x_2+a)\quad\forall a\in\mathbb{R}^4, \tag{2.54}$$

then all terms $A_\mu,...,H_\mu$ in the potentials are necessarily of the form

$$C_1\cdot e^{c_{i,\nu}x^\nu}+C_2\cdot e^{-c_{i,\nu}x^\nu} \tag{2.55}$$

for some $C_1, C_2\in\mathbb{C}$ and $c_i\in\mathbb{C}^4$, where $x=x_1-x_2$. In the case of A_0 and E_0, a constant term $-m_1$ resp. $-m_2$ is added.

Proof of Lemma 2.8. After the gauge transformation, W_i, X_i, Y_i and Z_i are functions of x_i only. If we assume that the potentials are translation invariant, it follows that these functions have to be constants. Therefore, we can derive second order differential equations for the functions A to H. We show the steps for B_μ and D_μ, the other cases are analogous. Since $V_k\in C^1(\Omega,\mathbb{C}^{K\times K})$, every scalar function $A_\mu, B_\mu, ..., H_\mu, W_{i,\mu}, ..., Z_{i,\mu}$ in the potentials has to be continuously differentiable. Equations (2.33e) to (2.33l) imply that the terms A to H are in fact two times continuously differentiable, because the first derivatives are expressible as a sum of continuously differentiable functions.
Therefore, we may differentiate equation (2.33f) once more. Inserting (2.33e) and (2.33g), we obtain

$$\frac{1}{4}\partial_{2,\nu}\partial_{2,\lambda}B_\mu=(Z_{2,\lambda}Z_{2,\nu}-Y_{2,\lambda}Y_{2,\nu})B_\mu+(Y_{2,\nu}Z_{2,\lambda}-Y_{2,\lambda}Z_{2,\nu})D_\mu. \tag{2.56}$$

Similarly for D_μ:

$$\frac{1}{4}\partial_{2,\nu}\partial_{2,\lambda}D_\mu = (Z_{2,\lambda}Z_{2,\nu} - Y_{2,\lambda}Y_{2,\nu})D_\mu + (Y_{2,\nu}Z_{2,\lambda} - Y_{2,\lambda}Z_{2,\nu})B_\mu \tag{2.57}$$

Although the derivatives $\partial_{2,\nu}$ and $\partial_{2,\lambda}$ need to commute, the right hand side of these equations is apparently not invariant under exchange of ν and λ. This implies that

$$B_\mu = D_\mu = 0 \ \vee \ Y_{2,\nu}Z_{2,\lambda} - Y_{2,\lambda}Z_{2,\nu} = 0. \tag{2.58}$$

In the first case, we are already done (the potentials are of the desired form, with the constants being equal to zero). So we go on with the second case, where the differential equation becomes

$$\partial_{2,\nu}\partial_{2,\lambda}B_\mu = 4(Z_{2,\lambda}Z_{2,\nu} - Y_{2,\lambda}Y_{2,\nu})B_\mu, \tag{2.59}$$

and the same for D_μ. Using $Y_{2,\nu}Z_{2,\lambda} = Y_{2,\lambda}Z_{2,\nu}$, it can be rewritten as

$$\partial_{2,\nu}\partial_{2,\lambda}B_\mu = 2\sqrt{Z_{2,\nu}^2 - Y_{2,\nu}^2}\cdot 2\sqrt{Z_{2,\lambda}^2 - Y_{2,\lambda}^2}\cdot B_\mu. \tag{2.60}$$

The square root is also defined for negative radicand as $\sqrt{x} := i\sqrt{|x|}$. This has the general solution

$$B_\mu = C_\mu^+ \exp\left(2\sqrt{Z_{2,\alpha}^2 - Y_{2,\alpha}^2}x_2^\alpha\right) + C_\mu^- \exp\left(-2\sqrt{Z_{2,\alpha}^2 - Y_{2,\alpha}^2}x_2^\alpha\right), \tag{2.61}$$

with free constants $C_\mu^\pm$ that may depend on x_1. Since the potential must be translation independent, the constants must be such that B_μ has the form (2.55).
We thus have the required form for B and D, and the other terms work analogously. In the case of A and E, one should derive the differential equations for the functions $(m_1\delta_{0\mu} + A_\mu)$ and $(m_2\delta_{0\nu} + E_\nu)$ instead. Then, the consistency condition poses several additional constraints, eqs. (2.33m)–(2.33p) amongst others, that were not considered so far. But we will not elucidate on that because we only want to show that the form (2.55) is *necessary*. □

CHAPTER
THREE

RIGOROUS FORMULATION OF THE QED MODEL BY DIRAC, FOCK, AND PODOLSKY

> *Dirac, Fock und Podolsky haben vor einiger Zeit die Quantenelektrodynamik in einer eleganten Weise formuliert, die gegenüber früheren Darstellungen vor allem den grossen Vorzug hat, die relativistische Invarianz der Theorie durch die Invarianz der Gleichungen selbst zum Ausdruck zu bringen.* – Felix Bloch, 1934 [14] [1].

The results of this chapter will be published in an article currently under preparation by Dirk-A. Deckert and the present author.
Having clarified that interaction potentials are not compatible with multi-time systems, we now turn to another idea, namely interaction of fermions through a second-quantized field. In this chapter, we rigorously treat a model of N interacting electrons that marked the birth of quantum electrodynamics in the 1930s. Remarkably, in their formulation of the model [4], Dirac, Fock and Podolsky take the concept of a multi-time wave function more or less for granted and plainly write

> This obvious relativistic invariance is achieved by the introduction of [a] separate time variable for each particle.

Our focus in this chapter is to show how their ideas can be cast into a mathematically rigorous form and that the model is well-posed, i.e. solutions exist and are unique. We give an informal introduction to the model in the following Section 3.1, the mathematical definition of the model is then given in Sec. 3.2. The main theorem about existence and uniqueness of solutions to the model formulated at the end of that section is derived in the proof Section 3.3. We put the results in perspective in Sec. 3.4.

3.1 The multi-time QED model

In a system of N Hamiltonian equations

$$i\partial_{t_j}\psi(x_1,...,x_N) = \mathcal{H}_j\psi(x_1,...,x_N), \quad j=1,...,N, \tag{3.1}$$

[1] Translation to English: *Some time ago, Dirac, Fock, and Podolsky have formulated quantum electrodynamics in an elegant way that has the advantage in comparison to former presentations, that the relativistic invariance of the theory is expressed through the invariance of the equations themselves.*

with a suitable partial Hamiltonian $\mathcal{H}_j$ for each particle, the consistency condition (1.20) poses a serious obstacle for the introduction of a non-trivial interaction. We discussed in the previous chapter that interaction potentials make the system inconsistent. Already in 1932, Paul Dirac pointed out a smart way to circumvent this problem [1]: by second quantization. He observed that in case the "potential" is not a multiplication operator, but a field operator on Fock space, the consistency condition can be retained although interaction is present. In order to neglect complications connected to the gauge freedom of the electromagnetic interaction, we explain the basic idea for a scalar field φ only. The Hamiltonians in question are

$$\mathcal{H}_j = \mathcal{H}_j^0 + \varphi(t_j, \mathbf{x}_j), \tag{3.2}$$

all containing one and the same second quantized scalar field φ on space-time $\mathbb{R}^4$, fulfilling

$$\Box\varphi(x) = \left(\partial_t^2 - \triangle\right)\varphi(t, \mathbf{x}) = 0, \tag{3.3}$$

as well as the canonical commutation relation

$$[\varphi(x_j), \varphi(x_k)] = i\Delta(x_j, x_k), \tag{3.4}$$

with Δ being the Pauli-Jordan function [15, 16] (see also (3.70)). It is well-known that (3.4) implies

$$[\varphi(x_j), \dot{\varphi}(x_k)]_{t_j=t_k} = i\delta^{(3)}(\mathbf{x}_j - \mathbf{x}_k). \tag{3.5}$$

This ensures the consistency of the system of equations since

$$\Delta(x_j, x_k) = 0 \quad \text{if } x_j, x_k \text{ are space-like related.} \tag{3.6}$$

The multi-time wave-function $\psi(x_1, ..., x_N)$ now takes values in a bosonic Fock space. This way, we can also explain how the seemingly "free" equations in fact mediate interaction. All particles create and annihilate field modes and even though those travel only at the speed of light, leading to (3.6), they will "reach" and influence the other particles at later times. The fact that

$$[\varphi(x_j), \varphi(x_k)] \neq 0 \quad \text{if } x_j, x_k \text{ are not space-like related} \tag{3.7}$$

leads to an informal equation derived (for the Maxwellian case) by Dirac, Fock and Podolsky in [4]. For the field operator $\varphi_H(t, \mathbf{x}) = U(t)^\dagger\varphi(0, \mathbf{x})U(t)$ in the Heisenberg picture, where $U(t)$ is the time evolution operator for the single-time model of N particles, one obtains informally

$$\Box\varphi_H(t, \mathbf{x}) = -\sum_{j=1}^{N} \delta^{(3)}\left(\hat{\mathbf{x}}_j(t) - \mathbf{x}\right), \tag{3.8}$$

a wave equation with source term, which demonstrates the arising interaction. A rigorous version for our model is derived in section 3.3.5. In view of introducing relativistic interaction, the paper by Dirac, Fock and Podolsky [4] was probably one of the most important contributions to the then young theory of quantum electrodynamics. In particular, Tomonaga and Schwinger based their work on Dirac's observation and generalized it to a varying number of particles in their formulation with wave-functions on space-like hypersurfaces [15, 16]. This laid the foundation for the relativistic S-matrix formalism of QED. The purpose of this chapter is to study Dirac's idea in mathematically rigorous terms in order to provide a result on existence and uniqueness of solutions.

Differences to the original proposal by Dirac, Fock, Podolsky A problem that all those QED-type models share are the ultraviolet divergences. In our treatment with multi-time wave functions, we are able to use a cut-off on the fields to avoid these difficulties, although it comes at the price of complicating the model, as explained in section 3.2, and breaking Lorentz invariance at an arbitrarily small, but finite spatial scale δ.
There is a further difference to the original formulation of Dirac, Fock, Podolsky, namely that the multi-time wave function ψ of N particles has N time arguments and not an additional "field time" argument. This is because we formulate the field degrees of freedom in momentum space and in the Dyson picture, leading to a time-dependent $\varphi(t, \mathbf{x})$ but no free field Hamiltonian in $\mathcal{H}_j$. The choice of a field time as in [4] corresponds to choosing a space-like hypersurface Σ (in that paper, only equal-time hypersurfaces Σ_t are considered) on which the field degrees of freedom are evaluated. Our formulation is mathematically convenient since the Hilbert space is fixed and not hypersurface-dependent. It is always possible to choose a hypersurface and perform the Fourier transformation to obtain field modes in position space.

Mathematical challenges There are three main difficulties we have to overcome for a mathematical solution theory of the model.

(a) The introduction of the cut-off, which can be thought of as considering small balls of diameter δ instead of point-like particles, makes the model well-defined, but not fully Lorentz covariant anymore. It furthermore forces us to take the domain $\mathscr{S}_\delta$, defined in (3.17), for the multi-time wave function, which is not an open set in $\mathbb{R}^{4N}$ and therefore does not allow for a simple notion of differentiability.

(b) We need to coalesce the functional analytic treatment of the Hamiltonian equations with the multi-time system with cut-offs, which requires a point-wise evaluation of functions. It is decisive for our proofs that we find a dense set $\mathscr{D}$ of smooth functions which is left invariant by the time evolution of each particle. The usual methods employed in the literature on Schrödinger Hamiltonians (see e.g. [43]) rely on boundedness from below, so they do not work here. A further difficulty is that the domain $\mathrm{dom}(\mathcal{H}_j(t))$ is not explicitly known and we do not even know if it depends on t.
One could also consider a different free Hamiltonian which is bounded from below, like the "pseudo-relativistic" $\sqrt{\mathbf{p}^2 + m^2}$, but this does not have finite propagation speed any more [44], which is indispensable for the consistency of the multi-time equations.

(c) Since we add unbounded and time-dependent interaction terms to the Dirac Hamiltonians, even the study of the single-time equations and self-adjointness of $\mathcal{H}_j(t)$ from (3.2) is not too easy. The usual abstract theorems of Kato [34] or Yosida [35, ch. XIV] about the existence of a propagator $U(t, s)$ require time-independence of the domain $\mathrm{dom}(\mathcal{H}_j(t))$. In our case, an explicit transformation which takes care of the time-dependence of the Hamiltonians can be found.

3.2 Definition of the model and main results

We now introduce the model described by the informal equations (3.1), (3.2), (3.4) in a mathematical context and formulate a solution sense for our existence and uniqueness results. The model describes the interaction of N electrons with a scalar field, an operator on Fock space, so there are two main ingredients we need to define: the field operator and the multi-time evolution equations.

Field operator with Cut-off We follow the standard quantization procedure. The Fock space is constructed with the symmetric tensor product, denoted by $\odot$,

$$\mathscr{F} = \bigoplus_{n=0}^{\infty} L^2(\mathbb{R}^3, \mathbb{C})^{\odot n}. \tag{3.9}$$

We think of $\mathbb{R}^3$ as momentum space in this definition. The total Hilbert space, in which the wave function $\psi(t_1, \cdot, ..., t_N, \cdot)$ is contained for fixed times, is

$$\mathscr{H} = L^2(\mathbb{R}^{3N}, \mathscr{F}^K) \cong L^2(\mathbb{R}^{3N}, \mathbb{C}^K) \otimes \mathscr{F} \cong L^2(\mathbb{R}^{3N}, \mathbb{C}) \otimes \mathscr{F}^K, \tag{3.10}$$

with $K = 4^N$ the dimension of spinor space. We will frequently use the identification of the isomorphic Hilbert spaces, for example in the regularity arguments in which we consider Sobolev spaces of the form $H^j(\mathbb{R}^{3N}, \mathbb{C}^K) \otimes \mathscr{F}$. This also allows to write

$$\begin{aligned} &\text{for a.e. } (x_1, ..., x_N) : \psi(x_1, ..., x_N) = \left(\psi^{(n)}(x_1, ..., x_N)\right)_{n \in \mathbb{N}_0}, \\ &\text{so that } \left((\mathbf{k}_1, ..., \mathbf{k}_n) \mapsto \psi^{(n)}(x_1, ..., x_N; \mathbf{k}_1, ..., \mathbf{k}_n)\right) \in \mathbb{C}^K \otimes L^2(\mathbb{R}^3, \mathbb{C})^{\odot n}. \end{aligned} \tag{3.11}$$

A dense set in $\mathscr{F}$ are the finite particle vectors $\mathscr{F}_{fin}$. On this set, we can define for square integrable f, as in Nelson's paper [45], the annihilation

$$\left(\int d^3\mathbf{k}\ f(\mathbf{k}) a(\mathbf{k}) \psi\right)^{(n)} (\mathbf{k}_1, ..., \mathbf{k}_n) = \sqrt{n+1} \int d^3\mathbf{k}\ f(\mathbf{k}) \psi^{(n+1)}(\mathbf{k}, \mathbf{k}_1, ..., \mathbf{k}_n) \tag{3.12}$$

and creation operators

$$\left(\int d^3\mathbf{k}\ f(\mathbf{k}) a^\dagger(\mathbf{k}) \psi\right)^{(n)} (\mathbf{k}_1, ..., \mathbf{k}_n) = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} f(\mathbf{k}_j) \psi^{(n-1)}(\mathbf{k}_1, ..., \widehat{\mathbf{k}_j}, ..., \mathbf{k}_n), \tag{3.13}$$

where a variable with hat is omitted. The field mass is $\mu \geq 0$ and the energy $\omega(\mathbf{k}) = \sqrt{\mathbf{k}^2 + \mu^2}$, which allows to define the free field Hamiltonian

$$\left(\mathcal{H}_f \psi\right)^{(n)} (\mathbf{k}_1, ...\mathbf{k}_n) = \sum_{j=1}^{n} \omega(\mathbf{k_j}) \psi(\mathbf{k}_1, ..., \mathbf{k}_n), \tag{3.14}$$

as self-adjoint operator on its domain $\mathrm{dom}(\mathcal{H}_f) \subset \mathscr{F}$; see [46]. One often encounters the expression $\mathcal{H}_f = \int_{\mathbb{R}^3} d^3\mathbf{k}\ \omega(\mathbf{k}) a^\dagger(\mathbf{k}) a(\mathbf{k})$ that is valid in the sense of quadratic forms. We will later use the notation $\mathrm{dom}(\mathcal{H}_f^\infty) := \bigcap_{j=0}^{\infty} \mathrm{dom}(\mathcal{H}_f^j)$.
For the scalar field, the final ingredient is the cut-off. Let $B_r(\mathbf{x})$ denote the open ball in $\mathbb{R}^3$ of radius r around $\mathbf{x}$. We use a real-valued bump function

$$\rho \in C_c^\infty(\mathbb{R}^3, \mathbb{R}) \text{ such that } \mathrm{supp}\,(\rho) \subset B_{\delta/2}(\mathbf{0}) \tag{3.15}$$

which has the direct physical interpretation of treating the particles as small balls of diameter δ. The Fourier transform $\hat{\rho}(\mathbf{k})$ is an element of the Schwartz space (but does not have compact support). For each particle index $j = 1, ..., N$, define the time-dependent scalar field

$$\varphi_j(t)\psi := \int d^3\mathbf{k}\ \left[\left(\frac{\hat{\rho}(\mathbf{k})}{\sqrt{\omega(\mathbf{k})}} e^{-i\omega(\mathbf{k})t} e^{i\mathbf{k}\cdot\hat{\mathbf{x}}_j} a(\mathbf{k}) + \frac{\hat{\rho}^\dagger(\mathbf{k})}{\sqrt{\omega(\mathbf{k})}} e^{i\omega(\mathbf{k})t} e^{-i\mathbf{k}\cdot\hat{\mathbf{x}}_j} a^\dagger(\mathbf{k})\right)\psi\right] \tag{3.16}$$

Here, $\hat{\mathbf{x}}_j$ is the position operator of the j-th particle which acts on a multi-time wave function by $\hat{\mathbf{x}}_j \psi(t_1, \mathbf{x}_1, ..., t_j, \mathbf{x}_j, ...) = \mathbf{x}_j \psi(t_1, \mathbf{x}_1, ..., t_j, \mathbf{x}_j, ...)$. If we had $\hat{\rho} \equiv 1$, the domain of the second summand in $\varphi_j(t)$ would be $\{0\}$, a manifestation of the ultraviolet problem. With a square integrable $\hat{\rho}$, the field operator is self-adjoint on a dense domain; see [46]. An equivalent definition is possible by direct fiber integrals, see [47, 48]. Despite the notation, one should not think of the φ_j as being N different fields, the index just denotes in a brief way that the single scalar field is evaluated at the coordinates of particles j, i.e. at $\mathbf{x}_j$.

Multi-Time Evolution Equations The domain in configuration space-time are those events which are at equal times or have a space-like distance of at least δ, i.e.

$$\mathscr{S}_\delta := \left\{ (x_1, ..., x_N) \in \mathbb{R}^{4N} \middle| \forall j \neq k : t_j = t_k \text{ or } \|\mathbf{x}_j - \mathbf{x}_k\| > |t_j - t_k| + \delta \right\}. \tag{3.17}$$

The multi-time wave function is represented as a map $\psi : \mathscr{S}_\delta \to \mathscr{F}^K$. Let $\mathcal{H}_j^0 = -i\gamma_j^0 \boldsymbol{\gamma}_j \cdot \boldsymbol{\nabla}_j + \gamma_j^0 m$ be the free Dirac operator acting on particle j, with the usual gamma matrices γ_j^μ. The Hamiltonians are given by

$$\mathcal{H}_j(t) = \mathcal{H}_j^0 + \varphi_j(t), \quad j = 1, ..., N. \tag{3.18}$$

To make sense of the evolution equations of the form (3.1) containing those Hamiltonians, we need the following further considerations.

Notion of solution The direct notion of a solution to our multi-time system would be a smooth function mapping from the space-like configurations $\mathscr{S}_\delta$ to the Fock space $\mathscr{F}^K$. However, the above introduced Hilbert space $\mathscr{H}$ simplifies the mathematical analysis considerably and it is helpful to define a solution as a map $\psi : \mathbb{R}^N \to \mathscr{H}, (t_1, ..., t_N) \mapsto \psi(t_1, ..., t_N)$, at first. This ψ is then required to solve the system (3.1) in $\mathscr{S}_\delta$ in an appropriate sense defined below.

We now treat the further difficulty that the domain $\mathscr{S}_\delta$ is not an open set in $\mathbb{R}^{4N}$, so that partial derivatives with respect to time coordinates cannot be straightforwardly defined in this set.

We adapt a method to define partial derivatives in $\mathscr{S}_\delta$ that was also employed by Petrat and Tumulka [3, sec. 4]. If all times are different, the usual partial derivatives exist, but not at points where for some $j \neq k$, $t_j = t_k$ while $\|\mathbf{x}_j - \mathbf{x}_k\| \leq \delta$. The idea is to only take the derivative with respect to the common time coordinate in that case. We implement it as follows: Each point $x = (x_1, ..., x_N) \in \mathscr{S}_\delta$ defines a partition of $\{1, ..., N\}$ into non-empty disjoint subsets $P_1, ..., P_L$ by the equivalence relation that is the transitive closure of the relation that holds between j and k exactly if[2] $\|\mathbf{x}_j - \mathbf{x}_k\| \leq |t_j - t_k| + \delta$. We call this the *corresponding partition* to x. By (3.17), all particles in one set P_i of the partition necessarily have the same time coordinate, i.e. $\forall i \in \{1, ..., L\}\ \forall j, k \in P_i$, we have $t_j = t_k$. We write this common time coordinate as t_{P_i} for each $i = 1, ..., L$.

The partial derivative with respect to t_{P_i} can now be defined for a differentiable function $\psi : \mathbb{R}^N \to \mathscr{H}$ as

$$\left(\frac{\partial}{\partial t_{P_i}} \psi(t_1, ..., t_N)\right)(\mathbf{x}_1, ..., \mathbf{x}_N) := \sum_{j \in P_i} \left(\frac{\partial}{\partial t_j} \psi(t_1, ..., t_N)\right)(\mathbf{x}_1, ..., \mathbf{x}_N), \tag{3.19}$$

provided that the expression on the right-hand side is well-defined. By this definition, $\frac{\partial}{\partial t_{P_i}} \psi$ can be obtained solely by limits inside $\mathscr{S}_\delta$, so changing the function ψ outside of the

[2] This gives exactly the partition called FP_{q^4} by Petrat and Tumulka.

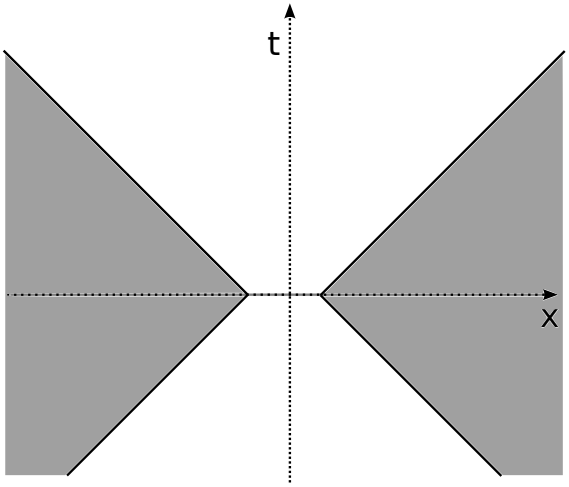

Figure 3.1: The set $\mathscr{S}_\delta$ is depicted in grey, for two particles in relative coordinates. Because of the line at $t = t_1 - t_2 = 0$, this is obviously not an open set in configuration space-time. At the origin, for example, the partial derivative ∂_{t_1} cannot be computed inside the set.

relevant domain $\mathscr{S}_\delta$ will not matter for the derivative, and thus also not for its status of being a solution.

Definition: (Notion of solution.) For each set $A \subset \{1, ..., N\}$, define the respective Hamiltonian

$$\mathcal{H}_A(t) := \sum_{j \in A} \left(\mathcal{H}_j^0 + \varphi_j(t) \right). \tag{3.20}$$

A *solution of the multi-time system* is a function $\psi : \mathbb{R}^N \to \mathscr{H}, (t_1, ..., t_N) \mapsto \psi(t_1, ..., t_N)$ such that the following hold:

i) ψ is differentiable.

ii) Pointwise evaluation: For every $(t_1, \mathbf{x}_1, ..., t_N, \mathbf{x}_N) \in \mathscr{S}_\delta$, and for all $j = 1, ..., N$, the following pointwise evaluations are well-defined:

$$\begin{aligned} \Big(\psi(t_1, ..., t_N)\Big)(\mathbf{x}_1, ..., \mathbf{x}_N), \\ \left(\partial_{t_j}\psi(t_1, ..., t_N)\right)(\mathbf{x}_1, ..., \mathbf{x}_N), \\ \Big(\mathcal{H}_j(t_j)\psi(t_1, ..., t_N)\Big)(\mathbf{x}_1, ..., \mathbf{x}_N). \end{aligned} \tag{3.21}$$

iii) Time derivatives: For every $x = (t_1, \mathbf{x}_1, ..., t_N, \mathbf{x}_N) \in \mathscr{S}_\delta$ with corresponding partition $P_1, ..., P_L$, the equations

$$\left(\frac{\partial}{\partial t_{P_j}}\psi(t_1, ..., t_N)\right)(\mathbf{x}_1, ..., \mathbf{x}_N) = \Big(\mathcal{H}_{P_j}(t_{P_j})\psi\left(t_1, ..., t_N\right)\Big)(\mathbf{x}_1, ..., \mathbf{x}_N), \quad j = 1, ..., L, \tag{3.22}$$

where the left hand side is defined by (3.19), are satisfied.

Due to the unfamiliar structure of the domain of definition $\mathscr{S}_\delta$ and our compact notation, this definition may look complicated at first sight. But it is only a rigorous way of restricting

the system (3.1) to those time directions in which taking the derivative is admissible in $\mathscr{S}_\delta$. It is helpful to look at Eq. (3.26) which shows the explicit form of the multi-time system for $N = 2$. We emphasize that with our notation in (3.20), the index of the Hamiltonian is actually a set, for example $\mathcal{H}_{\{1,2\}} = \mathcal{H}_1 + \mathcal{H}_2$ denoting the mutual Hamiltonian of particles 1 and 2. Now we can formulate our main results on existence and uniqueness of solutions, using the set of regular functions

$$\mathscr{D} := C_c^\infty(\mathbb{R}^{3N}, \mathbb{C}^K) \otimes \mathscr{F} \cap L^2(\mathbb{R}^{3N}, \mathbb{C}^K) \otimes \mathrm{dom}(\mathcal{H}_f^\infty). \tag{3.23}$$

Theorem 3.1. ***(Existence.)*** *Let $\psi^0 \in \mathscr{D}$. Then there is a solution ψ of the multi-time system in the sense of the above definition which satisfies $\psi(0, ..., 0) = \psi^0$ pointwise. In particular, there is such a solution ψ with*

$$\psi(t_1, \cdot, ..., t_N, \cdot) \in \mathscr{D}. \tag{3.24}$$

Theorem 3.2. ***(Uniqueness.)*** *Let $\psi^0 \in \mathscr{D}$. Let ψ_1 and ψ_2 be two solutions of the multi-time system in the sense of the above definition which both satisfy $\psi_k(0, ..., 0) = \psi^0$ pointwise for $k = 1, 2$. Then we have for all $(t_1, \mathbf{x}_1, ..., t_N, \mathbf{x}_N) \in \mathscr{S}_\delta$*

$$\Big(\psi_1(t_1, ..., t_N)\Big)(\mathbf{x}_1, ..., \mathbf{x}_N) = \Big(\psi_2(t_1, ..., t_N)\Big)(\mathbf{x}_1, ..., \mathbf{x}_N). \tag{3.25}$$

These main results will be proven in Sec. 3.3. We explain the concepts behind the proof of Theorem 3.1 in Sec. 3.3.1, then two main steps are performed: In Sec. 3.3.2, we construct the time evolution operators $U_A(t, s)$ for each single Hamiltonian $\mathcal{H}_A(t)$ and find an invariant dense subset $\mathscr{D}$, then we stitch together those operators for the multi-time evolution in Sec. 3.3.3. Uniqueness, i.e. Theorem 3.2, is treated separately in 3.3.4.
Finally, in Section 3.3.5, we demonstrate the presence of interaction in the considered model by deriving Eq. (3.116), a rigorous analogue of Eq. (3.8).

Remark: Multi-time wave functions are used for a manifestly Lorentz invariant formulation of quantum mechanics, so one might be worried about the apparent separation of time and space variables in our solution sense. But one should think about this only as a detour necessary for the mathematical proofs, the physical multi-time wave function is still $\psi(t_1, \mathbf{x}_1, ..., t_N, \mathbf{x}_N)$ on configuration space-time. Theorems 3.1 and 3.2 entail that a unique classical solution, a smooth function $\psi : \mathscr{S}_\delta \to \mathscr{F}^K$ exists *provided* that we can additionally prove that our solution is smooth in time directions not only in the sense of the Hilbert space $\mathscr{H}$, but pointwise. This is a hard exercise in functional analysis left for future work.

3.3 Proofs

3.3.1 Concept of the existence proof

Before treating the general case in the following sections, our methods are explained at the example of $N = 2$. In that case, we can demonstrate the concepts behind the existence proof without too many complications. The additional difficulties in the N particle case

are to some extent of notational nature.
We are looking for a pointwise evaluable solution $\psi : \mathbb{R}^2 \to \mathscr{H}$ to the system

$$\left.\begin{aligned}\Big(i\partial_{t_1}\psi(t_1,t_2)\Big)(\mathbf{x}_1,\mathbf{x}_2) &= \Big(\mathcal{H}_1(t_1)\psi(t_1,t_2)\Big)(\mathbf{x}_1,\mathbf{x}_2)\\ \Big(i\partial_{t_2}\psi(t_1,t_2)\Big)(\mathbf{x}_1,\mathbf{x}_2) &= \Big(\mathcal{H}_2(t_2)\psi(t_1,t_2)\Big)(\mathbf{x}_1,\mathbf{x}_2)\end{aligned}\right\} \text{ if } \|\mathbf{x}_1-\mathbf{x}_2\| > \delta + |t_1-t_2|,$$
$$\Big(i\partial_t\psi(t,t)\Big)(\mathbf{x}_1,\mathbf{x}_2) \;= \Big(\mathcal{H}_{\{1,2\}}(t)\psi(t,t)\Big)(\mathbf{x}_1,\mathbf{x}_2) \quad \text{if } t_1 = t_2 = t, \tag{3.26}$$

where $\mathcal{H}_{\{1,2\}} = \mathcal{H}_1 + \mathcal{H}_2$. Note that there is a little bit of redundancy in this system, since the second case is implied by the first if $t_1 = t_2$ and $\|\mathbf{x}_1 - \mathbf{x}_2\| > \delta + |t_1 - t_2|$. The relevance of the second case comes from the points where the times are equal, but the particles have smaller distance than δ, i.e. the line in figure 3.1.
The first step is to show that evolution operators $U_{\{1\}}, U_{\{2\}}, U_{\{1,2\}}$, one for each of the single equations in (3.26), exist. They should satisfy the usual properties of two-parameter propagators and, for all ψ in a suitable domain,

$$i\frac{\partial}{\partial t}U_A(t,s)\psi = \mathcal{H}_A(t)U_A(t,s)\psi, \quad A \in \{\{1\},\{2\},\{1,2\}\}. \tag{3.27}$$

A useful property of U_A, as common for Dirac propagators, is that the support of a wave function only grows into its future lightcone during the time evolution. The further necessary ingredient is the invariance of smooth functions under the time evolutions. This will be established by commutator theorems following Huang [49].
In the second step, a candidate for the solution can directly be constructed with the help of the evolution operators U_A. Given smooth initial values ψ^0 at $t_1 = t_2 = 0$, define

$$\psi(t_1,t_2) = U_{\{1\}}(t_1,t_2)U_{\{1,2\}}(t_2,0)\psi^0 \tag{3.28}$$

The idea is: First evolve both particles simultaneously up to time t_2 and then only evolve the first particle to t_1. If more times are added, we need to order them increasingly such that we do not "move back and forth" in the time coordinates. It is necessary, as mentioned above, to prove that the U_A operators keep functions sufficiently regular to be able to define ψ in a pointwise sense and obtain a differentiable function. By definition, $i\partial_{t_1}\psi(t_1,t_2) = \mathcal{H}_1(t_1)\psi(t_1,t_2)$ holds. If both times are equal, the equation $i\partial_t\psi(t,t) = \mathcal{H}_{\{1,2\}}(t)\psi(t,t)$ is also fulfilled. For the derivative with respect to t_2, one has

$$\Big(i\partial_{t_2}\psi(t_1,t_2)\Big)(\mathbf{x}_1,\mathbf{x}_2) = \Big(U_{\{1\}}(t_1,t_2)\mathcal{H}_2(t_2)U_{\{1,2\}}(t_2,0)\psi^0\Big)(\mathbf{x}_1,\mathbf{x}_2). \tag{3.29}$$

To show that ψ solves the multi-time equations, U_1 and $\mathcal{H}_2$ have to commute on the configurations with minimal space-like distance δ. By taking another derivative, and after treating some difficulties that originate in the unboundedness of $\mathcal{H}_2(t_2)$, we will be able to reduce this to the consistency condition

$$\left([\mathcal{H}_1(t_1),\mathcal{H}_2(t_2)]\,\psi(t_1,t_2)\right)(\mathbf{x}_1,\mathbf{x}_2) = 0. \tag{3.30}$$

The crucial ingredients in this step are that the commutators vanish at configurations inside our domain of definition $\mathscr{S}_\delta$, and the growth of supports with at most the speed of light.

3.3.2 Dynamics of the single-time equations

In this section, we consider a fixed set $A \subset \{1, ..., N\}$ with the respective Hamiltonian $\mathcal{H}_A(t)$ defined in (3.20) and construct a corresponding time evolution operator $U_A(t, s)$. This is contained in the following proposition, which uses the subsequent Lemmas 3.4 and 3.5. The subsection continues with important properties of the operator $U_A(t, s)$, namely the spreading of data with at most the speed of light (Lemma 3.6) and the invariance of certain smooth functions (Lemma 3.8, Corollary 3.9), namely those in the important set $\mathscr{D}$ defined in (3.23). We denote the identity map by $\mathbb{1}$.

Proposition 3.3. *There exists a unique two-parameter family of unitary operators $U_A(t, s) : \mathscr{H} \to \mathscr{H}$ with the properties that for all $t, s, r \in \mathbb{R}$,*

(a) $U_A(t, t) = \mathbb{1}$,

(b) $U_A(t, r) = U_A(t, s)U_A(s, r)$,

(c) If $\psi \in \mathscr{D}$, then $\frac{\partial}{\partial t} U_A(t, s)\psi\big|_{t=s} = -i\mathcal{H}_A(s)\psi$.

Remark: The third property in the proposition is slightly weaker than in the common case of time-independent Hamiltonians, where one can prove that the derivative exists for all functions in the domain of the Hamiltonian. But in our case, since we do not know how to prove that $\mathrm{dom}(\mathcal{H}(t))$ is independent of t, there is little hope to do more than prove things on a nice, common dense domain like $\mathscr{D}$.

Proof. We first prove the *existence* of U_A. Consider for a fixed $s \in \mathbb{R}$ the time-independent Hamiltonian

$$\widetilde{\mathcal{H}}_{A,s} := \mathcal{H}_f + \sum_{j \in A} \left(\mathcal{H}_j^0 + \varphi_j(s)\right). \tag{3.31}$$

It is proven below in Lemma 3.4 that this Hamiltonian is essentially self-adjoint on the dense domain $\mathscr{D}$. Therefore, there is a strongly continuous unitary one-parameter group $\widetilde{U}_{A,s}$ with the property that if $\psi \in \mathrm{dom}(\widetilde{\mathcal{H}}_{A,s})$, then $\frac{\partial}{\partial t}\widetilde{U}_{A,s}(t)\psi = -i\widetilde{\mathcal{H}}_{A,s}\psi$. We can transform back to the Hamiltonian without tilde by setting

$$U_A(t, s) := e^{i\mathcal{H}_f(t-s)}\widetilde{U}_{A,s}(t - s) \quad \forall t, s \in \mathbb{R}. \tag{3.32}$$

We have to check that the such defined two-parameter family of unitary operators satisfies the properties listed in the theorem.

(a) For all $t \in \mathbb{R}$, $U_A(t, t) = \mathbb{1}$ follows immediately by $\widetilde{U}_{A,s}(0) = \mathbb{1}$.

(b) We compute for any $t, s, r \in \mathbb{R}$,

$$\begin{aligned} U_A(t, s)U_A(s, r) &= e^{i\mathcal{H}_f(t-s)}\widetilde{U}_{A,s}(t - s)e^{i\mathcal{H}_f(s-r)}\widetilde{U}_{A,r}(s - r) \\ &= e^{i\mathcal{H}_f(t-r)} \underbrace{e^{i\mathcal{H}_f(r-s)}\widetilde{U}_{A,s}(t - s)e^{i\mathcal{H}_f(s-r)}}_{=\widetilde{U}_{A,r}(t-s) \text{ by Lemma 3.5, part 2}} \widetilde{U}_{A,r}(s - r) \\ &= e^{i\mathcal{H}_f(t-r)}\widetilde{U}_{A,r}(t - s)\widetilde{U}_{A,r}(s - r) \\ &= U_A(t, r). \end{aligned} \tag{3.33}$$

(c) Let $\psi \in \mathscr{D}$ and $t, s \in \mathbb{R}$, then also $\psi \in \mathrm{dom}(\mathcal{H}_f) \cap \mathrm{dom}(\widetilde{\mathcal{H}}_{A,s})$, and

$$\begin{aligned} & i\partial_t U_A(t,s)\psi(s)|_{t=s} \\ &= \Big[-\mathcal{H}_f e^{i\mathcal{H}_f(t-s)}\widetilde{U}_{A,s}(t-s)\psi(s) + e^{i\mathcal{H}_f(t-s)}\widetilde{\mathcal{H}}_{A,s}\widetilde{U}_{A,s}(t-s)\psi(s)\Big]_{t=s} \\ &= \Big[-\mathcal{H}_f\psi(t) + e^{i\mathcal{H}_f(t-s)}\widetilde{\mathcal{H}}_{A,s}e^{-i\mathcal{H}_f(t-s)}\psi(t)\Big]_{t=s} \\ &= \Big[-\mathcal{H}_f\psi(t) + \widetilde{\mathcal{H}}_{A,t}\psi(t)\Big]_{t=s} = \mathcal{H}_A(s)\psi(s), \end{aligned} \tag{3.34}$$

where we used in the last line the statement of Lemma 3.5, part 1. This establishes the third property and hence existence.

We now prove *uniqueness* of U_A. Assume there are two families $U_A(t,s)$ and $U'_A(t,s)$ with all required properties. Pick some $\psi^0 \in \mathscr{D}$, then $\psi(t) := U_A(t,0)\psi^0$ and $\psi'(t) := U'_A(t,0)\psi^0$ are differentiable w.r.t to t by the invariance of $\mathscr{D}$ (Corollary 3.9). By linearity, also $w(t) := \psi(t) - \psi'(t)$ satisfies the differential equation $i\partial_t w(t) = \mathcal{H}_A(t)w(t)$. Note that $w(0) = 0$. Because $\mathcal{H}_A(t)$ is self-adjoint for all times, the norm is preserved during time evolution:

$$i\partial_t \langle w(t), w(t)\rangle = -\langle \mathcal{H}_A(t)w(t), w(t)\rangle + \langle w(t), \mathcal{H}_A(t)w(t)\rangle = 0 \tag{3.35}$$

Therefore, also $w(t)$ must have norm zero, so $\psi(t) = \psi'(t)\ \forall t \in \mathbb{R}$, which proves that the families $U_A(t,s)$ and $U'_A(t,s)$ are in fact identical. □

We have used the statements of the following two lemmas:

Lemma 3.4. *Let $t, s \in \mathbb{R}$. The Hamiltonian $\mathcal{H}_A(t)$ and the operator $\widetilde{\mathcal{H}}_{A,s}$ defined in* (3.31) *are essentially self-adjoint on the domain $\mathscr{D}$ defined in* (3.23).

The following proof is a generalization of an argument by Arai [48] and a similar argument given in [50, app. C].

Proof. Let $t, s \in \mathbb{R}$. We want to prove essential self-adjointness of $\widetilde{\mathcal{H}}_{A,s}$ using the commutator theorem [36, theorem X.37], nicely proven in [51]. It is easy to see that the same argumentation can then also be applied to $\mathcal{H}_A(t)$, which just has one term less. Consider

$$K_A := \sum_{j\in A} -\triangle_j + \mathcal{H}_f + 1. \tag{3.36}$$

This operator is essentially self-adjoint on $\mathscr{D}$ due to well-known results (see e.g. [36]) and certainly satisfies $K_A \geq 1$. Therefore, to apply the commutator theorem, we need to prove:

1. $\exists c \in \mathbb{R}$ such that $\forall \phi \in \mathscr{D}$, $\|(\widetilde{\mathcal{H}}_{A,s})\phi\| \leq c\|K_A\phi\|$.
2. $\exists d \in \mathbb{R}$ such that $\forall \phi \in \mathscr{D}$, $|\left\langle \widetilde{\mathcal{H}}_{A,s}\phi, K_\alpha\phi\right\rangle - \left\langle K_A\phi, \widetilde{\mathcal{H}}_{A,s}\phi\right\rangle| \leq d\|K_A^{1/2}\phi\|$.

Proof of 1. We make use of the standard estimates (see e.g. [45]) valid for all $\psi \in \mathrm{dom}(\mathcal{H}_f^{1/2})$ and $f \in L^2(\mathbb{R}^3, \mathbb{C})$,

$$\left\|\int d^3\mathbf{k}\ f(\mathbf{k})a(\mathbf{k})\psi\right\| \leq \|f\|_2 \left\|\mathcal{H}_f^{1/2}\psi\right\|, \quad \left\|\int d^3\mathbf{k}\ f(\mathbf{k})a^\dagger(\mathbf{k})\psi\right\| \leq \|f\|_2 \left\|\mathcal{H}_f^{1/2}\psi\right\| + \|f\|_2\|\psi\|. \tag{3.37}$$

Now let $\phi \in \mathscr{D}$. We have by the triangle inequality

$$\left\|\widetilde{\mathcal{H}}_{A,s}\phi\right\| \leq \sum_{j\in A} \left(\left\|\mathcal{H}_j^0\phi\right\| + \|\varphi_j\phi\|\right) + \|\mathcal{H}_f\phi\|\,, \tag{3.38}$$

so we need to bound each of the summands on the right hand side. $\|\mathcal{H}_f\phi\| \leq \|K_A\phi\|$ is clear since 1 and $-\triangle$ are positive operators. Next we consider the free Dirac operator,

$$\left\|\mathcal{H}_j^0\phi\right\| \leq m\,\|\phi\| + \|-i(\boldsymbol{\alpha}_j\cdot\nabla_j)\phi\|\,. \tag{3.39}$$

The derivative term needs closer inspection,

$$\|-i(\boldsymbol{\alpha}_j\cdot\nabla_j)\phi\|^2 = \left\langle\phi, -(\boldsymbol{\alpha}_j\cdot\nabla_j)^2\phi\right\rangle = \langle\phi, -\triangle\phi\rangle\,, \tag{3.40}$$

where only the Laplacian survives because the α-matrices anticommute and the derivatives commute. Continuing with the Cauchy-Schwarz inequality and the elementary inequality $\sqrt{ab} \leq \frac{1}{2}(a+b)\ \forall a,b \geq 0$, we obtain

$$\|-i(\boldsymbol{\alpha}_j\cdot\nabla_j)\phi\| \leq \sqrt{\langle\phi,-\triangle\phi\rangle} \leq \sqrt{\|\phi\|\,\|-\triangle\phi\|} \leq \frac{1}{2}\left(\|\phi\| + \|-\triangle\phi\|\right). \tag{3.41}$$

Again, since all the summands in K_A are positive operators, this directly leads to

$$\left\|\mathcal{H}_j^0\phi\right\| \leq C\,\|K_A\phi\|\,. \tag{3.42}$$

In the whole thesis, C denotes an arbitrary positive constant that may be different each time. For the interaction term, we see that the factor $\frac{\hat{\rho}(\mathbf{k})}{\sqrt{\omega(\mathbf{k})}}$ is in L^2 since $\hat{\rho}$ being a Schwartz function ensures rapid decay at infinity and since the singularity at $k=0$ (present only for $\mu = 0$) is integrable. This allows the use of (3.37), giving

$$\|\varphi_j\phi\| \leq C\left(\left\|\mathcal{H}_f^{1/2}\phi\right\| + \|\phi\|\right), \tag{3.43}$$

and with one more application of Cauchy-Schwarz,

$$\left\|\mathcal{H}_f^{1/2}\phi\right\| \leq \|\phi\|^{1/2}\,\|\mathcal{H}_f\phi\|^{1/2} \leq \frac{1}{2}\left(\|\phi\| + \|\mathcal{H}_f\phi\|\right), \tag{3.44}$$

we are done with the proof that there is a constant c (not depending on ϕ) with $\|(\widetilde{\mathcal{H}}_{A,s})\phi\| \leq c\|K_A\phi\|$.

Proof of 2. As in the previous step, we can bound the summands in $\widetilde{\mathcal{H}}_{A,s}$ one by one. We first observe that $\mathcal{H}_f$ and $\mathcal{H}_j^0$ commute with K_A. For the interaction term, we have

$$[\varphi_j, K_A] = [\varphi_j, -\triangle_j] + [\varphi_j, \mathcal{H}_f]\,, \tag{3.45}$$

so let us compute

$$\begin{aligned}
\langle\varphi_j\phi, \triangle_j\phi\rangle - \langle\triangle_j\phi, \varphi_j\phi\rangle &= \sum_{a=1}^{3}\left\langle\frac{\partial}{\partial x_j^a}\phi, \frac{\partial}{\partial x_j^a}\varphi_j\phi\right\rangle - \left\langle\frac{\partial}{\partial x_j^a}\varphi_j\phi, \frac{\partial}{\partial x_j^a}\phi\right\rangle \\
&= 2i\sum_{a=1}^{3}\Im\left\langle\frac{\partial}{\partial x_j^a}\phi, \frac{\partial}{\partial x_j^a}\varphi_j\phi\right\rangle \\
&= 2i\sum_{a=1}^{3}\Im\left\langle\frac{\partial}{\partial x_j^a}\phi, \int d^3\mathbf{k}\,\left[\left(\frac{-ik_j^a\hat{\rho}(\mathbf{k})}{\sqrt{\omega(\mathbf{k})}}e^{-i\omega(\mathbf{k})t}e^{i\mathbf{k}\cdot\hat{\mathbf{x}}_j}a(\mathbf{k}) + c.c.\right)\phi\right]\right\rangle,
\end{aligned} \tag{3.46}$$

where the last equality holds since $\left\langle \frac{\partial}{\partial x_j^a}\phi, \varphi_j \frac{\partial}{\partial x_j^a}\phi \right\rangle$ is real; and "c.c" denotes the hermitian conjugate of the preceding term. Since $\hat{\rho}$ is a Schwartz function, also $-ik_j^a\hat{\rho}(\mathbf{k})$ is, so we get from the estimate (3.37)

$$|\langle\varphi_j\phi, \triangle_j\phi\rangle - \langle\triangle_j\phi, \varphi_j\phi\rangle| \leq C\left(\left\|\mathcal{H}_f^{1/2}\phi\right\| + \|\phi\|\right) \leq 2C\left\|K_A^{1/2}\phi\right\|. \tag{3.47}$$

For the second term in (3.45), we look at the commutator of φ_j and $\mathcal{H}_f$. This amounts to a time derivative of $\varphi_j(t)$, which gives an expression like in the last line of (3.46), but where the function $-ik_j^a\hat{\rho}(\mathbf{k})$ is replaced by $-i\omega(\mathbf{k})\hat{\rho}(\mathbf{k})$. This is again a Schwartz function. Using estimate (3.37) again for that function, we obtain

$$|\langle\varphi_j\phi, \mathcal{H}_f\phi\rangle - \langle\mathcal{H}_f\phi, \varphi_j\phi\rangle| \leq C\left\|K_A^{1/2}\phi\right\|. \tag{3.48}$$

This means we have shown that there is a constant d (independent of ϕ), such that

$$\left|\left\langle\widetilde{\mathcal{H}}_{A,s}\phi, K_A\phi\right\rangle - \left\langle K_A\phi, \widetilde{\mathcal{H}}_{A,s}\phi\right\rangle\right| \leq d\|K_A^{1/2}\phi\|. \tag{3.49}$$

This is the second necessary ingredient for the application of the commutator theorem, which gives the statement of the lemma. □

Lemma 3.5. *The self-adjoint Hamiltonian $\widetilde{\mathcal{H}}_{A,s}$ and the unitary group $\widetilde{U}_{A,s}$ it generates satisfy the following properties for all $r, s, t \in \mathbb{R}$:*

(a) $e^{i\mathcal{H}_f(t-s)}\widetilde{\mathcal{H}}_{A,s}^n e^{-i\mathcal{H}_f(t-s)} = \widetilde{\mathcal{H}}_{A,t}^n \quad \forall n \in \mathbb{N}$, *whenever both sides are well-defined.*

(b) $e^{i\mathcal{H}_f(r-s)}\widetilde{U}_{A,s}(t-s)e^{i\mathcal{H}_f(s-r)} = \widetilde{U}_{A,r}(t-s)$.

Proof. Let $r, s, t \in \mathbb{R}$.
Part (a). We have for $n = 1$

$$\begin{aligned} e^{i\mathcal{H}_f(t-s)}\widetilde{\mathcal{H}}_{A,s}e^{-i\mathcal{H}_f(t-s)} &= e^{i\mathcal{H}_f(t-s)}\left(\mathcal{H}_f + \sum_{j\in A}\mathcal{H}_j^0 + \varphi_j(s)\right)e^{-i\mathcal{H}_f(t-s)} \\ &= \mathcal{H}_f + \sum_{j\in A}\mathcal{H}_j^0 + e^{i\mathcal{H}_f(t-s)}\varphi_j(s)e^{-i\mathcal{H}_f(t-s)} \\ &= \widetilde{\mathcal{H}}_{A,t}. \end{aligned} \tag{3.50}$$

The statement for arbitrary $n \in \mathbb{N}$ follows directly from the $n = 1$ case, which can be seen by inserting the identity $\mathbb{1} = e^{-i\mathcal{H}_f(t-s)}e^{i\mathcal{H}_f(t-s)}$ between the factors of $\widetilde{\mathcal{H}}_{A,s}$,

$$e^{i\mathcal{H}_f(t-s)}\widetilde{\mathcal{H}}_{A,s}^n e^{-i\mathcal{H}_f(t-s)} = \prod_{k=1}^{n} e^{i\mathcal{H}_f(t-s)}\widetilde{\mathcal{H}}_{A,s}e^{-i\mathcal{H}_f(t-s)} = \widetilde{\mathcal{H}}_{A,t}^n. \tag{3.51}$$

Part (b). By the analytic vector theorem, the set $\mathscr{A}$ of analytic vectors for $\widetilde{\mathcal{H}}_{A,s}$ is dense. Hence its image under the unitary map $e^{i\mathcal{H}_f(r-s)}$ is also dense. Let $\psi \in e^{i\mathcal{H}_f(r-s)}(\mathscr{A})$. We can write

$$\begin{aligned} e^{i\mathcal{H}_f(r-s)}\widetilde{U}_{A,s}(t-s)e^{i\mathcal{H}_f(s-r)}\psi &= e^{i\mathcal{H}_f(r-s)}\sum_{n=0}^{\infty}\frac{i^n(t-s)^n}{n!}\widetilde{\mathcal{H}}_{A,s}^n e^{i\mathcal{H}_f(s-r)}\psi \\ &= \sum_{n=0}^{\infty}\frac{i^n(t-s)^n}{n!}e^{i\mathcal{H}_f(r-s)}\widetilde{\mathcal{H}}_{A,s}^n e^{i\mathcal{H}_f(s-r)}\psi \\ &= \sum_{n=0}^{\infty}\frac{i^n(t-s)^n}{n!}\widetilde{\mathcal{H}}_{A,r}^n\psi, \end{aligned} \tag{3.52}$$

where we used part 1 of the lemma in the last step. The series converges, so ψ is analytic for $\widetilde{\mathcal{H}}_{A,r}$, which proves

$$e^{i\mathcal{H}_f(r-s)}\widetilde{U}_{A,s}(t-s)e^{i\mathcal{H}_f(s-r)}\psi = \widetilde{U}_{A,r}(t-s)\psi, \quad \forall\psi \in e^{i\mathcal{H}_f(r-s)}(\mathscr{A}). \tag{3.53}$$

Equation (3.53) tells us that the bounded operators $e^{i\mathcal{H}_f(r-s)}\widetilde{U}_{A,s}(t-s)e^{i\mathcal{H}_f(s-r)}$ and $\widetilde{U}_{A,r}(t-s)$ agree on a dense set, which implies that they are equal. □

The next lemma is about the causal structure of our equations. It uses the usual definition of addition of sets,

$$M + R := \{m + r | m \in M, r \in R\}. \tag{3.54}$$

In order to simplify notation, it is implied that vectors in $\mathbb{R}^{3N}$ and $\mathbb{R}^3$ can be added by just changing the respective j-th coordinate, e.g. $(\mathbf{x}_1, \mathbf{x}_2) + \mathbf{y}_2 \equiv (\mathbf{x}_1, \mathbf{x}_2 + \mathbf{y}_2)$.

Lemma 3.6.

(a) The evolution operators U_A do not propagate data faster than light, i.e. if for $R \subset \mathbb{R}^{3N}$ we have $\operatorname{supp}\psi \subset R$*, then for all $t, s \in \mathbb{R}$,*

$$\operatorname{supp}\,(U_A(t,s)\psi) \subset R + \sum_{j\in A} B_{|t-s|}(\mathbf{x}_j). \tag{3.55}$$

(b) Let ψ be the solution of $i\partial_t\psi = \mathcal{H}_A(t)\psi$ with smooth initial values given as $\psi(0,...,0) = \psi^0$. Then for all $t \in \mathbb{R}$, $\psi(t, \mathbf{x}_1, ..., t, \mathbf{x}_N)$ is uniquely determined by specifying initial conditions on $\sum_{j\in A} \overline{B}_{|t|}(\mathbf{x}_j)$.

Proof. Part (a). This lightcone property of the free Dirac equation is well-known (compare [41, theorem 2.20]). The claim for our model is a direct generalization to the many-particle case of the functional analytic arguments in [47, theorem 3.4]. (Note that it is also feasible to adapt the arguments using current conservation in [3, lemma 14] since the continuity equation holds for our model, as well.)
Part (b). This follows directly from 1. since if ψ and ψ' are two solutions whose initial values ψ^0 and ψ'^0 agree on $\sum_{j\in A} \overline{B}_{|t|}(\mathbf{x}_j)$, then

$$\operatorname{supp}\,(\psi^0 - \psi'^0) \subset \mathbb{R}^{3N} \setminus \sum_{j\in A} \overline{B}_{|t|}(\mathbf{x}_j) \tag{3.56}$$

implies by (3.55)

$$\operatorname{supp}\,(U_A(t,0)(\psi^0 - \psi'^0)) \subset \mathbb{R}^{3N} \setminus \sum_{j\in A} \overline{B}_{|t|}(\mathbf{x}_j) + \sum_{j\in A} B_{|t|}(\mathbf{x}_j) = \mathbb{R}^{3N} \setminus \{(\mathbf{x}_1, ..., \mathbf{x}_N)\}, \tag{3.57}$$

which is the claim. □

Another necessary information is which domains stay invariant under the time evolutions we have just constructed. The idea is to exploit a theorem by Huang [49, thm. 2.3], which we cite here adopted to our notation.

Theorem 3.7. ***(Huang).*** *Let K be a positive self-adjoint operator and define $Z_j(t) = K^{j-1}\,[\mathcal{H}_A(t), K]\,K^{-j}$. Suppose that $Z_k(t)$ is bounded with $\|Z_k(\cdot)\| \in L^1_{loc}(\mathbb{R})$ for all $k \leq j$. Then $U_A(t,s)[\mathrm{dom}(K^j)] = \mathrm{dom}(K^j)$.*

We will use a family of comparison operators for $j \in \mathbb{N}$, abbreviating $\sum_{k=1}^{N} -\triangle_k =: -\triangle$,

$$K_n := (-\triangle)^n + (\mathcal{H}_f)^n + 1. \tag{3.58}$$

The operator K_n resembles the $n-th$ power of the operator K_A we defined in (3.36) for the commutator theorem. Its domain of self-adjointness is denoted by $\mathrm{dom}(K_n)$.

Lemma 3.8. *The family of operators $U_A(t,s)$ with $t,s, \in \mathbb{R}$ leaves the set $\mathrm{dom}(K_n)$ invariant for all $n \in \mathbb{N}$.*

Proof. Let $n \in \mathbb{N}$. It is known that K_n is self-adjoint and strictly positive. We prove the invariance of $\mathrm{dom}(K^n)$ using Thm. 3.7, hence we only need the case $j = 1$ and need to bound $Z_1(t) = [\mathcal{H}_A(t), K_n]K_n^{-1}$.
Note that, since K_n is positive, 0 is in its resolvent set. This means that $K_n : \mathrm{dom}(K_n) \to \mathscr{H}$ is bijective, so its inverse $K_n^{-1} : \mathscr{H} \to \mathrm{dom}(K_n)$ is bounded by the closed graph theorem. Because the Laplacian commutes with the free Dirac operator (in the sense of self-adjoint operators, which can e.g. be seen by their resolvents), this carries over to $(-\triangle)^n$ and the commutator gives

$$[\mathcal{H}_A(t), K_n] = \sum_{j\in A} [\varphi_j(t), (-\triangle)^n] + \sum_{j\in A} \left[\varphi_j(t), \mathcal{H}_f^n\right]. \tag{3.59}$$

The commutator terms give rise to derivatives of the field terms φ, similarly as in the calculation (3.46). It becomes apparent that arbitrary derivatives with respect to time or space variables lead to the multiplication of $\hat{\rho}(\mathbf{k})$ in (3.16) by a product of k^a and $\omega(\mathbf{k})$ factors, which still keep the rapid decay at infinity. Therefore, also the derivative is a quantum field with an L^2-function as cut-off function. This means that the bound (3.43) can analogously be applied to the commutator and we have some $C > 0$ with

$$\|[\mathcal{H}_A(t), K_n]\,\eta\| \leq C\left(\|\mathcal{H}_f\eta\| + \|\eta\|\right) \quad \forall \eta \in \mathrm{dom}(K). \tag{3.60}$$

By the inequality of arithmetic and geometric mean,

$$\|\mathcal{H}_f\eta\| = \|\sqrt[n]{\mathcal{H}_f^n}\eta\| \leq C(\|\mathcal{H}_f^n\eta\| + \|\eta\|) \leq C(\|K_n\eta\| + \|\eta\|) \tag{3.61}$$

Since $K_n^{-1}\psi \in \mathrm{dom}(K_n)$, we can apply this to $Z_1(t)$,

$$\|Z_1(t)\psi\| = \left\|([\mathcal{H}_A(t), K_n])\,K_n^{-1}\psi\right\| \leq C\left(\|K_nK_n^{-1}\psi\| + \|K_n^{-1}\psi\|\right) = C\left(1 + \|K_n^{-1}\|_{op}\right)|\psi\|, \tag{3.62}$$

which implies that $Z_1(t)$ is bounded with $\|Z_1(\cdot)\| \in L^1_{loc}(\mathbb{R})$. Hence, application of Theorem 3.7 yields the claim. □

Corollary 3.9. *The family of operators $U_A(t,s)$ with $t,s \in \mathbb{R}$ leaves the set $\mathscr{D}$, defined in* (3.23)*, invariant.*

Proof. By Lemma 3.8, $U_A(t,s)$ with $t,s \in \mathbb{R}$ leaves $\mathrm{dom}(K_n)$ invariant for each $n \in \mathbb{N}$. We claim that

$$\mathrm{dom}(K_n) = \mathrm{dom}((-\triangle)^n) \otimes \mathscr{F} \cap L^2(\mathbb{R}^{3N}, \mathbb{C}^K) \otimes \mathrm{dom}(\mathcal{H}_f^n). \tag{3.63}$$

The operator K_n is of the form $(-\triangle)^n \otimes \mathbb{1} + \mathbb{1} \otimes \mathcal{H}_f^n + 1$, where the bounded operator 1 is irrelevant for the domain. By [33, chap. VIII.10], an operator of this structure on a tensor product space is essentially self-adjoint on the domain $\mathrm{dom}((-\triangle)^n) \otimes \mathscr{F} \cap L^2(\mathbb{R}^{3N}, \mathbb{C}^K) \otimes \mathrm{dom}(\mathcal{H}_f^n)$. The domain of self-adjointness arises when we take the closure of that operator.

It is, however, known from [32, p. 160] that a sum of positive operators is already closed on the domain (3.63). Thus, (3.63) is actually the domain of self-adjointness of K_n.
Let $\psi \in \mathscr{D}$, then also $\psi \in \mathrm{dom}(K_n)$ for all $n \in \mathbb{N}$. Thus, $U_A(t,s)\psi \in \mathrm{dom}(K_n)$ for all $n \in \mathbb{N}$. For the Fock space part, this directly gives

$$U_A(t,s)\psi \in \bigcap_{n=1}^{\infty} L^2(\mathbb{R}^{3N}, \mathbb{C}^K) \otimes \mathrm{dom}(\mathcal{H}_f^n) = L^2(\mathbb{R}^{3N}, \mathbb{C}^K) \otimes \mathrm{dom}(\mathcal{H}_f^\infty). \tag{3.64}$$

In the L^2-part, we first note that Lemma 3.6 gives an upper bound on the growth of supports, so compactness of the support is preserved under the time evolution $U_A(t,s)$. Secondly, we have

$$U_A(t,s)\psi \in \bigcap_{n=1}^{\infty} \mathrm{dom}((-\triangle)^n) \otimes \mathscr{F} \subset C^\infty(\mathbb{R}^{3N}, \mathbb{C}^K) \otimes \mathscr{F}, \tag{3.65}$$

which follows from Sobolev's lemma as contained in the proposition in [36, chap. IX.7]. These two facts imply that the time evolution leaves C_c^∞ invariant and thus we infer $U_A(t,s)\psi \in \mathscr{D}$. □

Another result that will be helpful later is that not only the time evolutions leave the set $\mathscr{D}$ invariant, but also the terms in the Hamilton operators themselves.

Lemma 3.10. *The set $\mathscr{D}$ is left invariant by $\mathcal{H}_f$, $\mathcal{H}_j^0$ and $\varphi_j(t)$ for each $1 \leq j \leq N$ and $t \in \mathbb{R}$.*

Proof. 1. $\mathcal{H}_j^0$ only acts on the first tensor component and on that one, it leaves C_c^∞-functions invariant because it is a linear combination of partial derivatives and the identity.
2. $\mathcal{H}_f$ only acts on the second tensor component and on that one, it leaves $\mathrm{dom}(\mathcal{H}_f^\infty)$ invariant by definition.
3. First we note that φ_j does not increase supports. Now let $k \in \mathbb{N}, t \in \mathbb{R}$ and $\psi \in \mathrm{dom}(\mathcal{H}_f^{k+1})$. Then, using the same estimates as in the proof of Lemma 3.4,

$$\left\| \mathcal{H}_f^k \ \varphi_j(t)\psi \right\| \leq \left\| \varphi_j(t)\mathcal{H}_f^k\psi \right\| + \left\| \frac{\partial^k}{\partial t^k}\varphi_j(t)\psi \right\| \leq C \left(\left\| \mathcal{H}_f^{k+1}\psi \right\| + \|\mathcal{H}_f\psi\| + \|\psi\| \right) < \infty, \tag{3.66}$$

which shows that $\varphi_j(t)\psi \in \mathrm{dom}(\mathcal{H}_f^k)$ for every $t \in \mathbb{R}$. An analogous argument can be done for the operators $\mathcal{H}_j^0$, which together implies that $\varphi_j(t)$ leaves $\mathscr{D}$ invariant. □

3.3.3 Construction of the multi-time evolution

The construction of the solution of our multi-time system (3.22) relies on the consistency condition which we prove now.

Lemma 3.11. *Let $\psi \in \mathscr{D}$ and A, B be disjoint subsets of $\{1, ..., N\}$, then the consistency condition*

$$[\mathcal{H}_A(t_A), \mathcal{H}_B(t_B)]\, \psi(\mathbf{x}_1, ..., \mathbf{x}_N) = 0 \tag{3.67}$$

is satisfied whenever $\forall j \in A, k \in B : \|\mathbf{x}_j - \mathbf{x}_k\| > \delta + |t_A - t_B|$.

Proof. Let $t_A, t_B \in \mathbb{R}$. The commutator reads

$$[\mathcal{H}_A(t_A), \mathcal{H}_B(t_B)] = \left[\sum_{j\in A} \mathcal{H}_j^0 + \varphi_j(t_A), \sum_{k\in B} \mathcal{H}_k^0 + \varphi_k(t_B)\right] = \sum_{j\in A, k\in B} [\varphi_j(t_A), \varphi_k(t_B)]\,, \tag{3.68}$$

since, by definition, the free Dirac Hamiltonians commute with the other terms. We will now show that each of the summands in the double sum applied to $\psi \in \mathscr{D}$ vanishes when evaluated at $(\mathbf{x}_1, ..., \mathbf{x}_N) \in \mathbb{R}^{3N}$ with $\forall j \in A, k \in B : \|\mathbf{x}_j - \mathbf{x}_k\| > \delta + |t_A - t_B|$.
It is well-known (e.g. [36, thm X.41]) that field operators as defined in (3.16) satisfy the CCR, which means

$$\begin{aligned}
&[\varphi_j(t_A), \varphi_k(t_B)]\,\psi(\mathbf{x}_1, ..., \mathbf{x}_N) \\
&= i\Im \int \frac{d^3\mathbf{k}}{\omega(\mathbf{k})}\, \hat{\rho}(\mathbf{k})^\dagger \hat{\rho}(\mathbf{k}) e^{i\omega(\mathbf{k})t_A - i\mathbf{k}\cdot\mathbf{x}_j} e^{-i\omega(\mathbf{k})t_B + i\mathbf{k}\cdot\mathbf{x}_k} \psi(\mathbf{x}_1, ..., \mathbf{x}_N) \\
&= \frac{1}{2}\int \frac{d^3\mathbf{k}}{\omega(\mathbf{k})} \left(\hat{\rho}(\mathbf{k})^\dagger \hat{\rho}(\mathbf{k}) e^{i\omega(\mathbf{k})(t_A - t_B)} e^{-i\mathbf{k}\cdot(\mathbf{x}_j - \mathbf{x}_k)} - \text{ c.c } \right) \psi(\mathbf{x}_1, ..., \mathbf{x}_N) \\
&= \frac{1}{2}\int \frac{d^3\mathbf{k}}{\omega(\mathbf{k})} \Bigg(\int d^3\mathbf{y}_1 d^3\mathbf{y}_2 \rho(\mathbf{y}_1) e^{-i\mathbf{k}\cdot\mathbf{y}_1} \rho(\mathbf{y}_2) e^{i\mathbf{k}\cdot\mathbf{y}_2} e^{i\omega(\mathbf{k})(t_A - t_B)} e^{-i\mathbf{k}\cdot(\mathbf{x}_j - \mathbf{x}_k)} \\
&\qquad\qquad - \text{c.c} \Bigg) \psi(\mathbf{x}_1, ..., \mathbf{x}_N),
\end{aligned} \tag{3.69}$$

upon insertion of the Fourier transforms. We compare this to the so-called Pauli-Jordan function [52, p. 88], i.e. the distribution

$$\Delta(x_j, x_k) := c \int \frac{d^3\mathbf{k}}{\omega(\mathbf{k})} \left(e^{i\omega(\mathbf{k})(t_j - t_k) - i\mathbf{k}\cdot(\mathbf{x}_j - \mathbf{x}_k)} - \text{ c.c. } \right), \tag{3.70}$$

where $c = \frac{i}{16\pi^3}$. It is known that $\Delta(x_1, x_2) = 0$ whenever x_1 is space-like to x_2 [52, p. 89]. We define a double convolution by

$$\begin{aligned}
(\rho * *\Delta)(t_j, \mathbf{x}_j, t_k, \mathbf{x}_k) &:= \int d^3\mathbf{y}_1 d^3\mathbf{y}_2\ \rho(\mathbf{y}_1)\rho(\mathbf{y}_2)\Delta(t_j, \mathbf{x}_j - \mathbf{y}_1, t_k, \mathbf{x}_k - \mathbf{y}_2) \\
&= c \int \frac{d^3\mathbf{k}}{\omega(\mathbf{k})} \int d^3\mathbf{y}_1 d^3\mathbf{y}_2 \rho(\mathbf{y}_1)\rho(\mathbf{y}_2) \left(e^{i\omega(\mathbf{k})(t_j - t_k) - i\mathbf{k}\cdot(\mathbf{x}_j - \mathbf{y}_1 - \mathbf{x}_k + \mathbf{y}_2)} - \text{ c.c. } \right),
\end{aligned} \tag{3.71}$$

which is a well-defined integral since $\rho \in C_c^\infty(\mathbb{R}^3)$. Comparison to (3.69) yields

$$\frac{2}{c}\,[\varphi_j(t_A), \varphi_k(t_B)]\,\psi(\mathbf{x}_1, ..., \mathbf{x}_N) = (\rho * *\Delta)(t_A, \mathbf{x}_j, t_B, \mathbf{x}_k)\psi(\mathbf{x}_1, ..., \mathbf{x}_N). \tag{3.72}$$

We know that $\|\mathbf{x}_j - \mathbf{x}_k\| > |t_A - t_B| + \delta$ and by (3.15), $\rho(\mathbf{y}) \neq 0$ only if $\|\mathbf{y}\| < \frac{\delta}{2}$. Thus the argument of the function Δ in the double convolution (3.71) satisfies

$$\begin{aligned}
\|\mathbf{x}_j - \mathbf{y}_1 - (\mathbf{x}_k - \mathbf{y}_2)\| &\geq \|\mathbf{x}_k - \mathbf{x}_j\| - \|\mathbf{y}_1\| - \|\mathbf{y}_2\| \\
&\geq \|\mathbf{x}_j - \mathbf{x}_k\| - \delta \\
&> |t_A - t_B|,
\end{aligned} \tag{3.73}$$

i.e. it is space-like, which implies that $(\rho * *\Delta)(t_A, \mathbf{x}_j, t_B, \mathbf{x}_k) = 0$ and hence also the commutator is zero. □

With all the previous results at hand, the existence of solutions can be treated constructively. We first prove a lemma which contains the crucial ingredient for the subsequent theorem.

Lemma 3.12. *Let $\zeta \in \mathscr{D}$. Let A, B be arbitrary subsets of $\{1, ..., N\}$ with $A \cap B = \emptyset$, let $t_B \geq s \geq t_A$, then*

$$\left([\mathcal{H}_A(t_A), U_B(t_B, s)]\,\zeta\right)(\mathbf{x}_1, ..., \mathbf{x}_N) = 0. \tag{3.74}$$

holds at every point $(\mathbf{x}_1, ..., \mathbf{x}_N) \in \mathbb{R}^{3N}$ for which $\forall j \in A, k \in B$, $\|\mathbf{x}_j - \mathbf{x}_k\| > \delta + t_B - t_A$.

The idea of the proof is to take the derivative of the commutator in (3.74) with respect to t_B to get an expression where the consistency condition proven in Lemma 3.11 becomes useful. However, it is not immediately clear if a term of the form $\mathcal{H}_A(t_A)U_B(t_B, s)$ is differentiable or even continuous in t_B because $\mathcal{H}_A$ is not a continuous operator. Therefore, we have to take a detour and approximate $\mathcal{H}_A$ by bounded operators. A similar approximation by bounded operators is used in the proof of the Hille-Yosida theorem in [53, ch. 7.4].

Proof. Let $A, B \subset \{1, ..., N\}$ with $A \cap B = \emptyset$, $s, t_A, t_B \in \mathbb{R}$ with $t_B \geq s \geq t_A$, $\zeta \in \mathscr{D}$ and $(\mathbf{x}_1, ..., \mathbf{x}_N) \in \mathbb{R}^{3N}$ such that $\forall j \in A, k \in B$: $\|\mathbf{x}_j - \mathbf{x}_k\| > \delta + t_B - t_A$.
We abbreviate $\sum_{k\in A} \varphi_k(t) =: \varphi_A(t)$ for $t \in \mathbb{R}$. First note that the free Dirac terms in $\mathcal{H}_A$ trivially commute, so

$$\left([U_B(t_B, s), \mathcal{H}_A(t_A)]\,\zeta\right)(\mathbf{x}_1, ..., \mathbf{x}_N) = \left([U_B(t_B, s), \varphi_A(t_A)]\,\zeta\right)(\mathbf{x}_1, ..., \mathbf{x}_N). \tag{3.75}$$

Now define for $\varepsilon > 0, t \in \mathbb{R}$ a family of auxiliary operators

$$\varphi_A^\varepsilon(t) := \frac{\varphi_A(t)}{1 + i\varepsilon\varphi_A(t)}, \tag{3.76}$$

which are well-defined since $\varphi_A(t)$ is self-adjoint for all t [46]. For $\lambda \in \mathbb{R}, \varepsilon > 0$,

$$\left|\frac{\lambda}{1 + i\varepsilon\lambda}\right| \leq \frac{1}{\varepsilon} \quad \Longrightarrow \|\varphi_A^\varepsilon(t)\| \leq \frac{1}{\varepsilon} \tag{3.77}$$

where the implication follows by the spectral theorem. The difference of field operator φ_A and its approximation φ_A^ε can be recast into

$$(\varphi_A(t_A) - \varphi_A^\varepsilon(t_A)) = \frac{\varphi_A(t_A) + i\varepsilon\varphi_A(t_A)^2}{1 + i\varepsilon\varphi_A(t_A)} - \frac{\varphi_A(t_A)}{1 + i\varepsilon\varphi_A(t_A)} = \frac{i\varepsilon}{1 + i\varepsilon\varphi_A(t_A)}\varphi_A(t_A)^2 \tag{3.78}$$

and we note the bound for all $\varepsilon > 0$:

$$\left\|\frac{1}{1 + i\varepsilon\varphi_A(t_A)}\right\| \leq 1. \tag{3.79}$$

Because $U_B(t_B, s)\zeta \in \mathscr{D}$ by corollary 3.9, we find the bound

$$\begin{aligned} \|[U_B(t_B, s), \varphi_A(t_A) - \varphi_A^\varepsilon(t_A)]\,\zeta\| \leq\ & \|(\varphi_A(t_A) - \varphi_A^\varepsilon(t_A))\zeta\| \\ +\ & \|(\varphi_A(t_A) - \varphi_A^\varepsilon(t_A))U_B(t_B, s)\zeta\| \\ \leq\ & \varepsilon\left(\|\varphi_A(t_A)^2\zeta\| + \|\varphi_A(t_A)^2U_B(t_B, s)\zeta\|\right). \end{aligned} \tag{3.80}$$

Since we can take $\varepsilon \to 0$, the norm of the left hand side has to vanish. Because we furthermore know that $[U_B(t_B, s), \varphi_A(t_A) - \varphi_A^\varepsilon(t_A)]$ is a continuous function, the following implication holds:

$$\left([U_B(t_B, s), \varphi_A^\varepsilon(t_A)]\,\zeta\right)(\mathbf{x}_1, ..., \mathbf{x}_N) = 0\ \forall\varepsilon > 0 \Rightarrow \left([U_B(t_B, s), \varphi_A(t_A)]\,\zeta\right)(\mathbf{x}_1, ..., \mathbf{x}_N) = 0. \tag{3.81}$$

Thus it remains to prove that the commutator defined for $t \in \mathbb{R}$,

$$\Omega_t := [U_B(t,s), \varphi_A^\varepsilon(t_A)]\,\zeta, \tag{3.82}$$

vanishes at $(\mathbf{x}_1, ..., \mathbf{x}_N)$. Note that Ω_t depends on ε, which we do not write for brevity. As a merit of our approximation, $t \mapsto \Omega_t$ is a continuous map $\mathbb{R} \to \mathscr{H}$. We proceed in four steps:

1. Construct an auxiliary function ϕ_t that solves for $\eta \in \mathscr{D}$

$$i\partial_t \langle \eta, \phi_t \rangle = \langle \eta, [\varphi_B(t), \varphi_A^\varepsilon(t_A)]\, U_B(t,s)\eta \rangle + \langle \mathcal{H}_B(t)\eta, \phi_t \rangle\,. \tag{3.83}$$

2. Show that $\forall \eta \in \mathscr{D} : i\partial_t \langle \eta, \phi_t - \Omega_t \rangle = \langle \mathcal{H}_B(t)\eta, \phi_t - \Omega_t \rangle$.

3. Show that the weak equation proven in step 2 has a unique solution, thus $\phi_t = \Omega_t$.

4. Investigate the support properties of ϕ_t and conclude that Ω_t vanishes at $(\mathbf{x}_1, ..., \mathbf{x}_N)$.

Step 1: We introduce the abbreviation for $t \in \mathbb{R}$

$$f_t := [\varphi_B(t), \varphi_A^\varepsilon(t_A)]\, U_B(t,s)\zeta \tag{3.84}$$

and recognize that the function $f : \mathbb{R} \to \mathscr{H}, t \mapsto f_t$ is bounded and measurable. Define

$$\phi_t := \int_s^t d\tau\; e^{i\mathcal{H}_f(t-s)} e^{-i(\mathcal{H}_f + \mathcal{H}_B(s))(t-\tau)} e^{-i\mathcal{H}_f(\tau-s)} f_\tau. \tag{3.85}$$

For $\eta \in \mathscr{D}, t \in \mathbb{R}$, we compute using Fubini's theorem,

$$\begin{aligned} i\partial_t \langle \eta, \phi_t \rangle =& i\partial_t \int_s^t d\tau \left\langle e^{i\mathcal{H}_f(\tau-s)} e^{i(\mathcal{H}_f + \mathcal{H}_B(s))(t-\tau)} e^{-i\mathcal{H}_f(t-s)} \eta, f_\tau \right\rangle \\ =& \left\langle e^{i\mathcal{H}_f(t-s)} e^{i(\mathcal{H}_f + \mathcal{H}_B(s))(t-t)} e^{-i\mathcal{H}_f(t-s)} \eta, f_t \right\rangle \\ &+ \int_s^t d\tau \left\langle e^{i\mathcal{H}_f(\tau-s)} \mathcal{H}_B(s) e^{i(\mathcal{H}_f + \mathcal{H}_B(s))(t-\tau)} e^{-i\mathcal{H}_f(t-s)} \eta, f_\tau \right\rangle \\ =& \langle \eta, f_t \rangle + \langle \mathcal{H}_B(t)\eta, \phi_t \rangle\,. \end{aligned} \tag{3.86}$$

Step 2: A calculation similar to the one above is now possible for $\Omega_t, t \in \mathbb{R}$:

$$\begin{aligned} i\partial_t \langle \eta, \Omega_t \rangle =& i\partial_t \left(\langle U_B(s,t)\eta, \varphi_A^\varepsilon(t_A)\zeta \rangle - \left\langle \varphi_A^\varepsilon(t_A)^\dagger \eta, U_B(t,s)\zeta \right\rangle \right) \\ =& \langle U_B(s,t)\mathcal{H}_B(t)\eta, \varphi_A^\varepsilon(t_A)\zeta \rangle - \left\langle \varphi_A^\varepsilon(t_A)^\dagger \eta, \mathcal{H}_B(t) U_B(t,s)\zeta \right\rangle \\ &- \langle \mathcal{H}_B(t)\eta, \varphi_A^\varepsilon(t_A) U_B(t,s)\zeta \rangle + \langle \mathcal{H}_B(t)\eta, \varphi_A^\varepsilon(t_A) U_B(t,s)\zeta \rangle \\ =& \langle \mathcal{H}_B(t)\eta, \Omega_t \rangle + \langle \eta, [\mathcal{H}_B(t), \varphi_A^\varepsilon(t_A)]\, U_B(t,s)\zeta \rangle \\ =& \langle \mathcal{H}_B(t)\eta, \Omega_t \rangle + \langle \eta, [\varphi_B(t), \varphi_A^\varepsilon(t_A)]\, U_B(t,s)\zeta \rangle + \sum_{k \in B} \left\langle \eta, \underbrace{[\mathcal{H}_k^0, \varphi_A^\varepsilon(t_A)]}_{=0} \zeta \right\rangle \\ =& \langle \eta, f_t \rangle + \langle \mathcal{H}_B(t)\eta, \Omega_t \rangle\,. \end{aligned} \tag{3.87}$$

This together with (3.86) yields that the difference $\phi_t - \Omega_t$ is a weak solution of the Dirac equation in the sense that $\forall \eta \in \mathscr{D}$:

$$i\partial_t \langle \eta, \phi_t - \Omega_t \rangle = \langle \mathcal{H}_B(t)\eta, \phi_t - \Omega_t \rangle \quad . \tag{3.88}$$

Step 3: For all $s \in \mathbb{R}$, $U_B(s,s) = \mathbb{1}$ implies $\Omega_s = 0$ and by definition, $\phi_s = 0$. To show that Ω_t and ϕ_t are actually equal for all times $t \in \mathbb{R}$, it thus suffices to prove uniqueness of solutions to Eq. (3.88).
To this end, let $\rho : \mathbb{R} \to \mathcal{H}, t \mapsto \rho_t$ be continuous and for every $\eta \in \mathscr{D}$ a solution to

$$i\partial_t \langle \eta, \rho_t \rangle = \langle \mathcal{H}_B(t)\eta, \rho_t \rangle . \tag{3.89}$$

We claim that then, for all $t \in \mathbb{R}$, $\rho_t = U_B(t,s)\rho_s$. To see this we consider $t \mapsto \langle U_B(t,s)\eta, \rho_t \rangle$, we prove that this is differentiable with zero derivative. For $h > 0$, we find

$$\begin{aligned}
&\frac{1}{h}\Big\| \langle U_B(t+h,s)\eta, \rho_{t+h}\rangle - \langle U_B(t,s)\eta, \rho_t\rangle \Big\| \\
&\leq \Big\| \frac{1}{h} \langle U_B(t+h,s)\eta - U_B(t,s)\eta, \rho_{t+h}\rangle - \langle i\mathcal{H}_B(t)U_B(t,s)\eta, \rho_{t+h}\rangle \Big\| \\
&\quad + \Big\| \frac{1}{h} \langle U_B(t,s)\eta, \rho_{t+h} - \rho_t\rangle - i\langle \mathcal{H}_B(t)U_B(t,s)\eta, \rho_{t+h}\rangle \Big\| \\
&\leq \Big\| \frac{1}{h}\big(U_B(t+h,s)\eta - U_B(t,s)\eta\big) - i\mathcal{H}_B(t)U_B(t,s)\eta \Big\| \|\rho_{t+h}\| \\
&\quad + \Big\| \frac{1}{h} \langle U_B(t,s)\eta, \rho_{t+h} - \rho_t\rangle - i\langle \mathcal{H}_B(t)U_B(t,s)\eta, \rho_t\rangle \Big\| + \Big\| \langle \mathcal{H}_B(t)U_B(t,s)\eta, \rho_{t+h} - \rho_t\rangle \Big\|.
\end{aligned} \tag{3.90}$$

The first term goes to zero as $h \to 0$ because $\eta \in \mathscr{D}$ and since ρ_t is continuous, the norm ρ_{t+h} is bounded in a neighbourhood of t. The second term vanishes using (3.89), noting that also $U_B(t,s)\eta \in \mathscr{D}$ by Corollary 3.9. The last term also goes to zero by continuity of ρ_t. We have thus proven that

$$\partial_t \langle U_B(t,s)\eta, \rho_t\rangle = 0 \;\Rightarrow \langle \eta, U_B(s,t)\rho_t\rangle = const. \tag{3.91}$$

This implies the desired uniqueness statement $\langle \eta, U_B(t,s)\rho_s - \rho_t\rangle = 0$ for all $\eta \in \mathscr{D}$. Since $\mathscr{D} \subset \mathscr{H}$ is dense, $\rho_t = U_B(t,s)\rho_s$ follows.
In the special case of (3.88), the initial value is $\rho_s = \phi_s - \Omega_s = 0$. Furthermore, $t \mapsto \Omega_t - \phi_t$ is continuous, hence

$$\forall t \in \mathbb{R} : \phi_t - \Omega_t = 0. \tag{3.92}$$

Step 4: Thanks to Eq. (3.85), we now have an explicit formula for Ω_t by means of $\Omega_t = \phi_t$. Next, we investigate its support.
To treat the commutator term in (3.84), we insert two identities:

$$\begin{aligned}
[\varphi_B(t), \varphi_A^\varepsilon(t_A)] &= \frac{1}{1+i\varepsilon\varphi_A(t_A)}(1+i\varepsilon\varphi_A(t_A))\varphi_B(t)\varphi_A(t_A)\frac{1}{1+i\varepsilon\varphi_A(t_A)} \\
&\quad - \frac{1}{1+i\varepsilon\varphi_A(t_A)}\varphi_A(t_A)\varphi_B(t)(1+i\varepsilon\varphi_A(t_A))\frac{1}{1+i\varepsilon\varphi_A(t_A)} \\
&= \frac{1}{1+i\varepsilon\varphi_A(t_A)}\,[\varphi_B(t), \varphi_A(t_A)]\,\frac{1}{1+i\varepsilon\varphi_A(t_A)}.
\end{aligned} \tag{3.93}$$

The operator $\frac{1}{1+i\varepsilon\varphi_A(t_A)}$ does not increase the domain of functions since it is the resolvent of $\varphi_A(t_A)$ that can be written as a direct fiber integral, compare [47, thm. 3.4] and [54, thm. XIII.85]. Hence, Lemma 3.11 guarantees that $f_t(\mathbf{x}_1, ..., \mathbf{x}_N) = 0$ whenever $\|\mathbf{x}_j - \mathbf{x}_k\| > \delta + |t - t_A|$ for all $j \in A, k \in B$.
The spatial support is not altered by the $\mathcal{H}_f$ operators and their exponentials, so we have

$$\operatorname{supp}\left(e^{-i\mathcal{H}_f(\tau-s)}f_\tau\right) \subset \left\{(\mathbf{x}_1, ..., \mathbf{x}_N) \in \mathbb{R}^{3N} \middle| \exists j \in A, k \in B : \|\mathbf{x}_j - \mathbf{x}_k\| \leq \delta + \tau - t_A\right\}. \tag{3.94}$$

Applying Lemma 3.6, this support can grow by at most $\sum_{k\in B} B_{t-\tau}(\mathbf{x}_j)$ when acted on by $e^{-i(\mathcal{H}_f+\mathcal{H}_B(s))(t-\tau)}$. So this implies

$$\operatorname{supp}\left(e^{-i(\mathcal{H}_f+\mathcal{H}_B(s))(t-\tau)}e^{-i\mathcal{H}_f(\tau-s)}f_\tau\right) \subset \left\{(\mathbf{x}_1,...,\mathbf{x}_N)\in\mathbb{R}^{3N}\,\middle|\, \begin{array}{c}\exists j\in A, k\in B:\\ \|\mathbf{x}_j-\mathbf{x}_k\|\leq \delta+t-t_A\end{array}\right\}. \tag{3.95}$$

Consider $\Omega_{t_B}=\phi_{t_B}$. By (3.95), the integrand in Eq. (3.85) vanishes whenever $\|\mathbf{x}_j-\mathbf{x}_k\|> \delta+t_B-t_A$. This is satisfied for $(\mathbf{x}_1,...,\mathbf{x}_N)$ by assumption, which yields

$$\Omega_t(\mathbf{x}_1,...,\mathbf{x}_N)=\left([U_B(t,s),\varphi_A^\varepsilon(t_A)]\,\zeta\right)(\mathbf{x}_1,...,\mathbf{x}_N)=0 \tag{3.96}$$

for every positive ε, and thus with (3.81) the claim of the lemma. □

We are now ready to prove the existence Theorem 3.1. In addition to the claim in Thm. 3.1 we also prove the following extended claim that states the form of the solution.

Theorem 3.13. *For each $\psi^0\in\mathscr{D}$, there exists a solution ψ of the multi-time system on $\mathscr{S}_\delta$ with initial data $\psi(0,...,0)=\psi^0$ and with $\psi(t_1,...,t_N)\in\mathscr{D}$.*
Let σ be a permutation on $\{1,...,N\}$ such that $t_{\sigma(1)}\geq t_{\sigma(2)}\geq\cdots\geq t_{\sigma(N)}$, then one such solution is given by

$$\begin{aligned}&\psi(t_1,...,t_N)\\ &=U_{\{\sigma(1)\}}(t_{\sigma(1)},t_{\sigma(2)})\ldots U_{\{\sigma(1),...,\sigma(N-1)\}}(t_{\sigma(N-1)},t_{\sigma(N)})U_{\{1,2,...,N\}}(t_{\sigma(N)},0)\psi^0.\end{aligned} \tag{3.97}$$

For the proof, it will be helpful to abbreviate formulas like (3.97) using the $\bigcirc$-symbol for the ordered product of operators, $\bigcirc_{k=1}^l A_k := A_1A_2...A_l$. In this notation, expression (3.97) reads

$$\left(\left(\bigcirc_{k=1}^{N-1}U_{\{\sigma(j)|j\leq k\}}(t_{\sigma(k)},t_{\sigma(k+1)})\right)U_{\{1,,...,N\}}(t_{\sigma(N)},0)\psi^0\right)(\mathbf{x}_1,...,\mathbf{x}_N). \tag{3.98}$$

Proof. Let $\psi^0\in\mathscr{D}$, and define $\psi:\mathbb{R}^N\to\mathscr{H}$ by Eq. (3.97). Property $U_A(t,t)=\mathbb{1}$ stated in Theorem 3.3 ensures $\psi(0,...,0)=\psi^0$, so the correct initial value is attained. $\psi^0\in\mathscr{D}$ implies that for all $t_1,...,t_N\in\mathbb{R}$, $\psi(t_1,...,t_N)\in\mathscr{D}$ since $\mathscr{D}$ is preserved by the operators U_A by virtue of Corollary 3.9.
We now show the three points from the definition of the solution sense.
i) Since $\psi:\mathbb{R}^N\to\mathscr{D}\subset\mathscr{H}$, we may infer by Theorem 3.3 part 3 that ψ is differentiable.
ii) Let $j\in\{1,...,N\}$. By Lemma 3.10 also $\mathcal{H}_j(t_j)\psi(t_1,...,t_N)\in\mathscr{D}$, so both expressions are pointwise evaluable. The same is true for $\partial_{t_j}\psi(t_1,...,t_N)$ since it amounts to a successive application of U_A operators and of $\mathcal{H}_j$, which all leave $\mathscr{D}$ invariant.
iii) We now have to check that ψ satisfies the respective equations (3.22) in S_δ. Given a set $A\subset\{1,...,N\}$ and a time $t_A\in\mathbb{R}$, consider a configuration $(t_1,\mathbf{x}_1,...,t_N,\mathbf{x}_N)\in\mathscr{S}_\delta$ where $t_j=t_A\ \forall j\in A$. We assume w.l.o.g. that the times are already ordered $t_1\geq t_2\geq\cdots\geq t_N$, so that the permutation in (3.97) is the identity. Let $a:=\min(A)$ and $b:=\max(A)$, then

$$\begin{aligned}\psi(t_1,...,t_N)=&\left(\bigcirc_{k=1}^{a-2}U_{\{j|j\leq k\}}(t_k,t_{k+1})\right)U_{\{j|j\leq a-1\}}(t_{a-1},t_A)U_{\{j|j\leq b\}}(t_A,t_{b+1})\\ &\left(\bigcirc_{k=b+1}^{N-1}U_{\{j|j\leq k\}}(t_k,t_{k+1})\right)U_{\{1,,...,N\}}(t_N,0)\psi^0\end{aligned} \tag{3.99}$$

We take the derivative of (3.99) with respect to t_A and use that for $\zeta\in\mathscr{D}$,

$$i\frac{d}{dt}U_B(s,t)\zeta=-U_B(s,t)\mathcal{H}_B(t)\zeta,\quad\forall\ s,t\in\mathbb{R},B\subset\{1,...,N\}, \tag{3.100}$$

which follows directly from the properties of the time evolution operators, in particular $U_B(t,s)U_B(s,t) = \mathbb{1}$. Abbreviating

$$\psi' := U_{\{j|j\leq b\}}(t_A, t_{b+1}) \left(\bigcirc_{k=b+1}^{N-1} U_{\{j|j\leq k\}}(t_k, t_{k+1})\right) U_{\{1,,...,N\}}(t_{\sigma(N)}, 0)\psi^0, \tag{3.101}$$

we obtain

$$\begin{aligned} & i\frac{\partial}{\partial t_A}\psi(t_1,...,t_N) \\ =& \left(\left(\bigcirc_{k=1}^{a-2} U_{\{j|j\leq k\}}(t_k,t_{k+1})\right) U_{\{j|j\leq a-1\}}(t_{a-1},t_A) \left(-\mathcal{H}_{\{j|j\leq a-1\}}(t_A) + \mathcal{H}_{\{j|j\leq b\}}(t_A)\right)\psi'\right) \\ =& \mathcal{H}_A(t_A)\psi(t_1,...,t_N) + \left(\left[\bigcirc_{k=1}^{a-1} U_{\{j|j\leq k\}}(t_k,t_{k+1}), \mathcal{H}_A(t_A)\right]\psi'\right). \end{aligned} \tag{3.102}$$

We rewrite the second term as

$$\begin{aligned} & \left[\bigcirc_{k=1}^{a-1} U_{\{j|j\leq k\}}(t_k,t_{k+1}), \mathcal{H}_A(t_A)\right]\psi' \\ =& \sum_{l=1}^{a-1} \left(\bigcirc_{k=1}^{l-1} U_{\{j|j\leq k\}}(t_k,t_{k+1})\right) \left[U_{\{j|j\leq l\}}(t_l,t_{l+1}), \mathcal{H}_A(t_A)\right] \left(\bigcirc_{k=l+1}^{a-1} U_{\{j|j\leq k\}}(t_k,t_{k+1})\right)\psi', \end{aligned} \tag{3.103}$$

where empty products such as $\bigcirc_{k=1}^{0}$ denote $\mathbb{1}$. Lemma 3.12 implies that for any $\zeta \in \mathscr{D}$ and $l < a$,

$$\text{supp}\left(\left[U_{\{j|j\leq l\}}(t_l,t_{l+1}), \mathcal{H}_A(t_A)\right]\zeta\right) \subset \{(\mathbf{x}_1,...,\mathbf{x}_N) | \exists k\in A, j\leq l : \|\mathbf{x}_j - \mathbf{x}_k\| \leq \delta + t_l - t_A\}. \tag{3.104}$$

The support properties of the evolution operators (Lemma 3.6) imply that if $\text{supp}\,(\xi) \subset R$, then $\text{supp}\left(\bigcirc_{k=1}^{l-1} U_{\{j|j\leq k\}}(t_k,t_{k+1})\xi\right)$ is a subset of

$$\left\{(\mathbf{y}_1,...,\mathbf{y}_N)\in\mathbb{R}^{3N} \middle| \exists(\mathbf{x}_1,...\mathbf{x}_N)\in R : \begin{array}{c} \mathbf{x}_j = \mathbf{y}_j \text{ if } j > l. \\ \|\mathbf{x}_j - \mathbf{y}_j\| \leq t_j - t_l \text{ if } j \leq l. \end{array}\right\} \tag{3.105}$$

Now we see that the support growth described by (3.105) is exactly such that the term $\left[\bigcirc_{k=1}^{a-1} U_{\{j|j\leq k\}}(t_k,t_{k+1}), \mathcal{H}_A(t_A)\right]\psi'(\mathbf{x}_1,...,\mathbf{x}_N) = 0$, whenever $\|\mathbf{x}_j - \mathbf{x}_k\| > \delta + |t_j - t_k|$ holds for all $j \in A, k \notin A$. Thus (3.102) evaluated inside of $\mathscr{S}_\delta$ becomes

$$\left(i\frac{\partial}{\partial t_A}\psi(t_1,...,t_N)\right)(\mathbf{x}_1,...,\mathbf{x}_N) = \left(\mathcal{H}_A(t_A)\psi(t_1,...,t_N)\right)(\mathbf{x}_1,...,\mathbf{x}_N), \tag{3.106}$$

which proves that ψ indeed is a solution of the multi-time system (3.22). □

3.3.4 Uniqueness of solutions

Uniqueness of solutions can be proven by induction over the particle number, using the key features of our multi-time system that the Hamiltonians $\mathcal{H}_k$ are self-adjoint and that the propagation speed is bounded by the speed of light (see Lemma 3.6).

Proof. Proof of Theorem 3.2 Let ψ_1, ψ_2 be solutions to (3.22) in the sense of our definition with $\psi_1(0,...,0) = \psi_2(0,...,0) = \psi^0$. Due to linearity, $\omega := \psi_1 - \psi_2$ is a solution to (3.22) with initial value $\omega(0,...,0) = \psi^0 - \psi^0 = 0$. In particular, the point-wise evaluations of ω as in (3.21) are also well-defined. By induction over $L \in \{1,...,N\}$, we prove the statement:

$\mathbf{A}(\mathbf{L})$: *At all points* $(t_1, \mathbf{x}_1, ..., t_N, \mathbf{x}_N) \in \mathscr{S}_\delta$ *with at most* L *different time coordinates, we have* $(\omega(t_1, ..., t_N))(\mathbf{x}_1, ..., \mathbf{x}_N) = 0$.
For the **base case** $\mathbf{A}(\mathbf{1})$, we consider configurations with all times equal, where ω satisfies

$$i\partial_t \omega(t, ..., t) = \mathcal{H}_{\{1,...,N\}}(t)\omega(t, ..., t). \tag{3.107}$$

By the uniqueness statement in Theorem 3.3, this implies

$$\omega(t, ..., t) = U_{\{1,...,N\}}(t, 0)\omega^0 = 0. \tag{3.108}$$

$\mathbf{A}(\mathbf{L}) \Longrightarrow \mathbf{A}(\mathbf{L+1})$: We assume that $A(L)$ holds, and let $(t_1, \mathbf{x}_1, ..., t_N, \mathbf{x}_N) \in \mathscr{S}_\delta$ with exactly $L+1$ different time coordinates. This means there is a unique partition of $\{1, ..., N\}$ into disjoint sets $\Pi_1, ..., \Pi_{L+1}$ which groups together particles with the same time coordinate in an ascending way:

$$\begin{aligned}
\Pi_1 &:= \left\{ j \in \{1, ..., N\} \middle| t_j = \min_{k \in \{1,...,N\}} t_k \right\} \\
\Pi_2 &:= \left\{ j \in \{1, ..., N\} \middle| t_j = \min_{k \in \{1,...,N\} \setminus \Pi_1} t_k \right\} \\
&\dots \\
\Pi_m &:= \left\{ j \in \{1, ..., N\} \middle| t_j = \min_{k \in \{1,...,N\} \setminus \cup_{i=1}^{m-1} \Pi_i} t_k \right\}.
\end{aligned} \tag{3.109}$$

Denote the largest time by t_{L+1} and the second largest one by t_L. We define the backwards lightcone with respect to the particles in Π_{L+1} as follows,

$$B := \left\{ (y_1, ..., y_N) \in \mathbb{R}^{4N} \;\middle|\; \begin{array}{c} y_j = x_j \text{ if } j \notin \Pi_{L+1} \\ \forall j \in \Pi_{L+1} : y_j^0 = \tau \text{ with } t_L \le \tau \le t_{L+1}, \\ |\mathbf{y}_j - \mathbf{x}_j| \le t_{L+1} - \tau \end{array} \right\}. \tag{3.110}$$

We show that $B \subset \mathscr{S}_\delta$. If $(y_1, ..., y_N) \in B$, consider $j \in \Pi_{L+1}$ and $k \notin \Pi_{L+1}$, then

$$\begin{aligned}
|y_k^0 - y_j^0| + \delta &= \tau - t_k + \delta = (\tau - t_{L+1}) + (t_{L+1} - t_k + \delta) \\
&< -|\mathbf{y}_j - \mathbf{x}_j| + |\mathbf{x}_k - \mathbf{x}_j| \le |\mathbf{x}_k - \mathbf{y}_j| = |\mathbf{y}_k - \mathbf{y}_j|.
\end{aligned} \tag{3.111}$$

Thus, all points in B are still in our domain $\mathscr{S}_\delta$. In particular, we have

$$\left(i\partial_\tau \omega(y_1^0, ..., y_N^0)\right)(\mathbf{y}_1, ..., \mathbf{y}_N) = \left(\mathcal{H}_{\Pi_{L+1}}(\tau)\omega(y_1^0, ..., y_N^0)\right)(\mathbf{y}_1, ..., \mathbf{y}_N) \quad \forall (y_1, ..., y_N) \in B. \tag{3.112}$$

Since B contains the domain of dependence, i.e. the set that uniquely determines the value of ω at $(t_1, \mathbf{x}_1, ..., t_N, \mathbf{x}_N)$ according to Lemma 3.6, Theorem 3.3 tells us that

$$\omega(x_1, ..., x_N) = \left(U_{\Pi_{L+1}}(t_{L+1}, t_L)\omega^{t_L}\right)(\mathbf{x}_1, ..., \mathbf{x}_N), \tag{3.113}$$

where ω^{t_L} denotes the function ω evaluated at the time coordinates as in $(t_1, ..., t_N)$ but where t_{L+1} is replaced by t_L. This only has L different times and is thus given according to the induction hypothesis $A(L)$ as $\omega^{t_L} = 0$ in the whole domain of dependence. Consequently,

$$(\omega(t_1, ..., t_N))(\mathbf{x}_1, ..., \mathbf{x}_N) = 0, \tag{3.114}$$

which concludes the uniqueness proof. □

3.3.5 Interaction

To illustrate that our model is indeed interacting, we now derive the precise version of Eq. (3.8) for our model, the Ehrenfest equation for the scalar field operator. It features a "source term" on the right hand side given by the smoothed delta-functions at the locations of the particles. The two factors of ρ in the double convolution $\rho * *\delta$ arise like this: One is included in (3.115) in the operator $\varphi(t, \mathbf{x})$ itself, the other one really comes from the smoothing by ρ of the interaction term in our model.

Lemma 3.14. *For every $t \in \mathbb{R}$ and $\mathbf{x} \in \mathbb{R}^3$, let us abbreviate the solution to given initial values $\psi^0 \in \mathscr{D}$ at equal times as $\psi^t := U_{\{1,...,N\}}(t, 0)\psi^0$ and $\mathcal{H}^t := \mathcal{H}_{\{1,...,N\}}(t)$ and write $\varphi(t, \mathbf{x})$ for the field operator acting as*

$$\varphi(t, \mathbf{x})\psi := \int d^3\mathbf{k} \left[\left(\frac{\hat{\rho}(\mathbf{k})}{\sqrt{\omega(\mathbf{k})}} e^{-i\omega(\mathbf{k})t} e^{i\mathbf{k}\cdot\mathbf{x}} a(\mathbf{k}) + \frac{\hat{\rho}^\dagger(\mathbf{k})}{\sqrt{\omega(\mathbf{k})}} e^{i\omega(\mathbf{k})t} e^{-i\mathbf{k}\cdot\mathbf{x}} a^\dagger(\mathbf{k}) \right) \psi \right]. \quad (3.115)$$

with field mass $\mu = 0$. Then, the following equation holds:

$$\Box \left\langle \psi^t, \varphi(t, \mathbf{x})\psi^t \right\rangle = -\sum_{k=1}^{N} \left\langle \psi^t, \rho * *\delta(\hat{\mathbf{x}}_k - \mathbf{x})\psi^t \right\rangle, \quad (3.116)$$

where $\Box := \partial_t^2 - \triangle_{\mathbf{x}}$, and the double convolution defined as in (3.71) *is here understood as a shorthand notation for*

$$\begin{aligned} \rho * *\delta(\hat{\mathbf{x}}_k - \mathbf{x}) &= \int d^3\mathbf{y}_1 \int d^3\mathbf{y}_2 \ \rho(\mathbf{y}_1)\rho(\mathbf{y}_2)\delta(\hat{\mathbf{x}}_k - \mathbf{y}_1 - (\mathbf{x} - \mathbf{y}_2)). \\ &= \int d^3\mathbf{y}_1 \ \rho(\mathbf{y}_1)\rho(\mathbf{x} - \hat{\mathbf{x}}_k + \mathbf{y}_1). \end{aligned} \quad (3.117)$$

Proof. Let $t \in \mathbb{R}$ and $\mathbf{x} \in \mathbb{R}^3$. The first step just uses that ψ^t solves the Dirac equation,

$$i\partial_t \left\langle \psi^t, \varphi(t, \mathbf{x})\psi^t \right\rangle = \left\langle -\mathcal{H}^t\psi^t, \varphi(t, \mathbf{x})\psi^t \right\rangle + \left\langle \psi^t, \varphi(t, \mathbf{x})\mathcal{H}^t\psi^t \right\rangle + \left\langle \psi^t, i\dot{\varphi}(t, \mathbf{x})\psi^t \right\rangle. \quad (3.118)$$

We already encountered $\dot{\varphi}$, the time-derivative of the operator φ, in the proof of Lemma 3.4. Since $\mathcal{H}^t$ and $\varphi(t, \mathbf{x})$ commute at equal times, only the third summand survives and the second derivative is

$$\begin{aligned} \partial_t^2 \left\langle \psi^t, \varphi(t, \mathbf{x})\psi^t \right\rangle &= -i\partial_t \left\langle \psi^t, i\dot{\varphi}(t, \mathbf{x})\psi^t \right\rangle \\ &= i \left\langle \mathcal{H}^t\psi^t, \dot{\varphi}(t, \mathbf{x})\psi^t \right\rangle - i \left\langle \psi^t, \dot{\varphi}(t, \mathbf{x})\mathcal{H}^t\psi^t \right\rangle + \left\langle \psi^t, \ddot{\varphi}(t, \mathbf{x})\psi^t \right\rangle \\ &= i \left\langle \psi^t, \left[\mathcal{H}^t, \dot{\varphi}(t, \mathbf{x})\right] \psi^t \right\rangle + \left\langle \psi^t, \triangle_{\mathbf{x}}\varphi(t, \mathbf{x})\psi^t \right\rangle. \end{aligned} \quad (3.119)$$

Hence,

$$\Box \left\langle \psi^t, \varphi(t, \mathbf{x})\psi^t \right\rangle = i \left\langle \psi^t, \left[\mathcal{H}^t, \dot{\varphi}(t, \mathbf{x})\right] \psi^t \right\rangle = i \sum_{k=1}^{N} \left\langle \psi^t, \left[\varphi_k(t), \dot{\varphi}(t, \mathbf{x})\right] \psi^t \right\rangle. \quad (3.120)$$

So we need to compute, with the integration variable $x = (\mathbf{x}_1, ..., \mathbf{x}_N)$,

$$\begin{aligned} & i \left\langle \psi^t, [\varphi_k(t), \dot{\varphi}(t, \mathbf{v})] \psi^t \right\rangle \\ &= i \int d^{3N}x \ \psi^{t\dagger}(x) \int \frac{d^3\mathbf{k}}{2\omega(\mathbf{k})} \left(\hat{\rho}^\dagger(\mathbf{k})\hat{\rho}(\mathbf{k})i\omega(\mathbf{k})e^{-i\mathbf{k}(\mathbf{x}_k - \mathbf{v})} - c.c. \right) \psi^t(x) \\ &= -\frac{1}{2} \int d^{3N}x \ \psi^{t\dagger}(x) \int d^3\mathbf{k} \ \hat{\rho}^\dagger(\mathbf{k})\hat{\rho}(\mathbf{k}) \left(e^{-i\mathbf{k}(\mathbf{x}_k - \mathbf{v})} + e^{i\mathbf{k}(\mathbf{x}_k - \mathbf{v})} \right) \psi^t(x). \end{aligned} \quad (3.121)$$

Denoting the function $\mathbf{y} \mapsto \rho(\mathbf{y} + \mathbf{v} - \mathbf{x}_k)$ by β, we have $\hat{\beta}(\mathbf{k}) = \hat{\rho}(\mathbf{k}) e^{i\mathbf{k}(\mathbf{x}_k - \mathbf{v})}$. Thus, the above formula can be rewritten with the help of the Plancherel theorem,

$$\begin{aligned}
&= -\frac{1}{2} \int d^{3N}x \; \psi^{t\dagger}(x) \left(\left\langle \hat{\rho}, \hat{\beta} \right\rangle_{L^2(\mathbb{R}^3)} + \left\langle \hat{\beta}, \hat{\rho} \right\rangle_{L^2(\mathbb{R}^3)} \right) \psi^t(x) \\
&= -\frac{1}{2} \int d^{3N}x \; \psi^{t\dagger}(x) \left(\langle \rho, \beta \rangle_{L^2(\mathbb{R}^3)} + \langle \beta, \rho \rangle_{L^2(\mathbb{R}^3)} \right) \psi^t(x) \\
&= - \int d^{3N}x \; \psi^{t\dagger}(x) \, \langle \rho, \beta \rangle_{L^2(\mathbb{R}^3)} \, \psi^t(x). \\
&= - \int d^{3N}x \; \psi^{t\dagger}(x) \int d^3\mathbf{y}_1 \rho(\mathbf{y}_1) \rho(\mathbf{v} - \mathbf{x_k} + \mathbf{y}_1) \psi^t(x)
\end{aligned} \tag{3.122}$$

We have used that ρ and β are real-valued. The result contains the term we wrote as $\rho * *\delta(\mathbf{x}_k - \mathbf{v})$ in (3.117). Inserting this into (3.120) gives

$$\begin{aligned}
\Box \left\langle \psi^t, \varphi(t, \mathbf{x}) \psi^t \right\rangle &= - \sum_{k=1}^{N} \int d^{3N}x \; \psi^{t\dagger}(x) \int d^3\mathbf{y}_1 \rho(\mathbf{y}_1) \rho(\mathbf{x} - \mathbf{x_k} + \mathbf{y}_1) \psi^t(x) \\
&\equiv - \sum_{k=1}^{N} \left\langle \psi^t, \rho * *\delta(\hat{\mathbf{x}}_k - \mathbf{x}) \psi^t \right\rangle,
\end{aligned} \tag{3.123}$$

which concludes the proof. □

3.4 Perspective

In this chapter, a rigorous formulation of the model by Dirac, Fock, and Podolsky [4] that was crucial for the development of QED is achieved. Although we use a scalar field for simplicity, the methods for proving well-posedness presented in this chapter are also applicable to an electromagnetic four-potential A^μ.

In our proofs, we demonstrate how cut-offs can be mathematically implemented in the multi-time formalism. One central step for the solution theory is to control the regularity of solutions to each single-time equation, which can be done in Lemma 3.8 by commutator methods, with the help of a theorem by Huang [49]. For future work, it would be desirable to also show that the solution ψ is differentiable in a classical sense in time directions, but there is surprisingly little literature on this point, even for the one-particle Dirac equation (one result is found in [41]).

Concerning the dynamics of multi-time wave functions, the model considered here is a logical step towards a Schrödinger picture formulation of QED. As explained in the previous chapter, the non-relativistic way of introducing interaction, i.e. by adding interaction potentials $V(x_j, x_k)$ to the Hamiltonian, is not suitable for multi-time wave functions because the consistency condition (1.20) usually fails there. In contrast, we have now seen that Dirac's idea of using a second quantized field indeed leads to consistent interacting dynamics. The consistency condition is satisfied on space-like configurations, but not on all configurations since the field modes traveling with (at most) the speed of light are able to entangle the electrons.

The way the interaction is introduced, however, forces us to introduce a cut-off function $\rho \in C_c^\infty(\mathbb{R}^3, \mathbb{R})$ that is not physical and leads to broken Lorentz invariance on the small scale δ. This kind of problems is common in relativistic quantum mechanics and new methods to overcome it should be probed, like the method of interior-boundary conditions employed e.g. in [55, 11]. This will be done in the subsequent chapter.

An alternative possibility is to use direct interaction models in the spirit of Wheeler-Feynman-electrodynamics, see [23, 24]. This seems to be a natural way of moving on from the model of Dirac, Fock, and Podolsky, since the "field" here is not much more than a mathematical tool to formulate the interaction of the electrons, and the probability density of electrons is supposed to be given by $\|\psi\|^2_{\mathscr{H}}$, which implies "integrating out" the field degrees of freedom.

CHAPTER
FOUR

INTERIOR-BOUNDARY CONDITIONS FOR MULTI-TIME WAVE FUNCTIONS

This chapter is based on the paper [11] by Matthias Lienert and the present author, with a slightly changed introduction and an additional new section 4.8. It treats another possibility to introduce interaction in multi-time systems, which is the formulation of creation and annihilation of particles by means of interior-boundary conditions. While we had to use an ultraviolet cut-off to make the model by Dirac, Fock, Podolsky in the preceding chapter well-defined, we now present a method which makes this unnecessary.

4.1 Introduction to interior-boundary conditions

The creation and annihilation of particles is a key feature of quantum field theory (QFT); however, it is usually connected with the problem of ultraviolet (UV) divergences. Interior-boundary conditions (IBCs) have been introduced as a possible way to circumvent the UV problem. In the IBC approach (see [5, 6] for an introduction), one uses a wave function φ on the configuration space $\mathcal{Q}$ of a variable number of particles. Consider, for example, $\mathcal{Q} = \mathcal{Q}^{(1)} \cup \mathcal{Q}^{(2)}$, where $\mathcal{Q}^{(1)} = \mathbb{R}^d$ and $\mathcal{Q}^{(2)} = \{(\mathbf{x}_1, \mathbf{x}_2) \in (\mathbb{R}^d)^2 : \mathbf{x}_1 \neq \mathbf{x}_2\}$. The wave function can be represented as $\varphi = \big(\varphi^{(1)}, \varphi^{(2)}\big)$ where $\varphi^{(1)}$ is a single-particle wave function and $\varphi^{(2)}$ a two-particle wave function. An IBC then is a condition relating the value of $\varphi^{(2)}$ at a boundary point of $\mathcal{Q}$, here $(\mathbf{x}, \mathbf{x}) \in \partial\mathcal{Q}$, with the value of $\varphi^{(1)}$ at a suitable interior point, here at $\mathbf{x}$.

The most important role of IBCs is to ensure that probability is transferred between the sectors of n and $n+1$ particles while the total probability remains conserved. Such a transfer of probability corresponds to particle creation and annihilation. In the example, it describes the process in which two particles can merge into a single one at a point $\mathbf{x}$ or, conversely, a single particle at $\mathbf{x}$ can split up into two. A certain notion of locality is important here: IBCs must only relate boundary points $q \in \partial\mathcal{Q}$ with interior points $q' \in \mathring{\mathcal{Q}}$ which canonically correspond to each other in the language of particle creation and annihilation. This means that $q = (\mathbf{x}, \mathbf{x})$ must be related with $q' = \mathbf{x}$, not with any other point.

Given this relation to particle creation and annihilation, it is not surprising that IBCs can also be derived from Hamiltonians which involve creation and annihilation operators. The creation operator can usually not be densely defined since it involves a delta function $\delta^{(d)}(\mathbf{x}_1 - \mathbf{x}_2)$. One can, however, obtain a boundary condition from the delta function by integrating over the Schrödinger equation, say in $\mathbf{x}_2$ around the point $\mathbf{x}_1$. In this way, one

arrives at a boundary condition that involves the wave function at points $(\mathbf{x}_1, \mathbf{x}_1) \in \partial\mathcal{Q}$ and $\mathbf{x}_1 \in \mathring{\mathcal{Q}}$, i.e., at an IBC.
Previous works [5, 6, 55, 56, 57, 58, 59] have focused on this relation of IBCs and non-relativistic Hamiltonians with particle creation and annihilation operators. It has been shown for simple models that understanding the creation part of the Hamiltonian as defining an IBC allows to make these models rigorous without the need for renormalization, which is usually required to treat the UV divergence. What is more, it is possible to explicitly state a well-defined version of the initial Hamiltonian, its domain being restricted by the IBC.
While these results seem promising, they have so far only been established for models with $-\triangle$ as the free Hamiltonian. To have an impact on more realistic QFTs, it is important to extend them to a relativistic setting. Among other things, this requires using the Dirac operator instead of the Laplacian to describe fermions. Besides, the idea of IBCs is intimately connected with the particle-position representation of the quantum state. We can draw a direct connection to the multi-time picture since this offers the covariant version of the particle-position representation that a relativistic treatment of IBCs requires. In the context of QFT, the multi-time wave function ψ can be represented as a sequence of n-particle wave functions, $\psi = (\psi^{(0)}, \psi^{(1)}, \psi^{(2)}, ...)$.
Of course, several important questions remain, such as:

(a) How can the idea of a probability flux between sectors of different particle numbers be formulated in the multi-time picture?

(b) How can IBCs be made compatible with the dynamics for a multi-time wave function?

To clarify this complex of questions constitutes the main goal of the chapter. As IBCs concern the rigorous formulation of QFTs, it is also crucial to prove the existence and uniqueness of solutions. There is, at present, no hope to do this in great generality, so we instead take an exploratory approach. We identify the simplest nontrivial model which still shows the main features we are setting out to treat (Dirac operators, multi-time wave functions, particle creation and annihilation): a system of a variable number of at most N indistinguishable, massless Dirac particles in 1+1 spacetime dimensions. At this example we explain how to rigorously address the above-mentioned questions a) and b) about the multi-time formulation of IBCs. We expect that the resulting developments can also be transferred to more general relativistic QFTs.

Structure of the chapter. We start by reviewing the basic concepts and results about multi-time wave functions with variable particle numbers which are relevant to our work (Sec. 4.2.1). Next (Sec. 4.2.2), we prove that a beautiful condition in terms of a differential form constructed from the multi-time wave function ensures (local) probability conservation on all Cauchy surfaces (Prop. 4.1). In Sec. 4.3, we introduce our model. This is done for the case of $N = 2$ sectors first, both for comprehensibility and because this case is used as a building block for the model with a general number $N \in \mathbb{N}$ of sectors of Fock space. We then identify a general class of IBCs which leads to local probability conservation (Thm. 4.3) and prove the existence and uniqueness of the model for $N = 2$ (Thm. 4.2). Sec. 4.4 deals with extending these results to general N. Our main result is the existence and uniqueness theorem 4.4. We then discuss the relation of our model with a Hamiltonian with creation and annihilation operators (Sec. 4.5). Lorentz invariance is briefly discussed in Sec. 4.6. Section 4.7 contains the proofs of our theorems. We conclude with a remark

on the existence of a Hamiltonian formulation of our model (Sec. 4.8) and a discussion of the results, including an outlook on possible future directions (Sec. 4.9).

4.2 Multi-time wave functions for variable particle numbers

4.2.1 Review of important concepts

For a variable number of particles, a multi-time wave function becomes a so-called *multi-time Fock function* (see [2] and also [60, 61]). It can be represented as a sequence of n-particle wave functions $\psi^{(n)}$,

$$\psi = (\psi^{(0)}, \psi^{(1)}, \psi^{(2)}, \psi^{(3)}, ...). \tag{4.1}$$

Since the no-particle amplitude $\psi^{(0)}$ has neither time nor space arguments in the multi-time formalism, it is a constant complex number. Thus, we shall disregard $\psi^{(0)}$ for the rest of the chapter and consider $\psi^{(n)}$ only for $n \in \mathbb{N}$, for which

$$\psi^{(n)} : \mathscr{S}^{(n)} \subset (\mathbb{R}^{1+d})^n \to \mathbb{C}^{k(n)}, \quad (x_1, ..., x_n) \mapsto \psi(x_1, ..., x_n). \tag{4.2}$$

Here, $k(n)$ denotes the number of spin components (e.g., $k(n) = 2^n$ for n Dirac particles in $d = 1$). The natural domain of $\psi^{(n)}$ is the set of spacelike configurations, $\mathscr{S}^{(n)}$, see (1.6). The total configuration space is then given by the set

$$\mathscr{S} = \bigcup_{n=1}^{\infty} \mathscr{S}^{(n)} \tag{4.3}$$

of spacelike configurations of a variable number of particles. Note that so far, this is a straightforward relativistic extension of the usual single-time wave function on Fock space with configuration space $\mathcal{Q} = \bigcup_{n=1}^{\infty} \mathbb{R}^{nd}_{\neq}$ where $\mathbb{R}^{nd}_{\neq} = \{(\mathbf{x}_1, ..., \mathbf{x}_n) \in \mathbb{R}^d : \mathbf{x}_i \neq \mathbf{x}_j \, \forall i \neq j)\}$. We will define the dynamics of ψ through a set of n PDEs for each $\psi^{(n)}$, and hence similarly to the multi-time systems treated above:

$$i\partial_{x_k^0} \psi^{(n)}(x_1, ..., x_n) = (\mathcal{H}_k \psi)^{(n)}(x_1, ..., x_n), \quad k = 1, ..., n. \tag{4.4}$$

It is understood that the multi-time equations (4.4) can be rewritten in a manifestly covariant form, as in the example of free Dirac particles:

$$\big(i\gamma_k^\mu \partial_{x_k^\mu} - m\big)\psi^{(n)}(x_1, ..., x_n) = 0, \quad k = 1, ..., n. \tag{4.5}$$

4.2.2 Probability conservation for arbitrary Cauchy surfaces

We explained that multi-time wave functions carry a physical meaning as a probability amplitude for particle detection in 1.4. It has recently been demonstrated [28] that for a wide class of QFTs with local interactions and finite propagation speed, there is a *Born rule for arbitrary Cauchy surfaces* $\Sigma \subset \mathbb{R}^{1+d}$. That means, a suitable quadratic expression $|\psi^{(n)}|^2_\Sigma$ in $\psi^{(n)}$ evaluated along Σ yields the probability density to detect n particles at locations $x_1, ..., x_n \in \Sigma$. For example, for Dirac particles, one has:

$$|\psi^{(n)}|^2_\Sigma(x_1, ..., x_n) \;=\; j^{\mu_1 \cdots \mu_n}(x_1, ..., x_n)\, n_{\mu_1}(x_1) \cdots n_{\mu_n}(x_n), \tag{4.6}$$

where n is the future-pointing normal vector field at Σ and

$$j^{\mu_1 \cdots \mu_n}(x_1, ..., x_n) \;=\; \overline{\psi}^{(n)}(x_1, ..., x_n)\gamma_1^{\mu_1} \cdots \gamma_n^{\mu_n} \psi^{(n)}(x_1, ..., x_n) \tag{4.7}$$

stands for the Dirac tensor current of the n-particle sector. Probability conservation then means

$$\sum_{n=1}^{\infty} \int_{\Sigma^n \cap \mathscr{S}^{(n)}} d\sigma_1(x_1) \cdots d\sigma_n(x_n)\; |\psi^{(n)}|^2_\Sigma(x_1, ..., x_n) = 1 \quad \text{independently of } \Sigma. \tag{4.8}$$

To emphasize that we are dealing with a configuration space with a boundary, we have written $\Sigma^n \cap \mathscr{S}^{(n)}$ for the domain of integration. This boundary $\partial\mathscr{S}^{(n)}$ consists of the light-like configurations of n particles. However, only a subset of $\partial\mathscr{S}^{(n)}$ plays a role for probability conservation here, namely the set of coincidence points,

$$\mathscr{C}^{(n)} = \{(x_1, ..., x_n) \in (\mathbb{R}^{1+d})^n : \exists\, i \neq j : x_i = x_j\}. \tag{4.9}$$

In fact, for $d = 1$ (the case we shall focus on later), the dimension of $\mathscr{C}^{(n)}$ is large enough that probability can get lost through $\mathscr{C}^{(n)}$. Accordingly, there must be conditions on the tensor currents $j^{\mu_1 \cdots \mu_n}$ which ensure that the probability lost in this way gets redistributed to a different sector of Fock space. We shall now work out a suitable local condition which guarantees exactly that.

Before coming to the main result of the section, we introduce for every $n \in \mathbb{N}$ a certain nd-form, the *current form* $\omega^{(n)}$ which is constructed from the tensor currents (see [20, 21] and [7, chap. 1.3]).

$$\omega^{(n)} = \sum_{\mu_1, ..., \mu_n = 0}^{d} (-1)^{\mu_1 + \cdots + \mu_n}\, j^{\mu_1 \cdots \mu_n}\, dx_1^0 \wedge \cdots \widehat{dx_1^{\mu_1}} \cdots \wedge dx_1^d \\ \wedge \cdots \wedge dx_n^0 \wedge \cdots \widehat{dx_n^{\mu_n}} \cdots \wedge dx_n^d, \tag{4.10}$$

where $\widehat{(\cdot)}$ denotes omission. Given $\omega^{(n)}$, the condition for probability conservation can be rewritten as follows.

$$\sum_{n=1}^{\infty} \int_{\Sigma^n \cap \mathscr{S}^{(n)}} \omega^{(n)} = 1 \quad \text{independently of } \Sigma. \tag{4.11}$$

Now we specialize to $d = 1$, denoting spacetime points by $x_i = (t_i, z_i)$. In $d = 1$, the configuration space $\mathscr{S}^{(n)}$ can be greatly simplified if one deals with a single species of indistinguishable particles (fermions). This is because we have

$$\mathscr{S}^{(n)} = \bigcup_{\sigma \in S_n} \mathscr{S}^{(n)}_\sigma, \quad \mathscr{S}^{(n)}_\sigma = \{(t_1, z_1, ..., t_n, z_n) \in \mathscr{S} : z_{\sigma(1)} < \cdots < z_{\sigma(n)}\}. \tag{4.12}$$

where S_n denotes the permutation group. Thus, it is sufficient to consider $\mathscr{S}^{(n)}_1 = \mathscr{S}^{(n)}_{\mathrm{id}}$ as the configuration space of n indistinguishable particles in $d = 1$, and $\mathscr{S}_1 = \bigcup_{n=1}^{\infty} \mathscr{S}^{(n)}_1$ for a variable particle number. The idea is to only formulate the multi-time equations (4.4) and the IBCs on $\mathscr{S}_1$ (and its boundary). Once a solution ψ on $\mathscr{S}_1$ is found, one can obtain an appropriately normalized multi-time wave function $\widetilde{\psi}$ on $\mathscr{S}$ by anti-symmetric extension and normalization of $\psi^{(n)}$ with a factor $\frac{1}{\sqrt{n!}}$. On each $\mathscr{S}^{(n)}_\sigma$:

$$\widetilde{\psi}^{(n)}_{s_1 \ldots s_n}(x_1, ..., x_n) = \frac{\mathrm{sgn}(\sigma)}{\sqrt{n!}}\, \psi^{(n)}_{s_{\sigma(1)} \ldots s_{\sigma(n)}}(x_{\sigma(1)}, ..., x_{\sigma(n)}), \tag{4.13}$$

where s_k is the spin index of the k-th particle.

We are now prepared to prove the condition for probability conservation. For technical reasons, we introduce a highest possible number $N \in \mathbb{N}$ of particles in the system.

Proposition 4.1 (Condition for local probability conservation.)**.** *Let $N \in \mathbb{N}$ and let $j = (j^\mu, j^{\nu\rho}, ..., j^{\mu_1 \cdots \mu_N})$ where each $j^{\mu_1 \cdots \mu_n}$ is given by (4.7). Moreover, for $n = 1, ..., N$, let $j^{\mu_1 \cdots \mu_n}$ be differentiable on $\mathscr{S}_1^{(n)}$ and continuous on $\overline{\mathscr{S}}_1^{(n)}$ such that $j^{\mu_1 \cdots \mu_n}$ is compactly supported on all sets of the form $(\Sigma)^n \cap \mathscr{S}_1^{(n)}$ where $\Sigma \subset \mathbb{R}^2$ is a smooth Cauchy surface. Let $\omega^{(n)}$ denote the n-form given by $j^{\mu_1 \cdots \mu_n}$ as in (4.10). Then:*

(a) Global probability conservation in the sense of

$$\sum_{n=1}^{N} \int_{\Sigma^n \cap \mathscr{S}_1^{(n)}} \omega^{(n)} = 1 \quad \textit{for all Cauchy surfaces } \Sigma \tag{4.14}$$

is ensured by the following condition (local probability conservation)*:*

$$\begin{cases} \mathrm{d}\omega^{(N)} = 0, \\ \mathrm{d}\omega^{(n)} = \sum_{k=1}^{n} {\Phi_k}^* \omega^{(n+1)}, \quad n = 1, ..., N-1. \end{cases} \tag{4.15}$$

Here, Φ_k is defined by ($k = 1, ..., n$):

$$\begin{aligned} \Phi_k : \mathscr{C}_k^{(n+1)} = \{(x_1, ..., x_{n+1}) \in \partial \mathscr{S}_1^{(n+1)} : x_k = x_{k+1}\} &\to \overline{\mathscr{S}}_1^{(n)}, \\ (x_1, ..., x_k, x_k, ..., x_{n+1}) &\mapsto (x_1, ..., x_k, ..., x_{n+1}), \end{aligned} \tag{4.16}$$

and $\big({\Phi_k}^* \omega^{(n+1)}\big)(\cdot) = \omega^{(n+1)}(\Phi_k(\cdot))$ denotes the pullback of $\omega^{(n+1)}$ by Φ_k. Moreover, evaluation of $\omega^{(n+1)}$ along $\mathscr{C}_k^{(n+1)} \subset \partial \mathscr{S}_1^{(n+1)}$ refers to the limit of $\omega^{(n+1)}(q)$ for $q \to \partial \mathscr{S}_1^{(n+1)}$ in $\mathscr{S}_1^{(n+1)}$.

(b) Let $\varepsilon_{\mu\nu}$ denote the Levi-Civita symbol. In terms of the tensor currents $j^{\mu_1 \cdots \mu_n}$, (4.15) is then equivalent to:

$$\begin{cases} \partial_{x_k^{\mu_k}} j^{\mu_1 \cdots \mu_k \cdots \mu_N} = 0 \quad \forall\, k = 1, ..., N \text{ on } \mathscr{S}_1^{(N)}, \\ \varepsilon_{\rho\sigma}\, j^{\mu_1 \cdots \mu_{k-1}\, \rho\, \sigma\, \mu_{k+1} \cdots \mu_n}(x_1, ..., x_k, x_k, ..., x_n) = (-1)^k \partial_{x_k^{\mu_k}} j^{\mu_1 \cdots \mu_k \cdots \mu_n}(x_1, ..., x_n) \\ \forall\, n = 1, ..., N-1,\ k = 1, ..., n,\ \forall\, (x_1, ..., x_n) \in \mathscr{S}_1^{(n)}. \end{cases} \tag{4.17}$$

The proof can be found in Sec. 4.7.1.

Remark. It is remarkable that probability conservation is ensured by the beautiful geometric condition (4.15). We call this condition *local probability conservation.* The adjective "local" deserves some explanation. One can see from the proof that we demand a certain detailed balance between the probability flux of the $(n+1)$-particle sector into the set where two of the $n+1$ points coincide and the global influx into the n-particle sector. We then mean by "local current conservation" that this redistribution of probability happens only between configurations $(x_1, ..., x_k, x_k, ..., x_n) \in \partial \mathscr{S}^{(n+1)}$ and $(x_1, ..., x_k, ..., x_n) \in \mathscr{S}^{(n)}$, i.e., configurations which correspond to each other canonically in the particle-position representation: $(x_1, ..., x_k, x_k, ..., x_n)$ refers to a configuration of $n+1$ particles on a certain Cauchy surface where two of the particles meet, and $(x_1, ..., x_k, ..., x_n)$ is the configuration where just n particles are present but at the same locations. So the straightforward interpretation of what happens here is that two of the particles merge to form a single one, i.e., one of the particles gets annihilated. It becomes evident that (4.15) regulates the way particle creation and annihilation can happen. In more sophisticated theories with exchange particles, we

expect that an analogous condition will hold on configurations where an exchange particle reaches one of the other particles.
In the following, we shall define IBCs as linear relations between the spin components of $\psi^{(n+1)}(x_1, ..., x_k, x_k, ..., x_n)$ and $\psi^{(n)}(x_1, ..., x_k, ..., x_n)$ (for $n \geq 1$) which ensure local probability conservation (4.15). We shall identify a large class of local IBCs for the simplest case of $N = 2$ sectors of Fock space (i.e., a model including the 1 and 2 particle sectors). The results will serve as a building block for the case of a general N.

4.3 A building block: the case $N = 2$

4.3.1 The model

We consider a variable number of at most $N = 2$ indistinguishable Dirac particles in one spatial dimension. This is the simplest non-trivial case where particle creation and annihilation is possible. We have $\psi = (\psi^{(1)}, \psi^{(2)})$ where $\psi^{(1)}$ and $\psi^{(2)}$ are maps of the form (4.2) with $k(n) = 2^n$ spin components on the reduced configuration spaces $\mathscr{S}_1^{(n)}$, see (4.12). Explicitly, we write:

$$\begin{aligned} \psi^{(1)} : \mathbb{R}^2 \to \mathbb{C}^2, \quad \psi^{(1)}(t,z) &= \begin{pmatrix} \psi^{(1)}_-(t,z) \\ \psi^{(1)}_+(t,z) \end{pmatrix}, \\ \psi^{(2)} : \mathscr{S}_1^{(2)} \to \mathbb{C}^4, \quad \psi^{(2)}(t_1,z_1,t_2,z_2) &= \begin{pmatrix} \psi^{(2)}_{--}(t_1,z_1,t_2,z_2) \\ \psi^{(2)}_{-+}(t_1,z_1,t_2,z_2) \\ \psi^{(2)}_{+-}(t_1,z_1,t_2,z_2) \\ \psi^{(2)}_{++}(t_1,z_1,t_2,z_2) \end{pmatrix}. \end{aligned} \tag{4.18}$$

The dynamics is defined as follows. $\psi^{(2)}$ obeys the free multi-time Dirac equations on $\mathscr{S}_1^{(2)}$ (here in Hamiltonian form):

$$i\partial_{t_k}\psi^{(2)}(t_1,z_1,t_2,z_2) = \mathcal{H}_k^{\mathrm{Dirac}}\psi^{(2)}(t_1,z_1,t_2,z_2), \quad k = 1,2, \tag{4.19}$$

where $\mathcal{H}_k^{\mathrm{Dirac}} = -i\gamma_k^0\gamma_k^1\partial_{z_k} + m\gamma_k^0$ is the Dirac Hamiltonian acting on the variables of the k-th particle. $\psi^{(1)}$ evolves according to:

$$i\partial_t\psi^{(1)}(t,z) = \mathcal{H}^{\mathrm{Dirac}}\psi^{(1)}(t,z) - A\psi^{(2)}(t,z,t,z), \tag{4.20}$$

where A is a 2×4 matrix. The term $A\psi^{(2)}(t,z,t,z)$ creates a coupling between the two sectors and allows for a global gain/loss of probability in the 1-particle sector. The matrix A is constrained by current conservation as will be explained in Sec. 4.3.2.
Furthermore, $\psi^{(1)}$ and $\psi^{(2)}$ need to obey the following IBC:

$$\psi^{(2)}_{-+}(t,z,t,z) - e^{i\theta}\psi^{(2)}_{+-}(t,z,t,z) = B\psi^{(1)}(t,z), \tag{4.21}$$

for some $\theta \in [0, 2\pi)$. B is a 1×2 matrix which can be expressed in terms of A (see Sec. 4.3.2). Expressions involving ψ on boundary points $q \in \partial\mathscr{S}$, such as $\psi^{(2)}_{+-}(t,z,t,z)$, denote limits of ψ within $\mathscr{S}_1$ towards the boundary, e.g., $\psi^{(2)}_{+-}(t,z,t,z) = \lim_{\varepsilon\to 0}\psi^{(2)}_{+-}(t,z-\varepsilon,t,z+\varepsilon)$. These limits can be understood in the literal sense; we will consider only continuously differentiable and bounded wave functions in this chapter.
The form of the IBC (4.21) can be motivated as follows.[1] In the case of no coupling between the two sectors (i.e., $A = 0$, $B = 0$), the model corresponds to free motion for the

[1] We have learned about this possibility from Roderich Tumulka (private communication).

1-particle sector and pure delta interactions for the 2-particle sector. Such relativistic delta interactions for multi-time wave functions have been treated in [20] and the appropriate form of the boundary conditions is known from there. If a coupling between the 1-particle and 2-particle sectors is desired, it is natural to include a linear term $B\psi^{(1)}(t,z)$ on the right hand side. Moreover, because a transfer of probability between the two sectors is expected, we need to add a source term to the free Dirac equation for the 1-particle sector. This source term should be linear and can only depend on (t,z). This suggests that it should have the form $A\psi^{(2)}(t,z,t,z)$. (A similar approach has been used in [6, Sec. 2.3] to introduce IBCs in a non-relativistic context.)
In order to simplify the problem, we choose the representation $\gamma^0 = \sigma_1$ and $\gamma^1 = \sigma_1\sigma_3$ where $\sigma_i,\ i = 1,2,3$ denote the Pauli matrices. If we then set $m = 0$, the Dirac Hamiltonian becomes diagonal:

$$\mathcal{H}^{\text{Dirac}} = \begin{pmatrix} -i\partial_z & 0 \\ 0 & i\partial_z \end{pmatrix}. \tag{4.22}$$

This diagonal form makes it possible to use a generalized method of characteristics which has been developed in [20, 21]. For this reason we shall indeed focus on the massless case. As probability conservation does not depend on the value of m, we expect that all of our results also hold for $m > 0$. However, this shall not be shown here.
Our goal is to prove the existence and uniqueness of solutions of the system of equations (4.19), (4.20) and (4.21) with $m = 0$ and initial values given by

$$\psi^{(1)}|_{t=0} = \psi_0^{(1)} \in C_b^1(\mathbb{R}, \mathbb{C}^2), \quad \psi^{(2)}|_{t_1=t_2=0} = \psi_0^{(2)} \in C_b^1(\{(z_1,z_2) \in \mathbb{R}^2, z_1 < z_2\}, \mathbb{C}^4). \tag{4.23}$$

Here, C_b^1 denotes the space of continuously differentiable functions which are bounded and have bounded derivatives. In addition, we require the following compatibility conditions between initial values and the IBC:

$$\psi_{0,-+}^{(2)}(z,z) - e^{i\theta}\psi_{0,+-}^{(2)}(z,z) = B\psi_0^{(1)}(z) \tag{4.24}$$

$$B\left(\mathcal{H}^{\text{Dirac}}\psi_0^{(1)}(z) - A\psi_0^{(2)}(z,z)\right) = i\left(\partial_{z_2} - \partial_{z_1}\right)\left(\psi_{0,-+}^{(2)}(z_1,z_2) + e^{i\theta}\psi_{0,+-}^{(2)}(z_1,z_2)\right)\Big|_{z_1=z_2=z} \tag{4.25}$$

The first condition expresses that the initial data must satisfy the IBC at time $t = 0$. The second condition turns out to be necessary to obtain a C^1 solution.

Theorem 4.2. *Let $T > 0$ and $A \in \mathbb{C}^{2\times 4}, B \in \mathbb{C}^{1\times 2}$ be arbitrary. Then for $t_1, t_2 \in [-T,T]$ there exists a unique C_b^1-solution ψ of the initial boundary value problem* (4.19), (4.20), (4.21) *with given initial values as in* (4.23) *that satisfy* (4.24) *and* (4.25). *We call such a ψ a* global solution.

We shall now determine which matrices A and B lead to local probability conservation. (The existence and uniqueness theorem holds for arbitrary constant matrices A, B.) After that, we compare the form of the equations and IBCs with a more familiar Hamiltonian involving creation and annihilation operators and establish a relation between the two formulations (Sec. 4.3.2). The proof of Thm. 4.2 is given is Sec. 4.7.3.

4.3.2 Probability conservation and IBCs

We need to check which matrices A and B ensure the condition (4.15) (or equivalently (4.17)) for local probability conservation. Eq. (4.17) yields the following two conditions.

For $n = 2$:

$$\partial_{x_1^\mu} j^{\mu\nu}(x_1, x_2) = 0 = \partial_{x_2^\nu} j^{\mu\nu}(x_1, x_2) \quad \text{on } \mathscr{S}_1^{(2)}, \tag{4.26}$$

and for $n = 1$:

$$(j^{01} - j^{10})(x, x) = -\partial_\mu j^\mu(x) \quad \forall\, x \in \mathbb{R}^2, \tag{4.27}$$

where $j^{\mu\nu} = \overline{\psi}^{(2)} \gamma_1^\mu \gamma_2^\nu \psi^{(2)}$ and $j^\mu = \overline{\psi}^{(1)} \gamma^\mu \psi^{(1)}$.
Now, (4.26) is already ensured by the free multi-time Dirac equations (4.19) for the two-particle sector. (This can be verified easily using (4.19) and its adjoint equation.) It will be the role of the IBC (4.21) to ensure (4.27). We now calculate both sides of (4.27) in detail to see which relation the matrices A and B need to satisfy. On the one hand, we have:

$$\partial_\mu j^\mu(x) \overset{(4.20)}{=} -2\Im \left(\psi^{(1)\dagger}(x)\, A \psi^{(2)}(x, x) \right). \tag{4.28}$$

On the other hand, the two-particle flow out of the set of coincidence points is given by [20]:

$$\big(j^{01} - j^{10}\big)(x, x) = 2\big(|\psi^{(2)}_{+-}|^2 - |\psi^{(2)}_{-+}|^2\big)(x, x). \tag{4.29}$$

Thus, condition (4.27) becomes:

$$\Im \left(\psi^{(1)\dagger}(x)\, A\, \psi^{(2)}(x, x) \right) = \big(|\psi^{(2)}_{+-}|^2 - |\psi^{(2)}_{-+}|^2\big)(x, x). \tag{4.30}$$

It is the content of the following theorem to identify a general class of IBCs which ensure this condition.

Theorem 4.3. *The most general translation invariant class of IBCs of the form* (4.21) *which ensures* (4.30) *(and hence* (4.17)*) for the model* (4.19), (4.20) *is given by a phase* $\theta \in [0, 2\pi)$ *and constant matrices* A, B *defined as follows.*

$$A^\dagger = \begin{pmatrix} 0 & 0 \\ \widetilde{A} & \\ 0 & 0 \end{pmatrix}, \tag{4.31}$$

where $\widetilde{A}$ *is a complex* 2×2 *matrix of the form*

$$\widetilde{A} = \begin{pmatrix} w_1 & w_2 \\ w_1 e^{i\phi} & w_2 e^{i\phi} \end{pmatrix} \tag{4.32}$$

with $w_1, w_2 \in \mathbb{C}$ *and* $\phi \in [0, 2\pi)$.
Furthermore, B *can be expressed completely in terms of* $\widetilde{A}$ *as*

$$B = \frac{1}{2i}\, (1, e^{i\theta}) \widetilde{A}. \tag{4.33}$$

The proof is given in Sec. 4.7.2.

Remarks.

(a) The IBC (4.21) describes the interaction effect of the annihilation of two particles into one if they meet (and conversely the creation of two particles out of one). It seems reasonable that the interaction between any two particles should be of the same form, regardless of which two particles $k, k+1$ meet, of the particle number of the sectors

that are considered and of the total number of sectors. Using this principle, the form of the matrices A and B from Theorem 4.3 will be a crucial building block for a model with N sectors.

(b) *Spin index notation.* It is helpful to write the matrices A and B using spin index notation. We have:

$$B = B^s \quad \text{and} \quad A = A^{tu}_s, \tag{4.34}$$

where $s, t, u = \pm 1$. An upper index means that the respective matrix will be contracted with a respective lower spin index of ψ. A lower index indicates that the matrix times ψ will have that spin index in addition to the spin indices which do not get summed over.

4.4 The case of $N > 2$ sectors of Fock space

We now generalize both the dynamics as well as the existence and uniqueness results to the case of $N > 2$ sectors of Fock space. The wave function then has the form $\psi = \big(\psi^{(1)}, ..., \psi^{(N)}\big)$, where each $\psi^{(n)}$ is a map

$$\psi^{(n)} : \mathscr{S}_1^{(n)} \to \mathbb{C}^{2^n}, \quad (t_1, z_1; ...; t_n, z_n) \mapsto \begin{pmatrix} \psi^{(n)}_{-...--}(t_1, z_1; ...; t_n, z_n) \\ \psi^{(n)}_{-...-+}(t_1, z_1; ...; t_n, z_n) \\ \psi^{(n)}_{-...+-}(t_1, z_1; ...; t_n, z_n) \\ \vdots \\ \psi^{(n)}_{+...++}(t_1, z_1; ...; t_n, z_n) \end{pmatrix}. \tag{4.35}$$

For readability, we shall sometimes use semicolons to divide space-time arguments associated with different particle indices. As evolution equations, we consider multi-time Dirac equations on $\mathscr{S}_1^{(n)}$ (here in Hamiltonian form)

$$\begin{aligned} i\partial_{t_k}\psi^{(n)} = H_k^{\text{Dirac}}\psi^{(n)} + f_k^{(n)} \quad &\Longleftrightarrow \quad i(\partial_{t_k} - s_k\partial_{z_k})\psi^{(n)}_{s_1...s_n} = f^{(n)}_{k,s_1...s_n}, \\ &n = 1, ..., N;\ k = 1, ..., n;\ s_1, ..., s_n = \pm 1. \end{aligned} \tag{4.36}$$

According to the remark at the end of the previous section, the source terms are given by:

$$f^{(n)}_{k,s_1...s_n}(x_1, ..., x_n) = (-1)^k \sum_{t,u=\pm 1} A^{tu}_{s_k} \psi^{(n+1)}_{s_1...s_{k-1}\, t\, u\, s_{k+1}...s_n}(x_1, ..., x_k, x_k, x_{k+1}, ..., x_n) \tag{4.37}$$

for $n = 1, ..., N-1$, and

$$f^{(N)}_{k,s_1...s_N} \equiv 0 \quad \forall k. \tag{4.38}$$

In addition, there is an interior-boundary condition. For some $\theta \in [0, 2\pi)$,

$$\begin{aligned} &\Big(\psi^{(n+1)}_{s_1...s_{k-1}-+s_{k+1}...s_n} - e^{i\theta}\psi^{(n+1)}_{s_1...s_{k-1}+-s_{k+1}...s_n}\Big)(x_1, ..., x_k, x_k, x_{k+1}, ..., x_n) \\ &= \sum_s B^s \psi^{(n)}_{s_1...s_{k-1}\, s\, s_{k+1}...s_n}(x_1, ..., x_n) \end{aligned} \tag{4.39}$$

for all $n = 1, ..., N-1$, all $k = 1, ..., n$, all spin components $s_1, ..., s_{k-1}, s_{k+1}, ..., s_n = \pm 1$ and all $x_1, ..., x_n \in \overline{\mathscr{S}}_1^{(n)}$. A, B are the same matrices as for $N = 2$ (see Thm. 4.3).
Initial data are given by

$$\psi^{(n)}|_{t_1=...=t_n=0} = \psi_0^{(n)} \in C_b^1(Z_n, \mathbb{C}^{2^n}), \tag{4.40}$$

where

$$Z_n := \{(z_1, ..., z_n) \in \mathbb{R}^n | z_1 < ... < z_n\} \tag{4.41}$$

and, as before, C_b^1 denotes the set of continuously differentiable functions which are bounded and have bounded partial derivatives.
The initial data have to satisfy the following compatibility conditions (which are the analogs of Eqs. (4.24) and (4.25) for general N).

$$\begin{aligned}
&\left(\psi^{(n+1)}_{0,s_1...s_{k-1}-+s_{k+1}...s_n} - e^{i\theta}\psi^{(n+1)}_{0,s_1...s_{k-1}+-s_{k+1}...s_n}\right)(z_1, ..., z_k, z_k, z_{k+1}, ..., z_n) \\
&= \sum_s B^s \psi^{(n)}_{0,s_1...s_{k-1}s\, s_{k+1}...s_n}(z_1, ..., z_n), \qquad (4.42) \\
&\sum_s B^s \left(H_k^{\text{Dirac}} \psi^{(n)}_{s_1...s_{k-1}ss_{k+1}...s_n} + f^{(n)}_{s_1...s_{k-1}ss_{k+1}...s_n}\right)(0, z_1; ...0, z_{k-1}; 0, z; 0, z_{k+2}...; 0, z_{n+1}) \\
&= i\left(\partial_{z_k} - \partial_{z_{k+1}}\right)\left(\psi^{(n+1)}_{0,s_1...s_{k-1}-+s_{k+1}...s_n} + e^{i\theta}\psi^{(n+1)}_{0,s_1...s_{k-1}+-s_{k+1}...s_n}\right)(z_1, ..., z_{n+1})\Big|_{z_k=z_{k+1}=z} \qquad (4.43)
\end{aligned}$$

The first condition ensures compatibility of initial data and IBC; the second condition is needed to obtain a C^1-solution.
We consider the following function spaces for $\psi^{(n)}$:

$$\mathscr{B}_n := C_b^1(\{(t_1, z_1; ...; t_n, z_n) \in \mathscr{S}_1^{(n)} : t_k \in [0, T], (z_1, ..., z_n) \in Z_n\}, \mathbb{C}^{2^n}), \tag{4.44}$$

In contrast to $N = 2$, we only admit positive times to avoid technical complications.
Accordingly, ψ is an element of

$$\mathscr{B} := \bigoplus_{n=1}^{N} \mathscr{B}_n. \tag{4.45}$$

Our main results are the following theorems, the first one about the existence and uniqueness of solutions and the second one about probability conservation.

Theorem 4.4 (Existence and uniqueness of solutions.)**.** *Let $T > 0$. Then for all initial data given by* (4.40) *with* (4.42) *and* (4.43) *and for all $0 \leq t_1, ..., t_N \leq T$, there exists a unique solution $\psi \in \mathscr{B}$ of the initial boundary value problem* (4.36)–(4.39)*.*

The proof is given in Sec. 4.7.4.

Theorem 4.5 (Local probability conservation.)**.** *Let $\theta \in [0, 2\pi)$ and let A and B be the matrices from Thm. 4.3. Then the IBCs* (4.39) *ensure local probability conservation in the sense of* (4.15)*.*

The proof can be found in Sec. 4.7.5.
Next, we establish a relation between the equations of our model and the usual formulation of QFTs via creation and annihilation operators.

4.5 Relation to creation/annihilation operators

Usually, one introduces QFTs using creation and annihilation operators. In the previous sections, we have chosen a different way. It is, therefore, important to connect the two approaches. In order to do this, we now consider the single-time version of our model,

so that the multi-time wave function ψ reduces to the single-time wave function φ in the Schrödinger picture of QFT. To obtain φ from ψ when ψ is defined only on $\mathscr{S}_1$, we need to combine (4.13) and (1.7). For $(t, z_1, ..., t, z_N) \in \mathscr{S}_\sigma^{(n)}$, we have:

$$\varphi^{(n)}_{s_1 \ldots s_n}(t; z_1, ..., z_n) = \frac{\mathrm{sgn}(\sigma)}{\sqrt{n!}}\, \psi^{(n)}_{s_{\sigma(1)} \ldots s_{\sigma(n)}}(t, z_{\sigma(1)}, ..., t, z_{\sigma(n)}). \tag{4.46}$$

This allows us to identify the interaction part of the Hamiltonian for the single-time version of our model. The annihilation terms in the n-th sector are obtained as the sum over the source terms in (4.37), i.e.

$$\begin{aligned}
&(\mathcal{H}^{\mathrm{ann}}_{\mathrm{int}}\varphi)^{(n)}_{s_1 \ldots s_n}(t; z_1, ..., z_n) \\
&= \sqrt{n+1} \sum_{k=1}^{n} \sum_{t,u=\pm 1} (-1)^k A^{tu}_{s_k} \varphi^{(n+1)}_{s_1 \ldots s_{k-1} t u s_{k+1} \ldots s_n}(t; z_1, ..., z_k, z_k, z_{k+1}, ..., z_n),
\end{aligned} \tag{4.47}$$

for $1 \leq n \leq N-1$. The factor of $\sqrt{n+1}$ comes from (4.46).
In the particle-position representation, the creation and annihilation operators are given as [62]

$$(a_r(z)\varphi)^{(n)}_{s_1 \ldots s_n}(t; z_1, ..., z_n) = \sqrt{n+1}\ \varphi^{(n+1)}_{r\, s_1 \ldots s_n}(t; z, z_1, ..., z_n), \tag{4.48}$$

$$(a^\dagger_r(z)\varphi)^{(n)}_{s_1 \ldots s_n}(t; z_1, ..., z_n) = \frac{1}{\sqrt{n}} \sum_{k=1}^{n} (-1)^{k+1} \delta_{s_k r} \delta(z - z_k) \varphi^{(n-1)}_{s_1 \ldots \widehat{s_k} \ldots s_n}(t; z_1, ..., \widehat{z_k}, ..., z_n). \tag{4.49}$$

This allows us to rewrite $\mathcal{H}^{\mathrm{ann}}_{\mathrm{int}}$ as follows:

$$(\mathcal{H}^{\mathrm{ann}}_{\mathrm{int}}\varphi)^{(n)} = \left(\int dz \sum_{rst} A^{st}_r a^\dagger_r(z) a_s(z) a_t(z) \varphi \right)^{(n)}, \quad 1 \leq n \leq N-1. \tag{4.50}$$

Now, the creation part $\mathcal{H}^{\mathrm{cre}}_{\mathrm{int}}$ of the Hamiltonian cannot be read off straightforwardly from (4.37). (As we will see, the reason is that it is not a well-defined quantity as it contains δ-functions.) We can, however, obtain $\mathcal{H}^{\mathrm{cre}}_{\mathrm{int}}$ by taking the adjoint of $\mathcal{H}^{\mathrm{ann}}_{\mathrm{int}}$:

$$\mathcal{H}^{\mathrm{cre}}_{\mathrm{int}} = (\mathcal{H}^{\mathrm{ann}}_{\mathrm{int}})^\dagger = \int dz \sum_{rst} (A^{st}_r)^* a^\dagger_t(z) a^\dagger_s(z) a_r(z). \tag{4.51}$$

The action of $\mathcal{H}^{\mathrm{cre}}_{\mathrm{int}}$ on wave functions is

$$\begin{aligned}
(\mathcal{H}^{\mathrm{cre}}_{\mathrm{int}}\varphi)^{(n)}_{s_1 \ldots s_n}(t; z_1, ..., z_n) = \sum_{\substack{j,k=1 \\ j \neq k}}^{n} \sum_{r} (-1)^{j+1} \frac{1}{\sqrt{n}} (A^{s_k s_j}_r)^* \delta(z_j - z_k) \times \\
\times \varphi^{(n-1)}_{s_1 \ldots \widehat{s_j} \ldots r \ldots s_n}(t; z_1, ..., \widehat{z_j}, ..., z_k, ..., z_n).
\end{aligned} \tag{4.52}$$

This is indeed not well-defined because the δ-distribution is not an element of L^2.
We shall now show that our model of Sec. 4.4 is a rigorous (and multi-time) version of the single-time model with Hamiltonian $\mathcal{H} = \mathcal{H}^{\mathrm{free}} + \mathcal{H}_{\mathrm{int}}$. Here, $\mathcal{H}^{\mathrm{free}}$ is the free Dirac Hamiltonian on the Fock space of a variable number of $1 \leq n \leq N$ particles and the interaction Hamiltonian is given by $\mathcal{H}_{\mathrm{int}} = \mathcal{H}^{\mathrm{ann}}_{\mathrm{int}} + \mathcal{H}^{\mathrm{cre}}_{\mathrm{int}}$, i.e.

$$\mathcal{H}_{\mathrm{int}} = \int dz \sum_{rst} \left(A^{st}_r a^\dagger_r(z) a_s(z) a_t(z) + (A^{st}_r)^* a^\dagger_t(z) a^\dagger_s(z) a_r(z) \right). \tag{4.53}$$

We have already seen that the annihilation parts of the two models agree. It remains to study the creation part. To treat this part, we now show at the example $n = 2$ that $\mathcal{H}^{\mathrm{cre}}_{\mathrm{int}}$ gives rise to the IBC when suitably interpreted (for $n > 2$ one proceeds analogously). For $n = 2$, (4.52) becomes:

$$(\mathcal{H}^{\mathrm{cre}}_{\mathrm{int}}\varphi)^{(2)}_{s_1 s_2}(t; z_1, z_2) = \frac{1}{\sqrt{2}} \sum_r \left(-(A_r^{s_1 s_2})^* + (A_r^{s_2 s_1})^*\right) \delta(z_1 - z_2)\varphi_r^{(1)}(z_1). \tag{4.54}$$

Considering (4.31), (4.32), one can see that the creation term vanishes for $s_1 = s_2$, and that $(A_r^{+-})^* = e^{i\phi}(A_r^{-+})^*$. Anti-symmetry of $(\mathcal{H}^{\mathrm{cre}}_{\mathrm{int}}\varphi)^{(2)}_{s_1 s_2}(t; z_1, z_2)$ dictates $\phi = \pi$. We specialize to $s_1 = -1, s_2 = +1$; the reversed case leads to the same conclusions. Then:

$$(\mathcal{H}^{\mathrm{cre}}_{\mathrm{int}}\varphi)^{(2)}_{-+}(t; z_1, z_2) = -\sqrt{2} \sum_r (A_r^{-+})^* \delta(z_1 - z_2)\varphi_r^{(1)}(t; z_1). \tag{4.55}$$

In order to give a proper interpretation to the δ-function, we will integrate the corresponding Dirac equation

$$i\partial_t \varphi^{(2)} = \mathcal{H}^{\mathrm{free}}\varphi^{(2)} + (\mathcal{H}_{\mathrm{int}}\varphi)^{(2)}. \tag{4.56}$$

in a small neighborhood of the set where $z_1 = z_2$. The component with $s_1 = -1, s_2 = +1$ reads

$$i\partial_t \varphi^{(2)}_{-+}(t; z_1, z_2) = \left(-i\partial_{z_1} + i\partial_{z_2}\right) \varphi^{(2)}_{-+}(t; z_1, z_2) + (\mathcal{H}_{\mathrm{int}}\varphi)^{(2)}_{-+}(t; z_1, z_2). \tag{4.57}$$

It is helpful to use relative coordinates $z = z_1 - z_2, Z = \frac{1}{2}(z_1 + z_2)$ because we expect by (4.21) a jump discontinuity of $\varphi^{(2)}$ exactly at $z = 0$. Note that $-i\partial_{z_1} + i\partial_{z_2} = -2i\partial_z$. We integrate (4.57) over dz from $-\varepsilon$ to $+\varepsilon$, and let ε go to zero. All terms vanish except the δ-function and the derivatives w.r.t. z, which means

$$0 = \lim_{\varepsilon \to 0} \int_{-\varepsilon}^{\varepsilon} dz \left(-2i\partial_z \varphi^{(2)}_{-+}(t, Z + z/2, Z - z/2) + (\mathcal{H}^{\mathrm{cre}}_{\mathrm{int}}\varphi)^{(2)}_{-+}(t, Z + z/2, Z - z/2)\right). \tag{4.58}$$

For $s_1 = -1, s_2 = +1$ this becomes, omitting the common time variable t,

$$-i \lim_{\varepsilon \to 0} \left(\varphi^{(2)}_{-+}(Z + \varepsilon, Z - \varepsilon) - \varphi^{(2)}_{-+}(Z - \varepsilon, Z + \varepsilon)\right) = -\frac{1}{\sqrt{2}} \sum_r (A_r^{-+})^* \varphi_r^{(1)}(Z). \tag{4.59}$$

Using the anti-symmetry of φ, we arrive at

$$\lim_{\varepsilon \to 0} \left(\varphi^{(2)}_{-+}(Z - \varepsilon, Z + \varepsilon) + \varphi^{(2)}_{+-}(Z - \varepsilon, Z + \varepsilon)\right) = -\frac{1}{\sqrt{2}i} \left(w_1\, \varphi^{(1)}_-(Z) + w_2\, \varphi^{(1)}_+(Z)\right). \tag{4.60}$$

We now compare this with the IBC (4.21) for $\phi = \pi$. Setting in addition $\theta = \pi$, we have:

$$\begin{aligned} \lim_{\varepsilon \to 0} \left(\varphi^{(2)}_{-+}(Z - \varepsilon, Z + \varepsilon) + \varphi^{(2)}_{+-}(Z - \varepsilon, Z + \varepsilon)\right) &= \frac{1}{\sqrt{2}} B\varphi^{(1)}(Z) = -\frac{1}{2\sqrt{2}i}(1, -1)\tilde{A}\varphi^{(1)}(Z) \\ &= -\frac{1}{\sqrt{2}i} \left(w_1\, \varphi^{(1)}_-(Z) + w_2\, \varphi^{(1)}_+(Z)\right), \end{aligned} \tag{4.61}$$

in agreement with (4.60). The fact that we obtain the IBC (4.21) only for $\theta = \pi$ can be explained as follows. In [21] it was shown that $\theta = \pi$ corresponds to the non-interacting case if no coupling between different sectors is present. This suggests that the creation and

annihilation operators only generate an interaction by particle exchange. Additional point interactions are not visible in $\mathcal{H}_{\text{int}}^{\text{ann}}$ and would have to be added manually to (4.53). The IBC approach, on the other hand, already incorporates the possibility of additional point interactions from the very start.
Next, we discuss the Lorentz invariance of our model.

4.6 Lorentz invariance

In this section we discuss the behavior of our model under (proper) Lorentz transformations. Lorentz invariance here concerns several aspects:

1. Covariance of the wave function,
2. Invariance of the domain,
3. Probability conservation in all Lorentz frames,
4. Invariance of the equations of motion,
5. Invariance of the boundary conditions.

Item 1. is clear because a multi-time wave function is a manifestly covariant object (see Eq. (4.62)). 2. is also ensured as the set $\mathscr{S}$ of spacelike configurations is invariant under Lorentz transformations. ($\mathscr{S}_1$ is also invariant under proper Lorentz transformations but not under reflections.) Concerning 3., we have already established in Prop. 4.1 that our model leads to probability conservation on all Cauchy surfaces which include the equal-time surfaces of all frames. We shall discuss 4. and 5. now.
An element Λ of the proper Lorentz group $\mathcal{L}_+^\uparrow$ in $d = 1$ is a boost in the only existing spatial direction, characterized by a parameter $\beta \in \mathbb{R}$. Under Λ, the multi-time wave function transforms according to

$$\psi^{(n)}(x_1, ..., x_n) \;\overset{\Lambda}{\longmapsto}\; S[\Lambda]^{\otimes n}\psi^{(n)}(\Lambda^{-1}x_1, ..., \Lambda^{-1}x_n), \tag{4.62}$$

where $S[\Lambda]$ are matrices forming a representation of the Lorentz group. In 1+1 dimensions and with our choice of the basis in spin space, we have:

$$S[\Lambda]_s^{s'} \;=\; \delta_s^{s'}\left(\cosh(\beta/2) - s\sinh(\beta/2)\right). \tag{4.63}$$

We shall check whether the transformed wave function ψ' solves the primed versions of Eqs. (4.36) and (4.39). Indeed, as a consequence of these equations one finds:

$$\begin{aligned} i(\partial_{t_k} - s_k\partial_{z_k})\psi'^{(n)}_{s_1\ldots s_n}(x_1, ..., x_n) \;&=\; \sum_{t,u=\pm1} (-1)^k \left(\cosh(\beta/2) - s_k\sinh(\beta/2)\right) A^{tu}_{s_k}\times \\ &\quad\times \psi'^{(n+1)}_{s_1\ldots s_{k-1}tus_{k+1}\ldots s_n}(x_1, ..., x_k, x_k, x_{k+1}, ..., x_n). \end{aligned} \tag{4.64}$$

Here we have used that $A^{tu}_{s_k} = 0$ whenever $t = u$ and that $S[\Lambda]^+_+ S[\Lambda]^-_-$ cancels out because $(\cosh(\beta/2) - \sinh(\beta/2))\,(\cosh(\beta/2) + \sinh(\beta/2)) = 1$. The equation would be Lorentz invariant if the matrix A transformed like a spinor, with its upper indices transformed via $S^{-1}[\Lambda]$, i.e.

$$\begin{aligned} A'^{tu}_{s_k} \;&=\; \left(\cosh(\beta/2) + t\sinh(\beta/2)\right)\left(\cosh(\beta/2) + u\sinh(\beta/2)\right)\left(\cosh(\beta/2) - s_k\sinh(\beta/2)\right) A^{tu}_{s_k} \\ &=\; \left(\cosh(\beta/2) - s_k\sinh(\beta/2)\right) A^{tu}_{s_k}. \end{aligned} \tag{4.65}$$

Since A does not transform in this way, but is a fixed matrix, Lorentz invariance is broken in this regard. The situation is similar for the IBC, where (4.39) implies

$$\begin{aligned}&\left(\psi'^{(n+1)}_{s_1\ldots s_{k-1}-+s_{k+1}\ldots s_n} - e^{i\theta}\psi'^{(n+1)}_{s_1\ldots s_{k-1}+-s_{k+1}\ldots s_n}\right)(x_1,...,x_k,x_k,x_{k+1},...,x_n)\\ &= \sum_s \left(\cosh(\beta/2) + s\sinh(\beta/2)\right) B^s \psi'^{(n)}_{s_1\ldots s_{k-1}\,s\,s_{k+1}\ldots s_n}(x_1,...,x_n)\end{aligned} \tag{4.66}$$

Thus, if B transformed like a spinor, the model would be manifestly Lorentz invariant. One can now clearly see that the only point where Lorentz invariance fails is the occurrence of the constant matrices A and B. This is due to the simplification that we only consider fermions. The matrix A, for example, needs to be introduced to match the number of spin components of $A\psi^{(n+1)}$ to the one of $\psi^{(n)}$. For more realistic QFTs with appropriate types of bosons (e.g. photons) as exchange particles this situation would not occur, and consequently there would be no issue with Lorentz invariance. To formulate such a multi-time IBC model with bosons as exchange particles as well as to address the problems that come along with it (such as the question of a suitable position representation for photons) is left as a task for future work.

4.7 Proofs

4.7.1 Proof of the condition for local probability conservation

Proof of proposition 4.1. We start with statement 1. The proof is based on a technique developed in [20, 21]. Let $\Sigma_1, \Sigma_2 \subset \mathbb{R}^2$ be smooth Cauchy surfaces given by time functions $\tau_i(z) : \mathbb{R} \to \mathbb{R}$, i.e.:

$$\Sigma_i = \{(t,z) \in \mathbb{R}^2 : t = \tau_i(z)\}, \quad i = 1,2. \tag{4.67}$$

We shall show (4.14) in the following form:

$$\sum_{n=1}^{N} \int_{\Sigma_1^n \cap \mathscr{S}_1^{(n)}} \omega^{(n)} = \sum_{n=1}^{N} \int_{\Sigma_2^n \cap \mathscr{S}_1^{(n)}} \omega^{(n)}. \tag{4.68}$$

To show (4.68), we consider each sector n separately and construct a closed surface $S^{(n)}$ to which we can apply Stokes' theorem. As j is compactly supported, we may choose $R > 0$ such that $j^{\mu_1\cdots\mu_n}(x_1,...,x_n) = 0$ if there is at least one $x_k = (t_k, z_k)$ with $|z_k| > R$ and all $n = 1,...,N$. We define

$$\Sigma_i^R = \{(t,z) \in \Sigma_i : |z| < R\}, \quad i = 1,2. \tag{4.69}$$

Then,

$$\int_{\Sigma_i^n \cap \mathscr{S}_1^{(n)}} \omega^{(n)} = \int_{(\Sigma_i^R)^n \cap \mathscr{S}_1^{(n)}} \omega^{(n)}. \tag{4.70}$$

Now consider the configuration spacetime volume

$$V_R^{(n)} = \left\{ (t_1,z_1;...;t_n,z_n) \in \overline{\mathscr{S}}_1^{(n)} \;\middle|\; \exists s \in [0,1] : \begin{array}{l} \forall i : t_i = \tau_1(z_i) + s(\tau_2(z_i) - \tau_1(z_i)) \\ \text{and } |z_i| \leq R \end{array} \right\}. \tag{4.71}$$

$V_R^{(n)}$ is a bounded and closed, hence compact (n+1)-dimensional submanifold of $\mathbb{R}^{2n}$. Its boundary $\partial V_R^{(n)}$ has the form

$$\partial V_R^{(n)} = (\Sigma_1^R)^n \cup (\Sigma_2^R)^n \cup M_R^{(n)} \tag{4.72}$$

and $M_R^{(n)} \subset \overline{\mathscr{S}}_1^{(n)}$ has the two parts

$$M_R^{(n)} = M_{1,R}^{(n)} \cup M_{2,R}^{(n)} \tag{4.73}$$

with

$$M_{1,R}^{(n)} = \{(t_1, z_1; ...; t_n, z_n) \in \partial V_R^{(n)} | \, \exists \, i : |z_i| = R\}, \tag{4.74}$$

hence $j^{\mu_1 \cdots \mu_n} = 0$ on $M_{1,R}^{(n)}$, and

$$M_{2,R}^{(n)} = \{(t_1, z_1; ...; t_n, z_n) \in \partial V_R^{(n)} \mid \exists \, i : (t_i, z_i) = (t_{i+1}, z_{i+1})\}. \tag{4.75}$$

In this situation, we can apply Stokes' theorem to obtain:

$$\int_{\partial V_R^{(n)}} \omega^{(n)} = \int_{V_R^{(n)}} d\omega^{(n)}, \tag{4.76}$$

and, as $j^{\mu_1 \cdots \mu_n} = 0$ on $M_{1,R}^{(n)}$, we find (considering (4.70) as well as orientation conventions):

$$\int_{\Sigma_1^n \cap \mathscr{S}_1^{(n)}} \omega^{(n)} - \int_{\Sigma_2^n \cap \mathscr{S}_1^{(n)}} \omega^{(n)} = \int_{V_R^{(n)}} d\omega^{(n)} - \int_{M_{2,R}^{(n)}} \omega^{(n)}. \tag{4.77}$$

Summation over $n = 1, ..., N$ yields:

$$\sum_{n=1}^{N} \int_{\Sigma_1^n \cap \mathscr{S}_1^{(n)}} \omega^{(n)} - \sum_{n=1}^{N} \int_{\Sigma_2^n \cap \mathscr{S}_1^{(n)}} \omega^{(n)} = \sum_{n=1}^{N} \left(\int_{V_R^{(n)}} d\omega^{(n)} - \int_{M_{2,R}^{(n)}} \omega^{(n)} \right). \tag{4.78}$$

We now show that condition (4.15) makes the right hand side vanish. To this end, note that $M_{2,R}^{(1)} = \emptyset$. Furthermore, we have $d\omega^{(N)} = 0$ by assumption. Thus, if we can show

$$\int_{V_R^{(n)}} d\omega^{(n)} = \int_{M_{2,R}^{(n+1)}} \omega^{(n+1)} \tag{4.79}$$

we obtain a telescoping sum and therefore, indeed

$$\sum_{n=1}^{N} \left(\int_{V_R^{(n)}} d\omega^{(n)} - \int_{M_{2,R}^{(n)}} \omega^{(n)} \right) = 0. \tag{4.80}$$

We turn to the proof of (4.79). Considering (4.75), we have:

$$M_{2,R}^{(n+1)} = \bigcup_{k=1}^{n} R_k^{(n+1)}, \tag{4.81}$$

where

$$R_k^{(n+1)} = \{(t_1, z_1; ...; t_{n+1}, z_{n+1}) \in M_{2,R}^{(n+1)} : (t_k, z_k) = (t_{k+1}, z_{k+1})\}. \tag{4.82}$$

Then, noting that $\Phi_k : (x_1, ..., x_k, x_k, x_{k+1}, ..., x_n) \mapsto (x_1, ..., x_n)$ from (4.16) defines a bijective map between $R_k^{(n+1)}$ and $V_R^{(n+1)}$, we obtain:

$$\begin{aligned} \int_{M_{2,R}^{(n+1)}} \omega^{(n+1)} &\overset{(4.81)}{=} \sum_{k=1}^{n} \int_{R_k^{(n+1)}} \omega^{(n+1)} \\ &= \sum_{k=1}^{n} \int_{V_R^{(n)}} {\Phi_k}^* \, \omega^{(n+1)}. \end{aligned} \tag{4.83}$$

This equals $\int_{V_R^{(n)}} d\omega^{(n)}$ if

$$\sum_{k=1}^{n} {\Phi_k}^* \, \omega^{(n+1)} = d\omega^{(n)}. \tag{4.84}$$

This, in turn, is ensured by condition (4.15). To summarize, if (4.15) holds, we obtain (4.79) and hence (4.68) which is equivalent to (4.14).

We now turn to point 2. To this end, we explicitly compute both sides of (4.15). Denoting omission by $\widehat{(\cdot)}$, we find:

$$\begin{aligned}
{\Phi_k}^* \, \omega^{(n+1)} = & \sum_{\mu_1,\ldots,\widehat{\mu}_k,\widehat{\mu}_{k+1},\ldots,\mu_{n+1}} (-1)^{\mu_1+\cdots+\widehat{\mu}_k+\widehat{\mu}_{k+1}+\cdots+\mu_{n+1}} \, j^{\mu_1\ldots\mu_{n+1}} \\
& dx_1^{1-\mu_1} \wedge \cdots \wedge (-dx_k^0 \wedge dx_k^1 - dx_k^1 \wedge dx_k^0) \wedge \cdots \wedge dx_n^{1-\mu_{n+1}} \\
= & \sum_{\mu_1,\ldots,\widehat{\mu}_k,\widehat{\mu}_{k+1},\ldots,\mu_{n+1}} (-1)^{1+\mu_1+\cdots+\widehat{\mu}_k+\widehat{\mu}_{k+1}+\cdots+\mu_{n+1}} \\
& \varepsilon_{\rho\,\sigma} \, j^{\mu_1\ldots\mu_{k-1}\,\rho\,\sigma\,\mu_{k+1}\ldots\mu_{n+1}} \, dx_1^{1-\mu_1} \wedge \cdots \wedge dx_k^0 \wedge dx_k^1 \wedge \cdots \wedge dx_n^{1-\mu_{n+1}}.
\end{aligned} \tag{4.85}$$

The left hand side of (4.15) is given by:

$$\begin{aligned}
d\omega^{(n)} \; = & \sum_{k=1}^{n} \sum_{\mu_1,\ldots,\mu_n} (-1)^{\mu_1+\cdots\mu_k+\cdots\mu_n} \, \partial_{k,\mu_k} j^{\mu_1\ldots\mu_k\ldots\mu_n} \\
& (-1)^{(k-1)+\mu_k} dx_1^{1-\mu_1} \wedge \cdots \wedge dx_k^0 \wedge dx_k^1 \wedge \cdots \wedge dx_n^{1-\mu_n}.
\end{aligned} \tag{4.86}$$

Thus, $\mathrm{d}\omega^{(N)} = 0$ is equivalent to the first line of (4.17). Comparing $\sum_{k=1}^{n} {\Phi_k}^* \, \omega^{(n+1)}$ and $d\omega^{(n)}$, we obtain the following condition (relabelling indices $\mu_{k+1}\ldots\mu_{n+1} \to \mu_k\ldots\mu_n$ in (4.85)):

$$\varepsilon_{\rho\,\sigma} \, j^{\mu_1\ldots\mu_{k-1}\,\rho\,\sigma\,\mu_{k+1}\ldots\mu_n}(x_1, ..., x_k, x_k, x_{k+1}, ..., x_n) = (-1)^k \partial_{k,\mu_k} j^{\mu_1\ldots\mu_k\ldots\mu_n}(x_1, ..., x_n). \tag{4.87}$$

This is identical to the second line of (4.17). □

4.7.2 Probability conserving IBCs in the 2-sector model

Proof of Theorem 4.3. We start from the balance condition (4.30). As we aim at a translation invariant model (4.20)-(4.21), it is clear that the phase θ and the matrices A, B must be constant.

For ease of notation, we omit the arguments of the wave function. In order to simplify (4.30), we eliminate the component $\psi_{-+}^{(2)}$ from the equation using the IBC (4.21). Introducing

$$\widetilde{\psi} = \begin{pmatrix} \psi_{-}^{(1)} \\ \psi_{+}^{(1)} \\ \psi_{+-}^{(2)} \end{pmatrix}, \tag{4.88}$$

(4.30) can be rewritten as:

$$\frac{1}{2i}\,\widetilde{\psi}^\dagger \left[\begin{pmatrix} \widetilde{A}^\dagger \begin{pmatrix} B \\ 0\;0 \end{pmatrix} & \widetilde{A}^\dagger \begin{pmatrix} e^{i\theta} \\ 1 \end{pmatrix} \\ 0 \quad 0 & 0 \end{pmatrix} - \begin{pmatrix} \begin{pmatrix} B^\dagger & 0 \\ 0 & \end{pmatrix} \widetilde{A} & 0 \\ (e^{-i\theta},1)\,\widetilde{A} & 0 \end{pmatrix} \right] \widetilde{\psi}$$
$$= \widetilde{\psi}^\dagger \begin{pmatrix} -B^\dagger B & -B^\dagger e^{i\theta} \\ -e^{-i\theta} B & 0 \end{pmatrix} \widetilde{\psi}. \tag{4.89}$$

As this equation has to hold for all $\widetilde{\psi}$, we obtain a condition for the matrices on l.h.s. and r.h.s. Evaluating this condition in detail, we obtain just two independent conditions: Firstly,

$$\frac{1}{2i}(e^{-i\theta},1)\,\widetilde{A} = e^{-i\theta} B \quad \Leftrightarrow \quad B = \frac{1}{2i}(1,e^{i\theta})\,\widetilde{A}. \tag{4.90}$$

This yields (4.33). Secondly,

$$\frac{1}{2i}\left[\widetilde{A}^\dagger \begin{pmatrix} B \\ 0\;0 \end{pmatrix} - \begin{pmatrix} B^\dagger & 0 \\ 0 & \end{pmatrix} \widetilde{A} \right] = -B^\dagger B. \tag{4.91}$$

Plugging (4.90) into this equation, we obtain the condition

$$\begin{aligned} & \frac{1}{4}\,\widetilde{A}^\dagger \begin{pmatrix} 2 & e^{i\theta} \\ e^{-i\theta} & 0 \end{pmatrix} \widetilde{A} \;=\; \frac{1}{4}\,\widetilde{A}^\dagger \begin{pmatrix} 1 & e^{i\theta} \\ e^{-i\theta} & 1 \end{pmatrix} \widetilde{A} \\ \Leftrightarrow \qquad & \widetilde{A}^\dagger \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \widetilde{A} \;=\; 0. \end{aligned} \tag{4.92}$$

Let $\widetilde{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. (4.92) then yields the three conditions (i)$|a| = |c|$, (ii) $|b| = |d|$ and (iii) $a^* b = c^* d$. These force $\widetilde{A}$ to be a rank-1 matrix of the form (4.32). □

4.7.3 Existence and uniqueness in the 2-sector model

We now prove Theorem 4.2 on the existence and uniqueness of solutions for our model with two sectors of Fock space. Later we shall also do this for N sectors; however, the case $N = 2$ is crucial to develop the technique of the proofs and makes the proof for a general $N > 2$ much more transparent.
The proof is divided into two steps. First we show that:

- Given the wave function in sector 1, we obtain a unique solution for sector 2.
- Given the wave function in sector 2, we obtain a unique solution for sector 1.

Second, we use a fixed point argument to find a combined solution of both sectors. The first step is carried out in the following lemmas.

Lemma 4.6. *Let*

$$\begin{aligned} \mathscr{B}_1 &:= C^1_b\left([-T,T]\times\mathbb{R}, \mathbb{C}^2\right), \\ \mathscr{B}_2 &:= C^1_b\left(\{(t_1,z_1,t_2,z_2)\in \mathscr{S}_1^{(2)} | t_1,t_2\in[-T,T], z_1<z_2\}, \mathbb{C}^4\right). \end{aligned} \tag{4.93}$$

Then, given a function $f^{(2)} \in \mathscr{B}_2$ and initial values $\psi^{(1)}(0,\cdot) \in C_b^1(\mathbb{R},\mathbb{C}^2)$, there exists a unique solution $\psi^{(1)} \in \mathscr{B}_1$ of

$$i\partial_t \psi^{(1)}(t,z) = H_1^{\text{Dirac}} \psi^{(1)}(t,z) - A f^{(2)}(t,z,t,z) \tag{4.94}$$

with these initial values.

Proof. Rewriting the system (4.94) gives

$$\begin{pmatrix} (\partial_t + \partial_z)\psi_-^{(1)}(t,z) \\ (\partial_t - \partial_z)\psi_+^{(1)}(t,z) \end{pmatrix} = iAf^{(2)}(t,z,t,z) =: \begin{pmatrix} \widetilde{f}_-(t,z) \\ \widetilde{f}_+(t,z) \end{pmatrix}, \tag{4.95}$$

with $\widetilde{f}_\pm \in C_b^1(\mathbb{R}^2,\mathbb{C})$. Now (4.95) can be directly integrated along characteristics. The solution is given by:

$$\psi^{(1)}(t,z) = \begin{pmatrix} \psi_-^{(1)}(0,z-t) \\ \psi_+^{(1)}(0,z+t) \end{pmatrix} + \int_0^t ds \begin{pmatrix} \widetilde{f}_-(s,z-t+s) \\ \widetilde{f}_+(s,z+t-s) \end{pmatrix}. \tag{4.96}$$

We have $\psi^{(1)} \in \mathscr{B}_1$ because the initial values and $\widetilde{f}$ are C_b^1-functions. □

Lemma 4.7. *Given a function $\psi^{(1)} \in \mathscr{B}_1$ and initial values*

$$\psi^{(2)}(0,\cdot,0,\cdot) = \psi_0^{(2)} \in C_b^1\left(\{(z_1,z_2) \in \mathbb{R}^2 | z_1 < z_2\}, \mathbb{C}^4\right) \tag{4.97}$$

which satisfy (4.24) *as well as the condition*

$$B\,\partial_t \psi^{(1)}(t,z)\Big|_{t=0} = (\partial_{z_2} - \partial_{z_1}) \left(\psi_{0,-+}^{(2)}(z_1,z_2) + e^{i\theta}\psi_{0,+-}^{(2)}(z_1,z_2)\right)\Big|_{z_1=z_2=z} \tag{4.98}$$

there exists a unique solution $\psi^{(2)} \in \mathscr{B}_2$ of (4.19) *with boundary condition* (4.21)*.*

Proof. The lemma is a special case of theorem 3.3. in [20], where the solution (eq. (32) in that paper) was given explicitly for a general class of initial boundary value problems. Adapted to our notation and with the characteristic variables $u_k := z_k - t_k$ and $v_k := z_k + t_k$ for $k = 1,2$, the solution of (4.19) on $\mathscr{S}_1^{(2)}$ with boundary condition (4.21) reads as follows:

$$\begin{aligned}
&\psi_{--}^{(2)}(t_1,z_1,t_2,z_2) = \psi_{0,--}^{(2)}(u_1,u_2), \\
&\psi_{-+}^{(2)}(t_1,z_1,t_2,z_2) = \begin{cases} \psi_{0,-+}^{(2)}(u_1,v_2) & \text{for } u_1 < v_2, \\ e^{i\theta}\psi_{0,+-}^{(2)}(v_2,u_1) + B\psi^{(1)}(\frac{v_2-u_1}{2},\frac{v_2+u_1}{2}) & \text{for } u_1 \geq v_2, \end{cases} \\
&\psi_{+-}^{(2)}(t_1,z_1,t_2,z_2) = \begin{cases} \psi_{0,+-}^{(2)}(v_1,u_2) & \text{for } v_1 < u_2, \\ e^{-i\theta}\left(\psi_{0,-+}^{(2)}(u_2,v_1) - B\psi^{(1)}(\frac{v_1-u_2}{2},\frac{v_1+u_2}{2})\right) & \text{for } v_1 \geq u_2, \end{cases} \\
&\psi_{++}^{(2)}(t_1,z_1,t_2,z_2) = \psi_{0,++}^{(2)}(v_1,v_2).
\end{aligned} \tag{4.99}$$

The such defined $\psi^{(2)}$ inherits the C_b^1-property from the initial and boundary values wherever $u_k \neq v_j$ for $j \neq k$. At $u_1 = v_2$ resp. $v_1 = u_2$, continuity of $\psi_{-+}^{(2)}$ resp. $\psi_{+-}^{(2)}$ amounts to condition (4.24). In order to check differentiability at those points, we compare the limits of the respective partial derivatives in the case differentiation in (4.99). We start with

comparing $\partial_{z_1}\psi^{(2)}_{-+}$ for $u_1 \nearrow v_2$ and $u_1 \searrow v_2$ at $v_2 = z$. Let D_k denote the derivative w.r.t. the k-th argument. The condition for the two limits to coincide then is:

$$D_1\psi^{(2)}_{0,-+}(z,z) \;=\; e^{i\theta}D_2\psi^{(2)}_{0,+-}(z,z) + \tfrac{1}{2}B(D_2\psi^{(1)}(0,z) - D_1\psi^{(1)}(0,z)). \tag{4.100}$$

Inserting the z-derivative of (4.24),

$$(D_1 + D_2)(\psi^{(2)}_{0,+-}(z,z) - e^{i\theta}\psi^{(2)}_{0,+-}(z,z)) = BD_2\psi^{(1)}(0,z), \tag{4.101}$$

this becomes (4.98). Similar computations show that all other partial derivatives exist and are continuous as a result of the same conditions. □

With the lemmas at hand, we now construct a fixed point map for our model. For given initial values (4.23) satisfying the compatibility conditions (4.24) and (4.25), we let

$$\mathcal{D} = \left\{(\psi^{(1)},\psi^{(2)}) \in \mathscr{B}_1 \oplus \mathscr{B}_2 \,\middle|\, \psi^{(1)}(0,z) = \psi^{(1)}_0(z), \psi^{(2)}(0,z_1,0,z_2) = \psi^{(2)}_0(z_1,z_2)\right\}. \tag{4.102}$$

Clearly, $\mathcal{D}$ is closed in $\mathscr{B}_1 \oplus \mathscr{B}_2$.

Definition: Let $F : \mathcal{D} \to \mathcal{D},\ (f^{(1)}, f^{(2)}) \mapsto (\psi^{(1)}, \psi^{(2)})$ be defined by the following procedure.

- Take $\psi^{(1)}$ to be the unique solution of (4.94) according to Lemma 4.6.
- Using the such constructed $\psi^{(1)}$ in the interior-boundary condition, $\psi^{(2)}$ is defined to be the unique solution of (4.19) and (4.21) according to Lemma 4.7.

Well-definedness of F: We need to check that F actually maps into $\mathcal{D}$. This is true if the following points hold.

- *C^1_b-property.* Lemma 4.7 gives a C^1_b-solution under the conditions (4.24) and (4.98). We require (4.24) and (4.25). Since $\psi^{(1)}$ solves (4.94),

$$B\partial_t\psi^{(1)}(t,z)|_{t=0} = -iB\left(H^{\text{Dirac}}\psi^{(1)}(0,z) - Af^{(2)}(0,z,0,z)\right) \tag{4.103}$$

  with $f^{(2)}(0,z,0,z) = \psi^{(2)}_0(z,z)$ and inserting (4.25) implies (4.98).
- *Initial values.* These are preserved under F as $(\psi^{(1)},\psi^{(2)}) = F(f^{(1)},f^{(2)})$ is constructed with respect to the same initial values.

By construction of F, we immediately obtain the following result.

Lemma 4.8. *Let $(\psi^{(1)},\psi^{(2)}) \in \mathscr{B}_1 \oplus \mathscr{B}_2$. Then the following statements are equivalent:*

i) $(\psi^{(1)},\psi^{(2)})$ is a C^1_b-solution of the initial boundary value problem (4.19),(4.20), (4.21) *with initial values given as in* (4.23).

ii) $(\psi^{(1)},\psi^{(2)})$ lies in $\mathcal{D}$ and is a fixed point of F.

The main work now lies in proving the following.

Lemma 4.9. *For every $T > 0$, F possesses a unique fixed point in $\mathcal{D}$.*

Proof. Let $\gamma \geq 0$. We equip $\mathscr{B}_1$ and $\mathscr{B}_2$ with the weighted norms

$$\begin{aligned}
\left\|f^{(1)}\right\|_{\mathscr{B}_1,\gamma} &:= \sup_{t\in[-T,T],z\in\mathbb{R}} \left(\left(|f^{(1)}(t,z)| + \max_{y\in\{t,z\}} |\partial_y f^{(1)}(t,z)| \right) e^{-\gamma|t|} \right), \\
\left\|f^{(2)}\right\|_{\mathscr{B}_2,\gamma} &:= \sup_{t_1,t_2\in[-T,T],(t_1,z_1,t_2,z_2)\in\mathscr{S}_1^{(2)}} \left(\left(|f^{(2)}(t_1,z_1,t_2,z_2)| \right.\right. \\
&\qquad \left.\left. + \max_{y\in\{t_1,z_1,t_2,z_2\}} |\partial_y f^{(2)}(t_1,z_1,t_2,z_2)| \right) e^{-\frac{\gamma}{2}(|t_1|+|t_2|)} \right),
\end{aligned} \tag{4.104}$$

where $|\cdot|$ denotes the maximum norm of $\mathbb{C}^2$ and $\mathbb{C}^4$, respectively. For $\gamma = 0$, the norms (4.104) reduce to the canonical norms on C_b^1-functions. In that case, one obtains complete spaces. As a consequence of the inequalities

$$\left\|f^{(k)}\right\|_{\mathscr{B}_k,0} e^{-\gamma T} \;\leq\; \left\|f^{(k)}\right\|_{\mathscr{B}_k,\gamma} \;\leq\; \left\|f^{(k)}\right\|_{\mathscr{B}_k,0}, \quad k = 1,2, \tag{4.105}$$

the norms (4.104) are equivalent for all $\gamma \geq 0$. This implies that $\mathscr{B}_1 \oplus \mathscr{B}_2$ equipped with the norm

$$\left\|(f^{(1)},f^{(2)})\right\|_{\mathscr{B}_1\oplus\mathscr{B}_2,\gamma} := \left\|f^{(1)}\right\|_{\mathscr{B}_1,\gamma} + \left\|f^{(2)}\right\|_{\mathscr{B}_2,\gamma} \tag{4.106}$$

is a Banach space. Recall that $\mathcal{D}$ is a closed subset of $\mathscr{B}_1 \oplus \mathscr{B}_2$. Our goal is to use Banach's fixed point theorem, so it remains to show that $F : \mathcal{D} \to \mathcal{D}$ is a contraction.
Let $f, g \in \mathcal{D}$ and $F\left(f^{(1)}, f^{(2)}\right) =: (\psi^{(1)}, \psi^{(2)})$ and $F\left(g^{(1)}, g^{(2)}\right) =: (\phi^{(1)}, \phi^{(2)})$. Moreover, we define $\widetilde{g}$ like $\widetilde{f}$ in (4.95) but with f replaced by g. Using the solution formula (4.96), we obtain:

$$\begin{pmatrix} (\psi_-^{(1)} - \phi_-^{(1)})(t,z) \\ (\psi_+^{(1)} - \phi_+^{(1)})(t,z) \end{pmatrix} = \int_0^t ds \begin{pmatrix} (\widetilde{f}_- - \widetilde{g}_-)(s, z-t+s) \\ (\widetilde{f}_+ - \widetilde{g}_+)(s, z+t-s) \end{pmatrix}. \tag{4.107}$$

This implies:

$$\begin{aligned}
\left|\psi_\pm^{(1)} - \phi_\pm^{(1)}\right|(t,z) \;&\leq\; \operatorname{sgn}(t) \int_0^t ds\; |(\widetilde{f}_\pm - \widetilde{g}_\pm)(s, z \pm t - \pm s)|\, e^{-\gamma|s|} e^{\gamma|s|} \\
&\leq \sup_{s\in[-|t|,|t|]} \left(|\widetilde{f}_\pm - \widetilde{g}_\pm|(s, z\pm t - \pm s)\, e^{-\gamma|s|} \right) \operatorname{sgn}(t) \int_0^t ds\; e^{\gamma|s|} \\
&\leq \sup_{s\in[-|t|,|t|],y\in\mathbb{R}} \left(|\widetilde{f}_\pm - \widetilde{g}_\pm|(s,y)\, e^{-\gamma|s|} \right) \frac{1}{\gamma} e^{\gamma|t|}.
\end{aligned} \tag{4.108}$$

Hence, recalling (4.95), we find:

$$\begin{aligned}
\left|\psi_\pm^{(1)} - \phi_\pm^{(1)}\right|(t,z)\, e^{-\gamma|t|} \;&\leq\; \frac{1}{\gamma} \sup_{s\in[-|t|,|t|],y\in\mathbb{R}} |\widetilde{f}_\pm - \widetilde{g}_\pm|(s,y)\, e^{-\gamma|s|} \\
&\leq\; \frac{1}{\gamma} \sup_{s\in[-|t|,|t|],y\in\mathbb{R}} \|A\|_\infty\, |f^{(2)} - g^{(2)}|(s,y,s,y)\, e^{-\gamma|s|}.
\end{aligned} \tag{4.109}$$

For the z-derivative, we obtain an analogous formula:

$$\begin{aligned}
\left|\partial_z(\psi_\pm^{(1)} - \phi_\pm^{(1)})(t,z)\right| e^{-\gamma|t|} \;&\leq\; \frac{1}{\gamma} \sup_{s\in[-|t|,|t|],y\in\mathbb{R}} \left|\partial_y(\widetilde{f}_\pm - \widetilde{g}_\pm)(s,y)\right| e^{-\gamma|s|} \\
&\leq\; \frac{1}{\gamma} \sup_{s\in[-|t|,|t|],y\in\mathbb{R}} \|A\|_\infty \left|\partial_y(f^{(2)} - g^{(2)})(s,y,s,y)\right| e^{-\gamma|s|}.
\end{aligned} \tag{4.110}$$

In the estimate for $\partial_t(\psi^{(1)}_\pm - \phi^{(1)}_\pm)(t,z)$ we obtain a similar expression as (4.110) (with ∂_y replaced by ∂_s) plus $(\widetilde{f}_\pm - \widetilde{g}_\pm)(t,z)$. The latter appears due to the time-dependent upper bound of the integral $\int_0^t ds$. We can bound it as follows.

$$\begin{aligned}\Big|(\widetilde{f}_\pm - \widetilde{g}_\pm)(t,z)\Big| &= |(\widetilde{f}_\pm - \widetilde{g}_\pm)(t,z) - (\widetilde{f}_\pm - \widetilde{g}_\pm)(0,z)| = \Big|\int_0^t \partial_s(\widetilde{f}_\pm - \widetilde{g}_\pm)(s,z)ds\Big| \\ &\leq \operatorname{sgn}(t)\int_0^t \big|\partial_s(\widetilde{f}_\pm - \widetilde{g}_\pm)(s,z)\big|e^{-\gamma|s|}e^{\gamma|s|}ds \\ &\leq \sup_{s\in[-|t|,|t|],y\in\mathbb{R}}\Big(|\partial_s(\widetilde{f}_\pm - \widetilde{g}_\pm)(s,z)|e^{-\gamma|s|}\Big)\frac{1}{\gamma}e^{\gamma|t|}. \end{aligned} \tag{4.111}$$

Here we have used that $\widetilde{f}_\pm(0,z) = \widetilde{g}_\pm(0,z)$ by definition of $\mathcal{D}$ and that $\widetilde{f}$ and $\widetilde{g}$ are C^1_b-functions. (4.111) implies:

$$\Big|(\widetilde{f}_\pm - \widetilde{g}_\pm)(t,z)\Big|e^{-\gamma|t|} \leq \frac{1}{\gamma}\|A\|_\infty \sup_{s\in[-|t|,|t|],y\in\mathbb{R}}\Big|\partial_s(f^{(2)} - g^{(2)})(s,z,s,z)\Big|e^{-\gamma|s|}. \tag{4.112}$$

Gathering the previous estimates (4.109), (4.110), (4.112) and considering (4.104), we obtain the bound

$$\Big\|\psi^{(1)} - \phi^{(1)}\Big\|_{\mathscr{B}_{1,\gamma}} \leq \frac{2}{\gamma}\|A\|_\infty\Big\|f^{(2)} - g^{(2)}\Big\|_{\mathscr{B}_{2,\gamma}}. \tag{4.113}$$

To bound the norm for the second sector, recall the solution formula (4.99). Since $\psi^{(2)}$ and $\phi^{(2)}$ have the same initial data $\psi^{(2)}_0$, their difference is given by (recall $u_k = z_k - t_k$, $v_k = z_k + t_k$):

$$(\psi^{(2)} - \phi^{(2)})(t_1,z_1,t_2,z_2) = \begin{pmatrix} 0 \\ \big[(B\psi^{(1)})_- - (B\phi^{(1)})_-\big](\frac{v_2-u_1}{2},\frac{v_2+u_1}{2})\,\mathbb{1}_{\{u_1\geq v_2\}}(t_1,z_1,t_2,z_2) \\ \big[(B\psi^{(1)})_+ - (B\phi^{(1)})_+\big](\frac{v_1-u_2}{2},\frac{v_1+u_2}{2})\,\mathbb{1}_{\{v_1\geq u_2\}}(t_1,z_1,t_2,z_2) \\ 0 \end{pmatrix} \tag{4.114}$$

where we write $B\alpha = \begin{pmatrix} (B\alpha)_- \\ (B\alpha)_+ \end{pmatrix}$ for two-component vectors α. For positive times $t_1, t_2 > 0$, only the third line of (4.114) is nonzero, and then the weight factors in the γ-norms satisfy

$$e^{-\frac{\gamma}{2}(|t_1|+|t_2|)} = e^{-\frac{\gamma}{2}(t_1+t_2)} \leq e^{-\frac{\gamma}{2}(t_1+t_2)}\underbrace{e^{-\frac{\gamma}{2}(z_1-z_2)}}_{\geq 1 \text{ as } z_1\leq z_2} = e^{-\gamma\frac{v_1-u_2}{2}} = e^{-\gamma\frac{|v_1-u_2|}{2}}. \tag{4.115}$$

If one time is positive and the other is negative, $(\psi^{(2)}-\phi^{(2)})(t_1,z_1,t_2,z_2) = 0$. For $t_1, t_2 < 0$, only the second line of (4.114) is nonzero, and the weight factors in the γ-norms satisfy

$$e^{-\frac{\gamma}{2}(|t_1|+|t_2|)} = e^{\frac{\gamma}{2}(t_1+t_2)} \leq e^{\frac{\gamma}{2}(t_1+t_2)}\underbrace{e^{\frac{\gamma}{2}(z_2-z_1)}}_{\geq 1 \text{ as } z_1\leq z_2} \leq e^{\gamma\frac{v_2-u_1}{2}} = e^{-\gamma\frac{|v_2-u_1|}{2}}. \tag{4.116}$$

Now $(v_1 - u_2)/2$ and $(v_2 - u_1)/2$ appear as the time arguments of $[(B\psi^{(1)})_\mp - (B\phi^{(1)})_\mp]$, respectively, in (4.114). It is therefore clear that

$$\Big\|\psi^{(2)} - \phi^{(2)}\Big\|_{\mathscr{B}_{2,\gamma}} \leq \Big\|B\psi^{(1)} - B\phi^{(1)}\Big\|_{\mathscr{B}_{1,\gamma}} \leq \|B\|_\infty\Big\|\psi^{(1)} - \phi^{(1)}\Big\|_{\mathscr{B}_{1,\gamma}}. \tag{4.117}$$

Together with (4.113), this implies that

$$\begin{aligned}\left\|\left(\psi^{(1)},\psi^{(2)}\right)-\left(\phi^{(1)},\phi^{(2)}\right)\right\|_{\mathscr{B}_1\oplus\mathscr{B}_2,\gamma} &\leq \left(\|B\|_\infty+1\right)\left\|\psi^{(1)}-\phi^{(1)}\right\|_{\mathscr{B}_1,\gamma}\\ &\leq \frac{2}{\gamma}\left(\|B\|_\infty+1\right)\|A\|_\infty\left\|f^{(2)}-g^{(2)}\right\|_{\mathscr{B}_2,\gamma}\\ &\leq C\left\|\left(f^{(1)},f^{(2)}\right)-\left(g^{(1)},g^{(2)}\right)\right\|_{\mathscr{B}_1\oplus\mathscr{B}_2,\gamma},\end{aligned}\tag{4.118}$$

with the constant $C=\frac{2}{\gamma}(\|B\|_\infty+1)\|A\|_\infty$.
Choosing, for example, $\gamma=10\,(\|B\|_\infty+1)\|A\|_\infty$ we have $C=\frac{1}{5}<1$. Thus, F is a contraction and Banach's fixed point theorem yields the claim. □

Together with Lemma 4.8 this proves Theorem 4.2, establishing that the multi-time IBC system has a unique global C_b^1-solution for all times.

4.7.4 Existence and uniqueness in the N-sector model

As a preparation, we prove the following statement which expresses a certain harmony of the interaction terms in the multi-time equations for different sectors of Fock space.

Lemma 4.10 (Consistency conditions.)**.** *The system of multi-time equations* (4.36)*,* (4.37) *and* (4.38) *satisfies the consistency conditions*

$$i(\partial_{t_k}-s_k\partial_{z_k})f^{(n)}_{l,s_1\ldots s_n}=i(\partial_{t_l}-s_l\partial_{z_l})f^{(n)}_{k,s_1\ldots s_n},\tag{4.119}$$

for all $k,l=1,...,n$*,* $s_1,...,s_n=\pm1$*,*

(i) for $n=N$ *(trivially).*

(ii) for $n=1,2,...,N-1$ *if* $\psi^{(n+1)}$ *satisfies the multi-time equations* (4.36)*.*

Proof. In the cases $n=1$ and $n=N$ there is nothing to show (note (4.38) for $n=N$). For $n=N-1$, we have, for all $k=1,...,N-1$:

$$\begin{aligned}&i(\partial_{t_k}-s_k\partial_{z_k})f^{(N-1)}_{l,s_1\ldots s_{N-1}}(x_1,...,x_{N-1})\\ &\overset{(4.37)}{=}\sum_{t,u=\pm1}(-1)^l A^{tu}_{s_l}i(\partial_{t_k}-s_k\partial_{z_k})\psi^{(N)}_{s_1\ldots s_{l-1}t\,u\,s_{l+1}\ldots s_{N-1}}(x_1,...,x_{l-1},x_l,x_l,x_{l+1},...,x_{N-1})\\ &=0\end{aligned}\tag{4.120}$$

because of (4.38). In particular, (4.119) follows.
For $2\leq n\leq N-2$, w.l.o.g. let $k<l$ and consider:

$$\begin{aligned}&i(\partial_{t_k}-s_k\partial_{z_k})f^{(n)}_{l,s_1\ldots s_n}(x_1,...,x_n)\\ &\overset{(4.37)}{=}\sum_{t,u=\pm1}(-1)^l A^{tu}_{s_l}i(\partial_{t_k}-s_k\partial_{z_k})\psi^{(n+1)}_{s_1\ldots s_{l-1}t\,u\,s_{l+1}\ldots s_n}(x_1,...,x_{l-1},x_l,x_l,x_{l+1},...,x_n)\\ &\overset{(4.36),(4.37)}{=}\sum_{t,u,v,w=\pm1}(-1)^{k+l}A^{tu}_{s_l}A^{vw}_{s_k}\psi^{(n+2)}_{s_1\ldots s_{k-1}v\,w\,s_{k+1}s_{l-1}t\,u\,s_{l+1}\ldots s_n}(x_1,...,x_{k-1},x_k,x_k,x_{k+1},...,\\ &\qquad\qquad x_{l-1},x_l,x_l,x_{l+1},...,x_n).\end{aligned}\tag{4.121}$$

A similar calculation yields:

$$i(\partial_{t_l} - s_l \partial_{z_l}) f^{(n)}_{k,s_1 \dots s_n}(x_1, ..., x_n)$$
$$= \sum_{t,u,v,w=\pm 1} (-1)^{k+l} A^{tu}_{s_k} A^{vw}_{s_l} \psi^{(n+2)}_{s_1 \dots s_{k-1} t\, u\, s_{k+1} s_{l-1} v\, w\, s_{l+1} \dots s_n}(x_1, ..., x_{k-1}, x_k, x_k, x_{k+1}, ...,$$
$$x_{l-1}, x_l, x_l, x_{l+1}, ..., x_n). \quad (4.122)$$

Relabeling $t, u \leftrightarrow v, w$ shows that (4.121), (4.122) agree; hence we obtain (4.119). □

The idea now is to prove Thm. 4.4 using a fixed point argument. In fact, it is possible to explicitly write down the solution of the model for a particular sector provided given the wave function of the neighboring sectors. First we explain how to do this heuristically. Then we define the fixed point map and show that it is, indeed, a contraction in a sequence of lemmas.

Heuristics. We now explain at the example $N = 3$ how to obtain a solution of the multi-time equations for a particular sector, given the solution on the neighboring sectors. In the paper [21], the solution for $A = 0$ was constructed such that probability is conserved for each sector, separately. This was done following the so-called *multi-time characteristics* back to the initial value surface at time zero. The multi-time characteristic associated with a certain component $\psi^{(3)}_{s_1 s_2 s_3}$ and a particular point $(t_1, z_1; t_2, z_2; t_3, z_3) \in \mathscr{S}^{(3)}_1$ is defined as the set that contains $(t_1, z_1; t_2, z_2; t_3, z_3)$ and along which that component would be constant by the homogeneous part of the multi-time equations (4.36), $(\partial_{t_k} - s_k \partial_{z_k}) \psi^{(3)}_{s_1 s_2 s_3} = 0$, $k = 1, 2, 3$. This leads to Cartesian products of three lines given by the characteristic variables $c_k := z_k + s_k t_k$ appearing in the multi-time equations. For example, for $\psi^{(3)}_{+-+}(t_1, z_1; t_2, z_2; t_3, z_3)$ the multi-time characteristic is given by the set

$$\left\{ (s_1, y_1; s_2, y_2; s_3, y_3) \in \mathbb{R}^6 \;\middle|\; s_1 + y_1 = c_1, s_2 - y_2 = c_2, s_3 + y_3 = c_3 \right\}. \quad (4.123)$$

Figure 4.1 shows two examples of multi-time characteristics with the three lines all drawn in one space-time diagram. In the left picture, for the component $\psi^{(3)}_{+++}$, the lines of the multi-

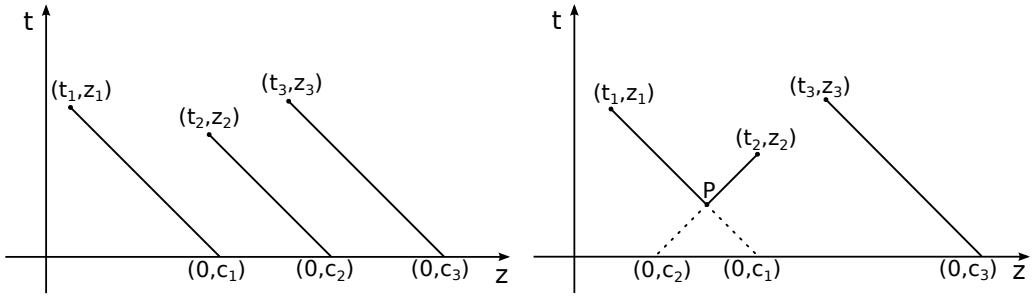

Figure 4.1: Two examples for multi-time characteristics. The one on the left, for the component $\psi^{(3)}_{+++}$, does not intersect the coincidence point set $\mathscr{C}$. The one the right, for $\psi^{(3)}_{+-+}$, intersects $\mathscr{C}$ (cf. point P).

time characteristic can be followed back to the initial value surface without intersecting the boundary $\partial \mathscr{S}^{(3)}$. "Following back" here means to choose a certain curve $\gamma(\tau)$ in the characteristic which connects $(t_1, z_1; t_2, z_2; t_3, z_3)$ with the point $(0, c_1; 0, c_2; 0, c_3)$. Along

the curve, the Dirac equation in the respective variables becomes an ordinary differential equation of the form $\frac{d}{d\tau}\psi(\gamma(\tau)) = f(\gamma(\tau))$ which can be integrated easily. Therefore, we will define an operator I below that just integrates the inhomogeneity along the characteristic lines and gives the solution. We will choose our curve $\gamma(\tau)$ corresponding to a certain path in the space of the time variables along the three lines which comprise the characteristic. In our case:

$$(0,0,0) \longrightarrow (t_1,0,0) \longrightarrow (t_1,t_2,0) \longrightarrow (t_1,t_2,t_3). \tag{4.124}$$

As it happens often in the study of multi-time equation, a change of this path in the time variables must not change the final result, which requires a certain integrability condition, called the *consistency condition*. This condition was shown in Lemma 4.10. In the right half of the picture, we additionally have to take the boundary condition into account because at the point $(P;P;t_3,z_3) \in \mathscr{C}$, the characteristic intersects the coincidence point set $\mathscr{C}$ and consequently leaves the domain. At this vertex point, the IBC has to be used, which we will implement via another operator V. The IBC then relates the value of the component $\psi^{(3)}_{s_1s_2s_3}$ with a different component which is associated with a different multi-time characteristic. One then follows this new multi-time characteristic back in time until either the boundary is reached again (then one repeats the process with a different component) or the initial surface $t_1 = t_2 = t_3 = 0$. This results in a formula where one uses the operators I and V alternatingly to obtain the solution from the initial data. The number of vertices in diagrams such as Figure 4.1 determines how many steps the process takes.

Solution formula. We construct the fixed point map sector-wise. To this end, let $n \in \mathbb{N}$ and assume that for all $k = 1,...,n$, $j = 1,...,n-1$, boundary functions $g^{(n)}_j \in D_{n-1}$, inhomogeneities $f^{(n)}_k \in D_n$ and initial values $\psi^{(n)}_0 \in C^1_b(Z_n, \mathbb{C}^{2^n})$ are given (see (4.41) for the definition of Z_n). We shall solve the following initial boundary value problem:

$$\begin{aligned} i\,(\partial_{t_k} - s_k\partial_{z_k})\,\psi^{(n)}_{s_1\ldots s_n} &= f^{(n)}_{k,s_1\ldots s_n}, \\ \Big(\psi^{(n)}_{s_1\ldots s_{j-1}-+s_{j+2}\ldots s_n} - e^{i\theta}\psi^{(n)}_{s_1\ldots s_{j-1}+-s_{j+2}\ldots s_n}\Big)&(x_1,...,x_j,x_j,x_{j+2},...,x_n) \\ &= g^{(n)}_{j,s_1\ldots s_{j-1}s_{j+2}\ldots s_n}(x_1,...,x_j,x_{j+2},...,x_n), \\ \psi^{(n)}|_{t_1=\ldots=t_n=0} &= \psi^{(n)}_0. \end{aligned} \tag{4.125}$$

Throughout the section we assume that the initial data are compatible with the boundary conditions in the sense of Eqs. (4.42) and (4.43).
The solution $\psi^{(n)}$ shall be constructed through repeated application of the operators I and V which we define now. For every $(t_1,...,t_n) \in \mathbb{R}^n$ and $t \in \mathbb{R}$, we define a map $I^{(t_1,\ldots,t_n)}_t$ with "I" for "integration along the characteristic".

$$\begin{aligned} &I^{(t_1,\ldots,t_n)}_t :\ C^1_b(Z_n,\mathbb{C}^{2^n}) \to C^1_b(Z_n,\mathbb{C}^{2^n}), \\ &\big(I^{(t_1,\ldots,t_n)}_t\phi\big)_{s_1\ldots s_n}(z_1,...,z_n) = \phi_{s_1\ldots s_n}(c_1,...,c_n) \\ &- i\sum_{k=1}^{n}\int_0^{t_k} ds\ f^{(n)}_{k,s_1\ldots s_n}(t+t_1,z_1;...;t+t_{k-1},z_{k-1};t+s,c_k-s_ks;t,c_{k+1};...;t,c_n). \end{aligned} \tag{4.126}$$

Here, $c_k = z_k + s_kt_k$ where t_k is the time variable in the upper index of I and z_k the spatial variable from the argument of $I^{(t_1,\ldots,t_n)}_t\phi$.

Moreover, for every $t \in \mathbb{R}, j \in \{1, ..., n-1\}$ we define maps $V_j^t : C_b^1(Z_n, \mathbb{C}^{2^n}) \to C_b^1(Z_n, \mathbb{C}^{2^n})$ with "V" for "switching indices at the vertex" (at time t) by:

$$\begin{aligned}
\left(V_j^t \phi\right)_{s_1 \dots s_j s_j \dots s_n}(z_1, ..., z_n) &= \phi_{s_1 \dots s_j s_j \dots s_n}(z_1, ..., z_n), \\
\left(V_j^t \phi\right)_{s_1 \dots - + \dots s_n}(z_1, ..., z_n) &= e^{i\theta} \phi_{s_1 \dots + - \dots s_n}(z_1, ...z_{j+1}, z_j, ..., z_n) \\
&\quad + g^{(n)}_{j, s_1 \dots s_n}(t, z_1; ...; \widehat{t, z_j}; ...; t, z_n), \\
\left(V_j^t \phi\right)_{s_1 \dots + - \dots s_n}(z_1, ..., z_n) &= e^{-i\theta} \big[\phi_{s_1 \dots - + \dots s_n}(z_1, ..., z_{j+1}, z_j, ..., z_n) \\
&\quad - g^{(n)}_{j, s_1 \dots s_n}(t, z_1; ...; \widehat{t, z_j}; ...; t, z_n) \big],
\end{aligned} \tag{4.127}$$

where $\widehat{(\cdot)}$ denotes omission.

Furthermore, for every point $(t_1, z_1; ...; t_n, z_n) \in \mathscr{S}_1^{(n)}$ and every spin index $s_1, ..., s_n$, we define a set of *collisions* according to the following rules. As previously, we let $c_k = z_k + s_k t_k$. A collision is a pair of indices (j, k) in the set

$$\mathsf{Collisions} := \left\{ (j,k) \in \{1, ..., n\}^2 : j < k \text{ but } c_j > c_k \right\}. \tag{4.128}$$

$\mathsf{Collisions}$ is a finite set with $L := |\mathsf{Collisions}| < n^2 - 1$. Its elements coincide with the index pairs of those lines which cross in the diagrams in Fig. 4.1. With each collision (j, k), we associate a *collision time*

$$\tau_{(j,k)} := \frac{1}{2}(c_j - c_k). \tag{4.129}$$

We label these collision times in increasing order[2], i.e. identify each $\tau_{(j,k)} =: \tau_a$ with $a = 1, ..., L$ such that $\tau_1 < \tau_2 < ... < \tau_L$. Set $\tau_0 := 0$. Moreover, each collision is assigned an index k_a according to the formula:

$$k_a((j,k)) := j + \left| \{ (j,l) \in \mathsf{Collisions} : l < k \} \right|, \quad a = 1, ..., L. \tag{4.130}$$

In the diagrams in Fig. 4.1, this number corresponds to the numbers of lines left of the vertex where the lines j, k cross plus one. The τ_a and k_a are functions of the space-time point and the spin index only. Most importantly, they allow us to write down an explicit solution formula which is obtained by following the characteristics from collision to collision, as motivated heuristically above:

$$\psi^{(n)}_{s_1 \dots s_n}(t_1, z_1; ...; t_n, z_n) = \left(I_{\tau_L}^{(t_1 - \tau_L, ..., t_n - \tau_L)} \left(\prod_{a=1}^{L} V_{k_a}^{\tau_a} I_{\tau_{a-1}}^{(\tau_a - \tau_{a-1}, ..., \tau_a - \tau_{a-1})} \right) \psi_0^{(n)} \right)_{s_1 \dots s_n}(z_1, ..., z_n). \tag{4.131}$$

It is understood that the factors in the product are written from right to left, i.e.:

$$\left(I_{\tau_L}^{(t_1 - \tau_L, ..., t_n - \tau_L)} V_{k_L}^{\tau_L} I^{(\tau_L - \tau_{L-1}, ..., \tau_L - \tau_{L-1})} \dots V_{k_1}^{\tau_1} I_0^{(\tau_1, ..., \tau_1)} \psi_0^{(n)} \right)_{s_1 \dots s_n}(z_1, ..., z_n). \tag{4.132}$$

Lemma 4.11. *Let $n \in \mathbb{N}$. The function $\psi^{(n)}$ defined by (4.131) for given $f_k^{(n)}$ and $g_j^{(n)}$ is the unique solution of the IBC system (4.125) in the n-th sector, provided the inhomogeneities satisfy, for all $j \neq k$, the consistency conditions*

$$i \left(\partial_{t_k} - s_k \partial_{z_k} \right) f^{(n)}_{j, s_1 \dots s_n} = i \left(\partial_{t_j} - s_j \partial_{z_j} \right) f^{(n)}_{k, s_1 \dots s_n}. \tag{4.133}$$

[2]There is a zero measure set of points for which this ordering is not possible since several collision times are equal. We omit these points in the upcoming considerations. Later, one can recover the value of the wave function at these points by continuation.

Proof. For $L \in \mathbb{N}_0$, we prove via induction over the number L of collisions the statement $\mathbf{A}(\mathbf{L})$: *A function $\psi^{(n)} \in \mathcal{D}$ solves the IBC system* (4.125) *at all points $(t_1, z_1; ...; t_n, z_n)$ and for all spin indices $s_1, ..., s_n$ for which $|\mathsf{Collisions}| \leq L$ if and only if it is given by* (4.131) *at those points.*

Base Case $\mathbf{A}(\mathbf{0})$. At points with $|\mathsf{Collisions}| = 0$, (4.131) yields

$$\begin{aligned}\psi^{(n)}_{s_1...s_n}(t_1, z_1; ...; t_n, z_n) \;&=\; \psi^{(n)}_{0,s_1...s_n}(c_1, ..., c_n) \\ &-i\sum_{k=1}^{n}\int_0^{t_k} ds\; f^{(n)}_{k,s_1...s_n}(t_1, z_1; ...; t_{k-1}, z_{k-1}; s, c_k - s_k s; 0, c_{k+1}; ...; 0, c_n). \end{aligned} \tag{4.134}$$

We first show that (4.134) indeed is a solution of the IBC system. As $L = 0$, the IBC does not come into play here. The initial conditions are satisfied by construction (see (4.131)). We now calculate the derivatives w.r.t. the n-th coordinates. Omitting spin indices, we find:

$$\begin{aligned} i(\partial_{t_n} - s_n\partial_{z_n})\psi^{(n)} \;&=\; f^{(n)}_n(t_1, z_1; ...; t_n, c_n - s_n t_n) + \int_0^{t_n} ds\, \underbrace{i(\partial_{t_n} - s_n\partial_{z_n}) f^{(n)}_k(...; s, c_n - s_n s)}_{=0} \\ &\quad + \sum_{k=1}^{n-1}\int_0^{t_k} ds\, \underbrace{i(\partial_{t_n} - s_n\partial_{z_n}) f^{(n)}_k(...; 0, c_n)}_{=0} \\ &=\; f^{(n)}_n(t_1, z_1; ...; t_n, z_n). \end{aligned} \tag{4.135}$$

Next, we consider an arbitrary particle index $j \neq n$. Because (4.134) is not symmetric in the particle indices, more work is required to see that the multi-time equation is satisfied. We shall use the consistency condition (4.133) to show this. We compute via a telescoping sum:

$$\begin{aligned} &\int_0^{t_j} ds\, f^{(n)}_j(t_1, z_1; ...; t_{j-1}, z_{j-1}; s, c_j - s_j s; 0, c_{j+1}; ...; 0, c_n) \\ &= \int_0^{t_j} ds\, f^{(n)}_j(t_1, z_1; ...; t_{j-1}, z_{j-1}; s, c_j - s_j s; t_{j+1}, z_{j+1}; ...; t_n, z_n) \\ &+ \sum_{l=j+1}^{n}\int_0^{t_j} ds \Big(f^{(n)}_j(t_1, z_1; ...; t_{j-1}, z_{j-1}; s, c_j - s_j s; 0, c_{j+1}; ...; 0, c_{l-1}; 0, c_l; t_{l+1}, z_{l+1}; ...; t_n, z_n) \\ &\qquad - f^{(n)}_j(t_1, z_1; ...; t_{j-1}, z_{j-1}; s, c_j - s_j s; 0, c_{j+1}; ...; 0, c_{l-1}; t_l, z_l; t_{l+1}, z_{l+1}; ...; t_n, z_n)\Big). \end{aligned} \tag{4.136}$$

Next, we use the fundamental theorem of calculus, abbreviating $(t_1, z_1; ...; t_{j-1}, z_{j-1})$ by $\star$

and $(t_{l+1}, z_{l+1}; ...; t_n, z_n)$ by $\sharp$,

$$\begin{aligned}
&\int_0^{t_j} ds \Big(f_j^{(n)}(\star; s, c_j - s_j s; 0, c_{j+1}; ...; 0, c_{l-1}; 0, c_l; \sharp) \\
&\qquad - f_j^{(n)}(\star; s, c_j - s_j s; 0, c_{j+1}; ...; 0, c_{l-1}; t_l, z_l; \sharp) \Big) \\
&= -\int_0^{t_j} ds \int_0^{t_l} dr \, \frac{d}{dr} f_j^{(n)}(\star; s, c_j - s_j s; 0, c_{j+1}; ...; 0, c_{l-1}; r, c_l - s_l r; \sharp) \\
&\overset{(4.133)}{=} -\int_0^{t_l} dr \int_0^{t_j} ds \, \frac{d}{ds} f_l^{(n)}(\star; s, c_j - s_j s; 0, c_{j+1}; ...; 0, c_{l-1}; r, c_l - s_l r; \sharp) \\
&= \int_0^{t_l} dr \Big(f_l^{(n)}(\star; 0, c_j; 0, c_{j+1}; ...; 0, c_{l-1}; r, c_l - s_l r; \sharp) \\
&\qquad - f_l^{(n)}(\star; t_j, z_j; 0, c_{j+1}; ...; 0, c_{l-1}; r, c_l - s_l r; \sharp) \Big),
\end{aligned} \tag{4.137}$$

where the consistency condition (4.133) has been used to obtain $\frac{d}{dr} f_j^{(n)}(\cdots) = \frac{d}{ds} f_l^{(n)}(\cdots)$ with the argument $(\cdots)$ as in (4.137). Inserting the result of (4.137) in all the summands in (4.136) leads to

$$\begin{aligned}
&\int_0^{t_j} ds \, f_j^{(n)}(\star; s, c_j - s_j s; 0, c_{j+1}; ...; 0, c_n) \\
&= \int_0^{t_j} ds \, f_j^{(n)}(\star; s, c_j - s_j s; t_{j+1}, z_{j+1}; ...; t_n, z_n) \\
&+ \sum_{l=j+1}^{n} \int_0^{t_l} ds \, \Big(f_l^{(n)}(\star; 0, c_j; 0, c_{j+1}; ...; 0, c_{l-1}; s, c_l - s_l s; \sharp) \\
&\qquad - f_l^{(n)}(\star; t_j, z_j; 0, c_{j+1}; ...; 0, c_{l-1}; s, c_l - s_l s; \sharp) \Big).
\end{aligned} \tag{4.138}$$

Inserting this expression into (4.134) yields:

$$\begin{aligned}
&\psi^{(n)}_{s_1 ... s_n}(t_1, z_1; ...; t_n, z_n) = \psi_0^{(n)}(c_1, ..., c_n) \\
&\quad -i \sum_{k<j} \int_0^{t_k} ds \; f^{(n)}_{k, s_1 ... s_n}(t_1, z_1; ...; t_{k-1}, z_{k-1}; s, c_k - s_k s; 0, c_{k+1}; ...; 0, c_n) \\
&\quad -i \int_0^{t_j} ds \; f^{(n)}_{j, s_1 ... s_n}(t_1, z_1; ...; t_{k-1}, z_{k-1}; s, c_j - s_j s; t_{k+1}, z_{k+1}; ...; t_n, z_n) \\
&\quad -i \sum_{k>j} \int_0^{t_k} ds \; f^{(n)}_{k, s_1 ... s_n}(t_1, z_1; ...; 0, c_j; ...; t_{k-1}, z_{k-1}; s, c_k - s_k s; 0, c_{k+1}; ...; 0, c_n).
\end{aligned} \tag{4.139}$$

Considering this expression, it becomes obvious that the same calculation as in (4.135) results in

$$i(\partial_{t_j} - s_j \partial_{z_j}) \psi^{(n)}_{s_1 ... s_n}(t_1, z_1, ..., t_n, z_n) = f^{(n)}_{j, s_1 ... s_n}(t_1, z_1, ..., t_n, z_n), \tag{4.140}$$

as desired. So (4.134) defines a solution of the IBC system for points without collisions. The fact that (4.134) gives the only solution of the IBC system on the points under consideration follows from the uniqueness of solutions of each single equation $(\partial_{t_k} - s_k \partial_{z_k}) \psi^{(n)} = f_k^{(n)}$ (see [53] and compare also [21, thm. 4.4]).

Induction step $\mathbf{A}(\mathbf{L}-\mathbf{1}) \Rightarrow \mathbf{A}(\mathbf{L})$. Let a point $(t_1, z_1, ..., t_n, z_n)$ and spin indices $(s_1, ..., s_n)$ be given such that $|\mathsf{Collisions}| = L$. The collision with the greatest time t_L must have the form $(k_L, k_L + 1)$ with k_L given by (4.130). (This can be seen from diagrams such as Fig. 4.1 and is easy to prove using the claim in the proof of [21, lemma 6.2].) W.l.o.g. we assume that $s_{k_L} = +1$ and $s_{k_L+1} = -1$. The reversed case is the only other possible one; it can be treated analogously.
We proceed in two steps: First we connect the value of $\psi^{(n)}$ at $(t_1, z_1, ..., t_N, z_N)$ with the value at the largest collision time τ_L via integration along the multi-time characteristics. Secondly, we implement the IBC at time τ_L via the operator V acting on a wave function component with only $L-1$ collisions (that is known due to the induction assumption).
For the first step, suppose the function $\psi^{(n)}_{\tau_L} = \psi^{(n)}|_{t_1=...=t_n=\tau_L} \in C^1_b(Z_n, \mathbb{C}^{2^n})$ is given. Analogously to the base case $L = 0$, it follows that the component $\psi^{(n)}_{s_1...s_n}$ solves the equations

$$i\left(\partial_{t_k} - s_k \partial_{z_k}\right) \psi^{(n)}_{s_1...s_n} \;=\; f^{(n)}_{k,s_1...s_n}, \quad k = 1, ..., n \tag{4.141}$$

if and only if it is given by

$$\psi^{(n)}_{s_1...s_n}(t_1, z_1, ..., t_n, z_n) \;=\; \left(I^{(t_1-\tau_L,...,t_n-\tau_L)}_{\tau_L} \psi^{(n)}_{\tau_L}\right)_{s_1...s_n}(z_1, ..., z_n). \tag{4.142}$$

In the second step, we want to find $\psi^{(n)}_{\tau_L}$. The multi-time characteristic associated with $\psi^{(n)}_{s_1...s_n}(t_1, z_1, ..., t_n, z_n)$ intersects the boundary of $\mathscr{S}^{(n)}_1$ in $P := (\tau_L, z_1 + s_1(\tau_L - t_1), ..., \tau_L, z_n + s_n(\tau_L - t_n))$. At that point, the component $\psi^{(n)}_{s_1...-+...s_n}$ (where $- = s_{k_L}$ and $+ = s_{k_L+1}$) has one collision less than $\psi^{(n)}_{s_1...+-...s_n}$, so it has $L-1$ collisions with the same times τ_a and indices k_a, $a = 1, ..., L-1$. By the induction hypothesis, it is then given by (4.131), i.e.

$$\begin{aligned}\psi^{(n)}_{s_1...-+...s_n}(P) = \Bigg(I^{(\tau_L-\tau_{L-1},...,\tau_L-\tau_{L-1})}_{\tau_{L-1}} \times \\ \times \left(\prod_{a=1}^{L-1} V^{\tau_a}_{k_a} I^{(\tau_a-\tau_{a-1},...,\tau_a-\tau_{a-1})}_{\tau_{a-1}}\right) \psi^{(n)}_0\Bigg)_{s_1...-+...s_n} (z_1 + s_1(\tau_L - t_1), ..., z_n + s_n(\tau_L - t_n)).\end{aligned} \tag{4.143}$$

By comparison of (4.127) with the IBC from (4.125), it becomes apparent that the latter evaluated at P is equivalent to

$$\psi^{(n)}_{s_1...+-...s_n}(P) = \left(V^{\tau_L}_{k_L} \psi^{(n)}(\tau_L, \cdot, ..., \tau_L, \cdot)\right)_{s_1...+-...s_n} (z_1 + s_1(\tau_L - t_1), ..., z_n + s_n(\tau_L - t_n)). \tag{4.144}$$

Therefore, combining both steps, we see that the system (4.125) is satisfied if and only if $\psi^{(n)}_{s_1...+-...s_n}(t_1, z_1, ..., t_n, z_n)$ is given by (4.131). This finishes the induction and thus the proof. □

The insight that Eq. (4.131) gives the solution in a specified sector is the basis of the following central definition.

Definition (fixed point map). Recall the definition (4.44) of the spaces $\mathscr{B}_n$ and let

$$\mathcal{D} := \left\{(\psi^{(1)}, ..., \psi^{(N)}) \in \mathscr{B}_1 \oplus \cdots \oplus \mathscr{B}_N : \psi^{(n)}\big|_{t_1=...=t_n=0} = \psi^{(n)}_0 \;\forall n = 1, ..., N\right\}. \tag{4.145}$$

Then the map $F : \mathcal{D} \to \mathcal{D}$, $(v^{(1)}, ..., v^{(N)}) \mapsto (\psi^{(1)}, ..., \psi^{(N)})$ is defined by the following procedure:

- Let $\psi^{(1)}$ be given by formula (4.131) for $n = 1$ with no boundary terms ($g^{(1)} = 0$ since $\partial\mathscr{S}^{(1)} = \emptyset$) and the inhomogeneity

$$f^{(1)}_{1,s}(x_1) \;= -\sum_{t,u=\pm 1} A^{tu}_s v^{(2)}_{tu}(x_1, x_1). \tag{4.146}$$

- Repeat the following for all $n = 2, ..., N-1$ in ascending order: $\psi^{(n)}$ is defined by formula (4.131) with boundary terms given, as in (4.39), by the already determined function $\psi^{(n-1)}$ and the inhomogeneity

$$f^{(n)}_{k,s_1\dots s_n}(x_1, ..., x_n) \;= \sum_{t,u=\pm 1} (-1)^k A^{tu}_{s_k} v^{(n+1)}_{s_1\dots s_{k-1}\, t\, u\, s_{k+1}\dots s_n}(x_1, ..., x_k, x_k, x_{k+1}, ..., x_n). \tag{4.147}$$

- Finally, $\psi^{(N)}$ is defined by (4.131) with boundary terms given, as in (4.39), by the already determined $\psi^{(N-1)}$ and the inhomogeneity $f^{(N)} = 0$.

Well-definedness of F. To show that F actually maps into $\mathcal{D}$, we have to check the C^1_b-property of the function defined by (4.131). Since for $n = 0$, the set Collisions is empty, $\psi^{(1)} \in C^1_b$ follows directly from the properties of the initial values. For $n \geq 2$, one has to consider those points separately where for some $j < k, c_j = c_k$. This occurs in $\mathscr{S}^{(n)}_1$ for positive times only if $s_j = +1$ and $s_k = -1$. When $c_j \searrow c_k$, the collision time $\tau_{(j,k)}$ approaches 0, so continuity at a point with $c_j = c_k$ amounts to:

$$\psi^{(n)}_{0,s_1\dots+-\dots s_n}(z_1, ...z_{j-1}, z, z, ..., z_n) \;=\; V^0_j \psi^{(n)}_{0,s_1\dots+-\dots s_n}(z_1, ...z_{j-1}, z, z, ..., z_n), \tag{4.148}$$

which follows from (4.42). By an argument analogous to the one given in (4.103) for $N = 2$, one can see that the z-derivative of (4.42) together with (4.43) implies the C^1_b-property of the functions $\psi^{(n)}$.
To continue with the fixed point argument, we endow the spaces $\mathscr{B}_n$ with weighted norms, similarly as in (4.104). For $\gamma \geq 0$,

$$\begin{aligned}\big\|f\big\|_{\mathscr{B}_n,\gamma} := \sup_{\substack{t_1,\dots,t_n\in[0,T]\\(z_1,\dots,z_n)\in Z_n}} \Bigg(\Big(&|f(t_1, z_1, ...t_n, z_n)| + \\ &+ \max_{y\in\{t_1,z_1,\dots t_n,z_n\}} |\partial_y f(t_1, z_1, ...t_n, z_n)|\Big) e^{-\frac{\gamma}{n}(t_1+\dots+t_n)}\Bigg).\end{aligned} \tag{4.149}$$

As in the case $N = 2$, $|\cdot|$ denotes the maximum norm in the finite dimensional spaces $\mathbb{C}^{2^n}$. Moreover, the norm on C^1_b is defined as

$$\big\|f\big\|_{C^1_b(Z_n,\mathbb{C}^{2^n})} := \sup_{(z_1,\dots,z_n)\in Z_n} \Big(|f(z_1, ..., z_n)| \;+ \max_{y\in\{z_1,\dots,z_n\}} |\partial_y f(z_1, ..., z_n)|\Big). \tag{4.150}$$

The constructive proof of the previous lemma directly yields a bound of the norm of $\psi^{(n)}$.

Lemma 4.12. *The function $\psi^{(n)}$ given by* (4.131) *satisfies the bound*

$$\Big\|\psi^{(n)}\Big\|_{\mathscr{B}_n,\gamma} \leq \; \|\psi_0\|_{C^1_b(Z_n,\mathbb{C}^{2^n})} + \frac{2n^4}{\gamma} \max_{1\leq k\leq n} \Big\|f^{(n)}_k\Big\|_{\mathscr{B}_n,\gamma} + n^2 \max_{1\leq j\leq n-1} \Big\|g^{(n)}_j\Big\|_{\mathscr{B}_{n-1},\gamma}. \tag{4.151}$$

Proof. Let $\phi \in C_b^1(Z_n, \mathbb{C}^{2^n})$. Then by (4.126),

$$\begin{aligned}
&\left|I_0^{(t_1,\dots,t_n)}\phi(z_1,\dots,z_n)\right| - |\phi(c_1,\dots,c_n)| \le \sum_{k=1}^{n}\left|\int_0^{t_k} ds\ f_k^{(n)}(t_1,z_1;\dots;s,c_k-s_ks;0,c_{k+1};\dots)\right| \\
&\le n \max_{k\in\{1,\dots,n\}}\left(\int_0^{t_k} ds\ \left|f_k^{(n)}(t_1,z_1;\dots;s,c_k-s_ks;0,c_{k+1};\dots)\right| e^{-\frac{\gamma}{n}s} e^{\frac{\gamma}{n}s}\right). \\
&\le n \max_{k\in\{1,\dots,n\}} \sup_{s\in[0,t_k]}\left|f_k^{(n)}(t_1,z_1;\dots;s,c_k-s_ks;0,c_{k+1};\dots)e^{-\frac{\gamma}{n}s}\right| \int_0^{t_k} ds\ e^{\frac{\gamma}{n}s} \\
&\le n \max_{k\in\{1,\dots,n\}} \sup_{s\in[0,t_k]}\left|f_k^{(n)}(t_1,z_1;\dots;s,c_k-s_ks;0,c_{k+1};\dots)e^{-\frac{\gamma}{n}(s+t_1+\dots+t_{k-1})}\right| \frac{n}{\gamma} e^{\frac{\gamma}{n}(t_k+t_1+\dots+t_{k-1})}.
\end{aligned} \tag{4.152}$$

This implies:

$$\begin{aligned}
&\left|I_0^{(t_1,\dots,t_n)}\phi(z_1,\dots,z_n)\right| e^{-\frac{\gamma}{n}(t_1+\dots+t_n)} \ \le\ |\phi(c_1,\dots,c_n)|\, e^{-\frac{\gamma}{n}(t_1+\dots+t_n)} + \\
&+ \frac{n^2}{\gamma} \max_{k\in\{1,\dots,n\}} \sup_{s\in[0,t_k]} \left|f_k^{(n)}(t_1,z_1;\dots;s,c_k-s_ks;0,c_{k+1};\dots)\right| e^{-\frac{\gamma}{n}(s+t_1+\dots+t_{k-1})}.
\end{aligned} \tag{4.153}$$

Changing the starting time from 0 to t changes only little,

$$\begin{aligned}
&\left|I_t^{(t_1,\dots,t_n)}\phi(z_1,\dots,z_n)\right| e^{-\frac{\gamma}{n}(t+t_1+\dots+t+t_n)} \ \le\ |\phi(c_1,\dots,c_n)|\, e^{-\frac{\gamma}{n}(t+t_1+\dots+t+t_n)} + \\
&+ \frac{n^2}{\gamma} \max_{k\in\{1,\dots,n\}} \sup_{s\in[0,t_k]} \left|f_k^{(n)}(t+t_1,z_1;\dots;t+s,c_k-s_ks;t,c_{k+1};\dots)\right| e^{-\frac{\gamma}{n}(t+s+t+t_1+\dots+t+t_{k-1})}.
\end{aligned} \tag{4.154}$$

Hence, together with a similar consideration for the derivatives (analogous to the one for $N=2$, $n=1$ in (4.111)) we find:

$$\left\|I_t^{(t_1,\dots,t_n)}\phi\right\|_{\mathscr{B}_n,\gamma} \ \le\ \|\phi\|_{C_b^1(Z_n,\mathbb{C}^{2^n})} + \frac{2n^2}{\gamma} \max_{1\le k\le n}\left\|f_k^{(n)}\right\|_{\mathscr{B}_n,\gamma}. \tag{4.155}$$

Similarly, it follows from the definition (4.127) that

$$\left\|V_j^t\phi\right\|_{C_b^1(Z_n,\mathbb{C}^{2^n})} e^{-\frac{\gamma}{n}nt} \ \le\ \|\phi\|_{C_b^1(Z_n,\mathbb{C}^{2^n})}\, e^{-\frac{\gamma}{n}nt} + \left\|g_j^{(n)}\right\|_{C_b^1(Z_{n-1},\mathbb{C}^{2^{n-1}})} e^{-\frac{\gamma}{n-1}(n-1)t}. \tag{4.156}$$

We see that each application of V_k^t leads to an additive contribution of at most

$$\max_{1\le j\le n}\left\|g_j^{(n)}\right\|_{C_b^1(Z_{n-1},\mathbb{C}^{2^{(n-1)}})} \times e^{-\frac{\gamma}{n-1}(n-1)t}, \tag{4.157}$$

where the last factor is the appropriate weight factor of the γ-norm. Thus,

$$\left\|V_k^t\phi\right\|_{\mathscr{B}_n,\gamma} \ \le\ \|\phi\|_{C_b^1(Z_n,\mathbb{C}^{2^n})} + \max_{1\le j\le n-1}\left\|g_j^{(n)}\right\|_{\mathscr{B}_{n-1},\gamma}. \tag{4.158}$$

We know that $\psi^{(n)}$ is given by the formula (4.131). If there are at most L collisions, the operators I and V are applied at most $L+1$ times. Each time, the terms from equations (4.155) and (4.158) add up. Therefore, we obtain the following bound of the norm of $\psi^{(n)}$:

$$\left\|\psi^{(n)}\right\|_{\mathscr{B}_n,\gamma} \le \left\|\psi_0^{(n)}\right\|_{C_b^1(Z_n,\mathbb{C}^{2^n})} + (L+1)\,\frac{2n^2}{\gamma} \max_{1\le k\le n}\left\|f_k^{(n)}\right\|_{\mathscr{B}_n,\gamma} + (L+1) \max_{1\le j\le n-1}\left\|g_j^{(n)}\right\|_{\mathscr{B}_{n-1},\gamma}. \tag{4.159}$$

Since the number of collisions is bounded by $L+1 < n^2$, this yields (4.151). □

Lemma 4.13. *Let $(\psi^{(1)}, ..., \psi^{(N)}) \in \mathscr{B}_1 \oplus \cdots \oplus \mathscr{B}_N$. Then the following statements are equivalent:*

(i) $(\psi^{(1)}, ..., \psi^{(N)})$ is a C_b^1-solution of the IBC system (4.36)–(4.39) with initial values given by (4.40).

(ii) $(\psi^{(1)}, ..., \psi^{(N)}) \in \mathcal{D}$ is a fixed point of F.

Proof. On the one hand, if (i) holds then we have $(\psi^{(1)}, ..., \psi^{(N)}) \in \mathcal{D}$ and F is constructed such that it does not change the functions $\psi^{(n)}$.
On the other hand, let $(\psi^{(1)}, ..., \psi^{(N)}) \in \mathcal{D}$ be a fixed point of F. Then the initial conditions are satisfied by definition of $\mathcal{D}$. It remains to check that $(\psi^{(1)}, ..., \psi^{(N)})$ also satisfies the multi-time equations and IBCs. To demonstrate this, we would like to apply Lemma 4.11. However, in order to show that the IBC system is satisfied for some sector $1 \leq n \leq N$, Lemma 4.11 requires the consistency conditions (4.133) for that n. Now for $n = N$, (4.133) is satisfied trivially as $f_k^{(N)} = 0 \, \forall k$. That means, Lemma 4.11 yields that $\psi^{(N)}$ satisfies the IBC system for the N-th sector. Now we can use Lemma 4.10 to conclude that the consistency condition (4.133) holds also for $n = N - 1$. Thus, Lemma 4.11 shows that the IBC system is satisfied for $n = N - 1$. Using the two lemmas alternatingly for descending $n = N - 2, ..., 1$ eventually yields (i). □

Lemma 4.14. *For every $T > 0$, F has a unique fixed point in $\mathcal{D}$.*

Proof. We shall use Banach's fixed point theorem. To this end, we first show that $\mathcal{D}$ is a closed subset of a Banach space. For $\gamma = 0$, the weighted norms (4.149) on $\mathscr{B}_n$ are the canonical norms on C_b^1-function, and it is well-know that this leads to complete spaces. Since we have the inequalities

$$\left\| f^{(n)} \right\|_{\mathscr{B}_n,0} e^{-\gamma T} \leq \left\| f^{(n)} \right\|_{\mathscr{B}_n,\gamma} \leq \left\| f^{(n)} \right\|_{\mathscr{B}_n,0} \quad \forall n \in \mathbb{N}, \tag{4.160}$$

the $\| \cdot \|_{\mathscr{B}_n,\gamma}$ norms are all equivalent for all $\gamma \geq 0$. Thus, $\mathscr{B}_1 \oplus \cdots \oplus \mathscr{B}_N$ equipped with the norm

$$\left\| (f^{(1)}, ..., f^{(N)}) \right\|_{\mathscr{B}_1 \oplus \cdots \oplus \mathscr{B}_N,\gamma} = \sum_{n=1}^{N} \left\| f^{(n)} \right\|_{\mathscr{B}_n,\gamma}. \tag{4.161}$$

is a Banach space. Moreover, it is easy to see that $\mathcal{D} \subset \mathscr{B}_1 \oplus \cdots \oplus \mathscr{B}_N$ is a closed subset. It remains to show that $F : \mathcal{D} \to \mathcal{D}$ is a contraction. To this end, let

$$\begin{aligned} &(v^{(1)}, ..., v^{(N)}), (w^{(1)}, ..., w^{(N)}) \in \mathcal{D}, \\ &(\psi^{(1)}, ..., \psi^{(N} := F(v^{(1)}, ..., v^{(N)}), (\phi^{(1)}, ..., \phi^{(N)}) := F(w^{(1)}, ..., , w^{(N)}). \end{aligned} \tag{4.162}$$

By linearity of the IBC system, $\psi^{(n)} - \phi^{(n)}$ is, for $n = 1, ..., N$, a solution of (4.125) with the inhomogeneities and boundary values replaced by the difference between the ones for $\psi^{(n)}$ and $\phi^{(n)}$ and initial data equal to zero. Keeping this in mind, we now prove the following (crude) estimate by induction over $n \leq N$:

$$\left\| \psi^{(n)} - \phi^{(n)} \right\|_{\mathscr{B}_n,\gamma} \leq \frac{1}{\gamma} (2N)^{4n} \|A\|_\infty \left(\sum_{k=2}^{\min\{n+1,N\}} \|B\|_\infty^{n+1-k} \left\| v^{(k)} - w^{(k)} \right\|_{\mathscr{B}_k,\gamma} \right). \tag{4.163}$$

Base case $n = 1$: Considering that for $n = 1$, there are no boundary conditions, the bound (4.151) directly leads to:

$$\begin{aligned}\left\|\psi^{(1)} - \phi^{(1)}\right\|_{\mathscr{B}_1,\gamma} &\leq 0 + \frac{2^4}{\gamma}\left\|A\left(v^{(2)}(x_1,x_1) - w^{(2)}(x_1,x_1)\right)\right\|_{\mathscr{B}_1,\gamma} \\ &\leq \frac{1}{\gamma}(2N)^4 \|A\|_\infty \left\|v^{(2)} - w^{(2)}\right\|_{\mathscr{B}_2,\gamma},\end{aligned} \tag{4.164}$$

i.e., (4.163) for $n = 1$.

Induction step $n \to n+1$: Assume that (4.163) holds for some $n \geq 1$. We first consider the case $n \leq N-2$. Using (4.151) for $n+1$ and then the induction hypothesis yields:

$$\begin{aligned}\left\|\psi^{(n+1)} - \phi^{(n+1)}\right\|_{\mathscr{B}_{n+1},\gamma} \leq\ & \frac{(2N)^4}{\gamma}\|A\|_\infty \left\|v^{(n+2)} - w^{(n+2)}\right\|_{\mathscr{B}_{n+2,\gamma}} \\ &+ N^2 \|B\|_\infty \left\|\psi^{(n)} - \phi^{(n)}\right\|_{\mathscr{B}_n,\gamma} \\ \leq\ & \frac{1}{\gamma}(2N)^{4(n+1)}\|A\|_\infty \left(\sum_{k=2}^{n+2}\|B\|_\infty^{n+2-k}\left\|v^{(k)} - w^{(k)}\right\|_{\mathscr{B}_k,\gamma}\right).\end{aligned} \tag{4.165}$$

In case $n = N-1$, the same calculation goes through, except that the term with $v^{(n+2)} - w^{(n+2)}$ is absent since $f_k^{(N)} = 0\,\forall k$. Thus, the sum ends at N. This proves (4.163).
Next, we determine an upper bound on the total norm,

$$\begin{aligned}&\left\|\left(\psi^{(1)},...,\psi^{(N)}\right) - \left(\phi^{(1)},...,\phi^{(N)}\right)\right\|_{\mathscr{B}_1\oplus\cdots\oplus\mathscr{B}_N,\gamma} \leq \sum_{n=1}^{N}\left\|\psi^{(n)} - \phi^{(n)}\right\|_{\mathscr{B}_n,\gamma} \\ &\leq \frac{1}{\gamma}(2N)^{4N}\|A\|_\infty \sum_{n=1}^{N}\sum_{k=2}^{\min\{n+1,N\}}\|B\|_\infty^{n+1-k}\left\|v^{(k)} - w^{(k)}\right\|_{\mathscr{B}_k,\gamma} \\ &\leq \frac{1}{\gamma}(2N)^{4N}\|A\|_\infty \sum_{n=1}^{N}\max\{1,\|B\|_\infty^N\}\sum_{k=1}^{N}\left\|v^{(k)} - w^{(k)}\right\|_{\mathscr{B}_k,\gamma} \\ &\leq \frac{1}{\gamma}(2N)^{4N+1}\|A\|_\infty \max\{1,\|B\|_\infty^N\}\left\|(v^{(1)},...,v^{(N)}) - (w^{(1)},...,w^{(N)})\right\|_{\mathscr{B}_1\oplus\cdots\oplus\mathscr{B}_N,\gamma}.\end{aligned} \tag{4.166}$$

If we now choose, for example, $\gamma = 5\,(2N)^{4N+1}\|A\|_\infty \max\{1,\|B\|_\infty^N\}$, then (4.166) yields:

$$\begin{aligned}&\left\|F(v^{(1)},...,v^{(N)}) - F(w^{(1)},...,w^{(N)})\right\|_{\mathscr{B}_1\oplus\cdots\oplus\mathscr{B}_N,\gamma} \\ \leq\ & \frac{1}{5}\left\|(v^{(1)},...,v^{(N)}) - (w^{(1)},...,w^{(N)})\right\|_{\mathscr{B}_1\oplus\cdots\oplus\mathscr{B}_N,\gamma}.\end{aligned} \tag{4.167}$$

Thus F is a contraction, and Banach's fixed point theorem yields the claim. □

Together with Lemma 4.13, this finishes the proof of Theorem 4.4, establishing that the multi-time IBC system has a unique C_b^1-solution for all positive times.

4.7.5 Proof of local probability conservation in the N-sector model

Proof of Theorem 4.5. We need to show that (4.17) holds. To this end, we consider both sides of the condition separately. For $n = N$, we just have the free evolution equations on $\mathscr{S}_1^{(N)}$, which ensure (4.17). We now turn to $1 \leq n < N$.
On the one hand, we have (writing out $j^{\mu_1 \cdots \mu_{n+1}}$ in components):

$$j^{\mu_1 \cdots \mu_{n+1}} = \sum_{s_1 \ldots s_{n+1} = \pm 1} |\psi^{(n+1)}_{s_1 \ldots s_{n+1}}|^2 \, (-s_1)^{\mu_1} \cdots (-s_{n+1})^{\mu_{n+1}}, \tag{4.168}$$

and therefore:

$$\begin{aligned} &j^{\mu_1 \ldots \mu_{k-1}\, 01\, \mu_{k+1} \ldots \mu_n} - j^{\mu_1 \ldots \mu_{k-1}\, 10\, \mu_{k+1} \ldots \mu_n} \\ &= 2 \sum_{s_1, \ldots, \widehat{s_k}, \ldots, s_n} (-s_1)^{\mu_1} \cdots \widehat{(-s_k)^{\mu_k}} \cdots (-s_n)^{\mu_n} \big(- |\psi^{(n+1)}_{s_1 \ldots s_{k-1} - + s_{k+1} \ldots s_n}|^2 + |\psi^{(n+1)}_{s_1 \ldots s_{k-1} + - s_{k+1} \ldots s_n}|^2 \big), \end{aligned} \tag{4.169}$$

where $\widehat{(\cdot)}$ denotes omission. On the other hand, we compute:

$$\begin{aligned} (-1)^k \partial_{k,\mu_k} j^{\mu_1 \ldots \mu_k \ldots \mu_n}(x_1, ..., x_n) &\overset{(4.168)}{=} \sum_{s_1, \ldots, s_n} (-s_1)^{\mu_1} \cdots \widehat{(-s_k)^{\mu_k}} \cdots (-s_n)^{\mu_n} \\ &\quad \times (-1)^k (\partial_{t_k} - s_k \partial_{z_k}) |\psi^{(n)}_{s_1 \ldots s_n}|^2 (x_1, ..., x_n). \end{aligned} \tag{4.170}$$

Using (4.36), we have:

$$\begin{aligned} &(-1)^k (\partial_{t_k} - s_k \partial_{z_k}) |\psi^{(n)}_{s_1 \ldots s_n}|^2 (x_1, ..., x_n) \\ &= -i \big(\psi^{(n)}\big)^*_{s_1 \ldots s_n} (x_1, ..., x_n) \left(\sum_{t,u} A^{tu}_{s_k} \psi^{(n+1)}_{s_1 \ldots s_{k-1}\, t\, u\, s_{k+1} \ldots s_n} (x_1, ..., x_{k-1}, x_k, x_k, x_{k+1}, ..., x_n) \right) + \text{c.c.} \\ &= 2 \Im \left(\big(\psi^{(n)}\big)^*_{s_1 \ldots s_n} (x_1, ..., x_n) \sum_{t,u} A^{tu}_{s_k} \psi^{(n+1)}_{s_1 \ldots s_{k-1}\, t\, u\, s_{k+1} \ldots s_n} (x_1, ..., x_{k-1}, x_k, x_k, x_{k+1}, ..., x_n) \right). \end{aligned} \tag{4.171}$$

Here "c.c." denotes the complex conjugate of the previous summand.
Now we compare (4.170) with (4.169). Demanding that both expressions be equal (condition (4.17)) yields

$$\begin{aligned} 0 \;=\; 2 \sum_{s_1, \ldots, \widehat{s_k}, \ldots, s_n} &(-s_1)^{\mu_1} \cdots \widehat{(-s_k)^{\mu_k}} \cdots (-s_n)^{\mu_n} \Big(\big(|\psi^{(n+1)}_{s_1 \ldots s_{k-1} - + s_{k+1} \ldots s_n}|^2 \\ &- |\psi^{(n+1)}_{s_1 \ldots s_{k-1} + - s_{k+1} \ldots s_n}|^2 \big)(x_1, ..., x_{k-1}, x_k, x_k, x_{k+1}, ..., x_n) \\ &+ \tfrac{1}{2} (-1)^k \sum_{s_k} (\partial_{t_k} - s_k \partial_{z_k}) |\psi^{(n)}_{s_1 \ldots s_n}|^2 (x_1, ..., x_n) \Big). \end{aligned} \tag{4.172}$$

This condition is certainly satisfied if all the summands vanish individually. Demanding that this be so and considering (4.171) gives the condition

$$\begin{aligned} &\Big(- |\psi^{(n+1)}_{s_1 \ldots s_{k-1} - + s_{k+1} \ldots s_n}|^2 + |\psi^{(n+1)}_{s_1 \ldots s_{k-1} + - s_{k+1} \ldots s_n}|^2 \Big) (x_1, ..., x_{k-1}, x_k, x_k, x_{k+1}, ..., x_n) \;= \\ &\sum_{s_k} \Im \left(\big(\psi^{(n)}\big)^*_{s_1 \ldots s_n} (x_1, ..., x_n) \sum_{t,u} A^{tu}_{s_k} \psi^{(n+1)}_{s_1 \ldots s_{k-1}\, t\, u\, s_{k+1} \ldots s_n} (x_1, ..., x_{k-1}, x_k, x_k, x_{k+1}, ..., x_n) \right). \end{aligned} \tag{4.173}$$

Now, except for additional spin indices and spacetime variables this condition has exactly the form of the condition for the model with only two sectors (see (4.30)). There the condition

$$\left(-|\psi^{(2)}_{-+}|^2 + |\psi^{(2)}_{+-}|^2\right)(x,x) = \sum_{s_1} \Im\left(\left(\psi^{(1)}\right)^*_{s_1}(x) \sum_{t,u} A^{tu}_{s_1} \psi^{(2)}_{tu}(x,x)\right) \tag{4.174}$$

was ensured by the IBC (4.21)

$$\psi^{(2)}_{-+} - e^{i\theta}\psi^{(2)}_{+-}(x,x) = \sum_s B^s \psi^{(1)}_s(x). \tag{4.175}$$

In an analogous way, we conclude that the IBCs (4.39) ensure the conditions (4.173) for $k = 1, ..., n$; $n = 1, ..., N-1$. □

4.8 Existence of a self-adjoint Hamiltonian

Previous articles on interior-boundary conditions such as [55] construct the dynamics by proving the self-adjointness of the respective Hamiltonian. In this chapter, we choose another way by direct treatment of the multi-time equations with methods from the theory of partial differential equations. This provides a multi-time evolution, but it is a natural question whether the corresponding single-time evolution is generated by a self-adjoint Hamiltonian. In other words: Have we proven as a side-product of our theorems above that a self-adjoint Hamiltonian exists describing the dynamics of $\psi(t, \mathbf{x}_1, ..., t, \mathbf{x}_N)$? The answer is Yes, and we explain it in this section, although we give only a motivation and do not provide a rigorous derivation.

Theorem 4.4 states that given initial data in $\psi^0 \in C^1_b$ that satisfy the conditions (4.42) and (4.43), we obtain a unique global solution ψ to the IBC model with N sectors. Call the set of all admissible initial values $\mathscr{A}$. If we restrict the solution to equal times, we obtain for $t \in \mathbb{R}$ a linear map

$$U(t) : \mathscr{A} \to \mathscr{H}, \left(U(t)\psi^0\right)(\mathbf{x}_1, ..., \mathbf{x}_N) := \psi(t, \mathbf{x}_1, ..., t, \mathbf{x}_N), \tag{4.176}$$

where $\mathscr{H}$ is the Hilbert space of our problem, a Fock space constructed over $L^2(\mathbb{R}, \mathbb{C}^2)$. This map satisfies by Thm. 4.5

$$\|U(t)\psi^0\|_{\mathscr{H}} = \|\psi^0\|_{\mathscr{H}}, \tag{4.177}$$

so it is bounded. We need to argue that $\mathscr{A} \subset \mathscr{H}$ is a dense set to obtain a unitary map $U(t) : \mathscr{H} \to \mathscr{H}$ on the whole Hilbert space by the bounded linear extension theorem. We do not give the argument for density of $\mathscr{A}$ in detail, but point out that it is heuristically clear since C^1_b is dense in L^2 and the additional constraints (4.42), (4.43) only fix the functions and their derivatives on a set of measure zero.

Since we additionally know from Thm. 4.4 that the solution $\psi(t, \mathbf{x}_1, ..., t, \mathbf{x}_N)$ is continuous in t, we have that $U(t)$ is actually a strongly continuous one-parameter group of unitary operators. Thus, by the theorem of Stone [36, thm. VIII.7], it can be written as

$$U(t) = e^{-i\mathcal{H}t}, \tag{4.178}$$

with a self-adjoint Hamiltonian operator $\mathcal{H} : \mathrm{dom}(\mathcal{H}) \subset \mathscr{H} \to \mathscr{H}$.

4.9 Discussion

By constructing an explicit model of massless Dirac particles in 1+1 dimensions, we have shown that interior-boundary conditions can be combined with Dirac's concept of multi-time wave functions. As the IBC approach uses the particle-position representation, multi-time wave functions are necessary to make the IBC approach manifestly covariant. We have then demonstrated rigorously the existence and uniqueness of solutions and identified a wide class of IBCs which ensure probability conservation on arbitrary Cauchy surfaces, in particular in all Lorentz frames. Overall, we have obtained a new rigorous and (almost fully) covariant model QFT. As discussed in Sec. 4.6, "almost fully covariant" here means that the model contains certain constant matrices A, B which would have to transform in a certain way for the model to be Lorentz invariant. We believe that this limitation is due to the fact that the model uses only fermions and does not involve exchange particles (e.g. photons) yet.

While we have explained how to obtain a multi-time formulation of the IBC approach only at the example of a particular model, we think that the basic insights can also be transferred to more general cases. For higher space-time dimensions, for example, we expect that a similar consideration as used in Sec. 4.7.1 to obtain the condition (4.15) for local probability conservation will lead to the correct IBCs. This consideration only requires that the respective free multi-time equations have a conserved tensor current $j^{\mu_1 \cdots \mu_n}(x_1, ..., x_n)$ with the right properties for each sector of Fock space.

For higher dimensions, though, one will have to take into account that the codimension of the set of coincidence points $\mathscr{C}$ is greater than for 1+1 dimensions. To determine the probability flow into $\mathscr{C}$, one then has to take integrals over balls with infinitesimal radii instead of mere limits towards $\mathscr{C}$. This will lead to the appearance of integrals also in the IBC, as well as in the term which then replaces $A\psi^{(n+1)}$ in the equation of motion for $\psi^{(n)}$. The integrals in the IBC will then force the wave function to become divergent, i.e., more singular than in the 1+1 dimensional case. It seems that such a strong singularity is related to the ultraviolet divergence problem. It would be a highly interesting question for future research to see whether the IBC approach can also help to overcome the UV problem in these cases. Previous works in the non-relativistic case (i.e., for the Laplacian as the free Hamiltonian) [5, 6, 57, 58] suggest that this may be so; however, in the relativistic case (and for the Dirac Hamiltonian) the question is completely open. An important obstacle is that point interactions do not seem to be possible for the Dirac operator in 1+3 dimensions [63].

As indicated before, it would be desirable to treat more realistic QFTs involving exchange bosons with the multi-time IBC approach in the future. The exchange particles would help to make the theory fully Lorentz invariant. However, one then requires a suitable particle-position representation of bosonic wave functions which is an open problem by itself. For photons, the difficulties have recently been discussed in [64] and a new wave equation for the photon has been suggested. It might be possible to combine a multi-particle (and multi-time) version of that equation with our model. As the probability current of [64] involves a preferred time-like vector field, the IBCs (and consequently the whole wave function dynamics) would then likely depend on that preferred vector field.

Finally, we believe that our results hold also in the massive case. After all, the mass term corresponds only to a bounded operator, so in the single-time model, adding it does not change the self-adjointness of the Hamiltonian. However, the simple solution formulas we have used for our fixed point scheme in the multi-time equations are not available then. Consequently, we expect the analysis to become substantially more complicated.

CHAPTER
FIVE

LINEAR MULTI-TIME SYSTEMS OF ARBITRARY ORDER: EXISTENCE AND UNIQUENESS RESULTS

The multi-time systems we looked at so far were always of first order in the time derivatives, but systems of Klein-Gordon equations or other higher-order equations are also possible. In this final mathematical chapter, we investigate a very general multi-time system of partial differential equations of the form

$$\frac{\partial^n}{\partial t_k^n}\psi(x_1,...,x_N) = L_k(x_1,...,x_N)\psi(x_1,...,x_N), \qquad k=1,...,N, \tag{5.1}$$

with $n \in \mathbb{N}$, for the unknown function $\psi : \mathbb{R}^{4N} \to \mathbb{C}^K$. This gives us an occasion to generalize the ideas of the solution theory to multi-time systems from above. It is useful to consider strong solutions in this general case – note that the case $n = 1$ relevant for the largest part of this thesis is also allowed here, but the focus is on the more general case $n > 1$.

Initial values are given at some fixed times $s_1, ..., s_N \in \mathbb{R}$, in most cases $s_1 = ... = s_N = 0$ for simplicity. Here, L_k is an arbitrary linear differential operator on the spatial coordinates, i.e. it contains no derivatives with respect to any t_j. The order of L_k is not relevant for our considerations. The order n of time derivatives, however, plays an important role. We present results on the existence and uniqueness in a multi-time system of the form (5.1) *given* the knowledge about respective properties for solutions of each single equation.

We write a single equation for some fixed k as an equation with one time coordinate t_k and fixed parameters $S = (s_1, ..., \widehat{s}_k, ..., s_N) \in \mathbb{R}^{N-1}$, where a hat denotes omission, as

$$\frac{\partial^n}{\partial t_k^n}\psi(t_k, \mathbf{x}_1, ..., \mathbf{x}_N; S) = L_k(t_k, \mathbf{x}_1, ..., \mathbf{x}_N; S)\psi(t_k, \mathbf{x}_1, ..., \mathbf{x}_N; S). \tag{5.2}$$

The solution of such an equation is supposed to be an element $\psi \in C^n(\mathbb{R}, \mathrm{dom}(L_k))$, where $\mathrm{dom}(L_k)$ denotes the domain of that operator in the sense of

$$\mathrm{dom}(L) := \left\{\phi : \mathbb{R}^{3N} \to \mathbb{C}^K \middle| L\phi \in C(\mathbb{R}^{3N}, \mathbb{C}^K)\right\}. \tag{5.3}$$

In the study of differential equations, one often confines oneself to first order equations and then demonstrates how to reduce higher order equations to first order ones. Although this is also possible in multi-time systems, it is far from obvious how to do so and actually complicates the presentation, so we only briefly comment on it at the end of this chapter. We first demonstrate at an example in 5.1 which initial values are necessary and then prove theorems about the uniqueness (Sec. 5.2) and existence (Sec. 5.3) of solutions to (5.1). We then derive an analogue of the consistency condition for higher order systems in Sec. 5.4.

5.1 Necessary initial values

It is a possibly surprising fact that n^N initial conditions are needed to ensure uniqueness of solutions to system (5.1). We explain this by the simple example of

$$\partial_{t_k}^n \psi(t_1, \mathbf{x}_1, ..., t_N, \mathbf{x}_N) = 0, \qquad k = 1, ..., N. \tag{5.4}$$

This system is solved by the function

$$\psi^\alpha(t_1, \mathbf{x}_1, ..., t_N, \mathbf{x}_N) = t_1^{\alpha_1} \cdot t_2^{\alpha_2} \cdot ... \cdot t_N^{\alpha_N}, \tag{5.5}$$

for every multi-index $\alpha = (\alpha_1, ..., \alpha_N) \in \{0, ..., n-1\}^N$. Since $(\partial_t)^j \, t^k\big|_{t=0} = k! \delta_{j,k}$, we have for each multi-index $\beta \in \mathbb{N}_0^N$,

$$\partial_{t_1}^{\beta_1} \dots \partial_{t_N}^{\beta_N} \psi^\alpha\big|_{t_1=...=t_N=0} = \prod_{j=1}^N (\delta_{\alpha_j,\beta_j} \alpha_j!) \equiv \delta_{\alpha,\beta} \alpha!, \tag{5.6}$$

which demonstrates that the ψ^α are n^N different solutions to (5.4) that are only uniquely determined if we know all derivatives

$$\partial_t^\beta \psi^\alpha\big|_{t_1=...=t_N=0} \equiv \partial_{t_1}^{\beta_1} \dots \partial_{t_N}^{\beta_N} \psi^\alpha\big|_{t_1=...=t_N=0} \tag{5.7}$$

for $\beta \in \{0, ..., n-1\}^N$. Thus, we need n^N initial conditions on ψ. As a further side-effect of this, the mixed time derivatives of ψ indeed have to be known to a high order. It thus seems like we need to require differentiability up to order $(n-1)N$ for $n > 1$. We use this, or even up to order nN for brevity, as a technical assumption in the uniqueness proof below.

5.2 Uniqueness of solutions

We now prove uniqueness of the multi-time solution given the uniqueness in the single equations. In essence, the theorem below states that uniqueness in multi-time systems on $\mathbb{R}^{4N}$ is rather direct, it follows without further assumptions except the higher regularity mentioned above, which is only required for technical reasons. We already remarked that this is common for multi-time systems because a solution to the system is a special case of a solution to a single equation. We formulate our assumption, which is (for the linear equation) a convenient formulation of the statement that each single equation has at most one solution for given initial values.

Assumption (U): For all $k = 1, ..., N$, all $S \in \mathbb{R}^{N-1}$, $s_k \in \mathbb{R}$, equation (5.2) with initial values given as

$$\psi(0, \mathbf{x}_1, ..., \mathbf{x}_N; S) = 0, \qquad \partial_{t_k}^m \psi(t_k, \mathbf{x}_1, ..., \mathbf{x}_N; S)\big|_{t_k=0} = 0 \;\; \forall m < n, \tag{5.8}$$

has only a single solution $\psi \in C^n(\mathbb{R}, \mathrm{dom}(L_k))$ given by

$$\psi(t_k, \mathbf{x}_1, ..., \mathbf{x}_N; S) = 0. \tag{5.9}$$

Theorem 5.1. *Consider a multi-time system* (5.1) *for which assumption* ***(U)*** *holds. Then there is at most one solution* $\psi \in C^{nN}(\mathbb{R}^N, \mathrm{dom}(L_1) \cap \cdots \cap \mathrm{dom}(L_N))$ *to* (5.1) *for initial values given as*

$$\partial_{t_1}^{\alpha_1} ... \partial_{t_N}^{\alpha_N} \psi(s_1, \mathbf{x}_1, ..., s_N, \mathbf{x}_N) = \varphi^\alpha(\mathbf{x}_1, ..., \mathbf{x}_N), \quad \varphi^\alpha \in \mathrm{dom}(L_1) \cap \cdots \cap \mathrm{dom}(L_N), \tag{5.10}$$

for all multi-indices $\alpha = (\alpha_1, ..., \alpha_N) \in \{0, ..., n-1\}^N$ *at some* $(s_1, ..., s_N) \in \mathbb{R}^N$.

Proof. The proof is by induction over N. For $N = 1$, assumption **(U)** directly implies uniqueness by the linearity of the equation.
So let $N \in \mathbb{N}$ and assume that the statement of the theorem holds for all $k \leq N$. Suppose that $\psi, \psi' \in C^{n(N+1)}(\mathbb{R}^{N+1}, \mathrm{dom}(L_1) \cap \cdots \cap \mathrm{dom}(L_{N+1}))$ are two solutions to (5.1) for $N+1$ with the same initial values φ^α. Define $\omega = \psi - \psi'$, hence

$$\partial_{t_1}^{\alpha_1} ... \partial_{t_{N+1}}^{\alpha_{N+1}} \omega(s_1, \mathbf{x}_1, ..., s_{N+1}, \mathbf{x}_{N+1}) = 0 \quad \forall \alpha \in \{0, ..., n-1\}^{N+1}. \tag{5.11}$$

If we treat $(s_{N+1}, \mathbf{x}_{N+1})$ as a fixed parameter, ω is a solution of system (5.1) for N variables, so the induction assumption implies

$$\omega(t_1, \mathbf{x}_1, ..., t_N, \mathbf{x_N}, s_{N+1}, \mathbf{x}_{N+1}) = 0 \quad \forall t_1, ..., t_N \in \mathbb{R}. \tag{5.12}$$

A similar derivation is now possible for the derivatives of ω: Let $m < n$. By the commutability of the derivatives of $\omega \in C^{n(N+1)}(\mathbb{R}^{N+1}, \mathrm{dom}(L_1) \cap \cdots \cap \mathrm{dom}(L_{N+1}))$,

$$\partial_{t_k}^n \left(\partial_{t_{N+1}}^m \omega \right) = L_k \left(\partial_{t_{N+1}}^m \omega \right) \quad \forall k = 1, ..., N. \tag{5.13}$$

The derivative $\partial_{t_{N+1}}^m \omega \in C^{nN}(\mathbb{R}^N, \mathrm{dom}(L_1) \cap \cdots \cap \mathrm{dom}(L_N))$ is thus a solution to the multi-time system of N equations and satisfies, by (5.11),

$$\partial_{t_1}^{\alpha_1} \ldots \partial_{t_N}^{\alpha_N} \left(\partial_{t_{N+1}}^m \omega \right) (s_1, \mathbf{x}_1, ..., s_{N+1}, \mathbf{x}_{N+1}) = 0 \quad \forall \alpha \in \{0, ..., n-1\}^{N+1}. \tag{5.14}$$

Hence, applying the induction assumption to those functions,

$$\partial_{t_{N+1}}^m \omega(t_1, \mathbf{x}_1, ..., t_N, \mathbf{x}_N, s_{N+1}, \mathbf{x}_{N+1}) = 0 \quad \forall m < n \ \forall t_1, ..., t_N \in \mathbb{R}. \tag{5.15}$$

Now we treat $(x_1, ..., x_N)$ as fixed. By the uniqueness property **(U)** of the equation $\partial_{t_{N+1}}^n \omega(x) = L_{N+1}(x)\omega(x)$, equations (5.12) and (5.15) imply

$$\omega(t_1, \mathbf{x}_1, ..., t_N, \mathbf{x}_N, t_{N+1}, \mathbf{x}_{N+1}) = 0. \tag{5.16}$$

Therefore, $\psi = \psi'$ and the solution is indeed unique. □

5.3 Existence conditions

Whereas uniqueness of the solution to our multi-time system of order n in time is direct, we need an additional condition for the existence of solutions, just as in the first order case discussed in Sec. 1.5. First, we formulate the requirement of the existence of solutions to each single equation:

Assumption (E): For each $k = 1, ..., N$, the respective equation (5.2) with initial values given as

$$\partial_{t_k}^m \psi(t_k, \mathbf{x}_1, ..., \mathbf{x}_N; S)\big|_{t_k = s_k} = f^m(\mathbf{x}_1, ..., \mathbf{x}_N), \quad 0 \leq m < n, \tag{5.17}$$

with $f^m \in \mathrm{dom}(L_k)$, possesses a solution $\psi \in C^n(\mathbb{R}, \mathrm{dom}(L_k))$.
This assumption entails the existence of *solution operators* $U_k^{(l)}(t_k, s_k; S)$, $l = 0, ..., n-1$, in the following sense: If **(E)** holds, given S and s_k, there is a solution ψ_{f^l} to (5.2) with the special initial values $f^m = 0$ for $m \neq l$, only f^l nonzero. Using this, define a map $U_k^{(l)}(t_k, s_k; S) : \mathrm{dom}(L_k) \to \mathrm{dom}(L_k)$ by

$$\left(U_k^{(l)}(t_k, s_k; S) f^l\right)(\mathbf{x}_1, ..., \mathbf{x}_N) := \psi_{f^l}(t_k, \mathbf{x}_1, ..., \mathbf{x}_N; S) \tag{5.18}$$

Since the differential equation is linear, it follows that $U_k^{(l)}(t_k, s_k; S)$ is linear. Furthermore, to satisfy the initial conditions,

$$\partial_{t_k}^m U_k^{(l)}(s_k, s_k; S) = \mathbb{1}_{\mathrm{dom}(L_k)} \delta_{m,l}. \tag{5.19}$$

With these operators at hand, we can construct a multi-time solution provided they commute.

Theorem 5.2. *We abbreviate*

$$S_{\hat{j},k} := (s_1, ..., \widehat{s_j}, ..., t_k, ..., s_N), \quad S_{\hat{j}} := (s_1, ..., \widehat{s_j}, ..., s_N). \tag{5.20}$$

Assume ***(E)*** *and that the solution operators satisfy*

$$\begin{aligned} &U_j^{(l)}(t_j, s_j; S_{\hat{j},k}) U_k^{(m)}(t_k, s_k; S_{\hat{k}}) f \\ &= U_k^{(l)}(t_k, s_k; S_{\hat{k},j}) U_j^{(m)}(t_j, s_j; S_{\hat{j}}) f \end{aligned} \tag{5.21}$$

for all $(s_1, ..., s_N), (t_1, ..., t_N) \in \mathbb{R}^N$, $j \neq k$, $f \in \mathrm{dom}(L_1) \cap \cdots \cap \mathrm{dom}(L_N)$ *and all* $l, m \in \{0, ..., n-1\}$.
Then, the multi-time system (5.1) *with initial values*

$$\partial_{t_1}^{\alpha_1} \dots \partial_{t_N}^{\alpha_N} \psi(s_1, \mathbf{x}_1, ..., s_N, \mathbf{x}_N) = f^\alpha(\mathbf{x}_1, ..., \mathbf{x}_N), \quad \alpha \in \{0, 1, ..., n-1\}^N, \tag{5.22}$$

where $f^\alpha \in \mathrm{dom}(L_1) \cap \cdots \cap \mathrm{dom}(L_N)$, *possesses a solution* $\psi \in C^n(\mathbb{R}^N, \mathrm{dom}(L_1)) \cap \cdots \cap \mathrm{dom}(L_N))$. *This solution is given by*

$$\begin{aligned} \psi(t_1, \mathbf{x}_1 ..., t_N, \mathbf{x}_N) = \Big(&\sum_\alpha U_1^{(\alpha_1)}(t_1, s_1; t_2, ..., t_N) U_2^{(\alpha_2)}(t_2, s_2; s_1, t_3, ..., t_N) \\ &\dots U_N^{(\alpha_N)}(t_N, s_N; s_1, ..., s_{N-1}) f^\alpha \Big)(\mathbf{x}_1, ..., \mathbf{x}_N), \end{aligned} \tag{5.23}$$

where the sum is over all multi-indices $\alpha = (\alpha_1, ..., \alpha_N) \in \{0, 1, ..., n-1\}^N$.

Proof. The function given in (5.23) is clearly in $C^n(\mathbb{R}^N, \mathrm{dom}(L_1)) \cap \cdots \cap \mathrm{dom}(L_N))$ by the properties of the solution operators $U_k^{(\alpha_k)}$. Let $\beta \in \{0, ..., n-1\}^N$. We first check the initial values, using (5.19),

$$\partial_{t_1}^{\beta_1} \dots \partial_{t_N}^{\beta_N} \psi(0, \mathbf{x}_1, ..., 0, \mathbf{x}_N) = \left(\sum_\alpha \delta_{\alpha_1, \beta_1} \dots \delta_{\alpha_N, \beta_N} f^\alpha \right)(\mathbf{x}_1, ..., \mathbf{x}_N) = f^\beta(\mathbf{x}_1, ..., \mathbf{x}_N). \tag{5.24}$$

Now we show that ψ satisfies (5.1). To this end, we prove via induction over $a \in \mathbb{N}, a \leq N$ the statement **I(a)**: Any permutation $\tilde{\sigma} \in S_a$ can be extended via the identity to a permutation $\sigma \in S_N$, and then it holds that

$$\begin{aligned} &U_1^{(\alpha_1)}(t_1, s_1; t_2, ..., t_N) \dots U_a^{(\alpha_a)}(t_a, s_a; s_1, ..., s_{a-1}, t_{a+1}, ..., t_N) \\ &= U_{\sigma(1)}^{(\alpha_{\sigma(1)})}(t_{\sigma(1)}, s_{\sigma(1)}; t_{\sigma(2)}, ..., t_{\sigma(N)}) \dots U_{\sigma(a)}^{(\alpha_{\sigma(a)})}(t_{\sigma(a)}, s_{\sigma(a)}; s_{\sigma(1)}, ..., s_{\sigma(a-1)}, t_{\sigma(a+1)}, ..., t_{\sigma(N)}). \end{aligned} \tag{5.25}$$

The base case **I(1)** is trivial since the only permutation in S_1 is the identity. So for the induction step, assume that **I(a-1)** is true and let $\tilde{\sigma} \in S_a$ be extended by the identity to $\sigma \in S_N$. If $\sigma(1) = 1$, **I(a)** follows directly from the induction assumption because only $a-1$ elements are actually permuted. So assume $\sigma(1) = b \neq 1$. We can certainly write $\sigma = \sigma_2 \circ \tau_{1b} \circ \sigma_1$, where τ_{1b} is the transposition of elements 1 and b and σ_1, σ_2 are extensions of permutations of only $a-1$ elements, with $\sigma_1(2) = b$. A possible way to do this is illustrated in the following:

$$\begin{array}{c} \big(1, 2, ...,b, ..., a\big) \\ \downarrow \sigma_1 \\ \big(1, b, 2, ..., b-1,b+1, ..., a\big) \\ \downarrow \tau_{1b} \\ \big(b, 1, 2, ..., b-1,b+1, ..., a\big) \\ \downarrow \sigma_2 \\ \big(b, \sigma(2), \sigma(3),..., \sigma(a)\big). \end{array} \tag{5.26}$$

By **I(a-1)**, we have

$$\begin{aligned} &U_1^{(\alpha_1)}(t_1, s_1; t_2, ..., t_N) \dots U_a^{(\alpha_a)}(t_a, s_a; s_1, ..., s_{a-1}, t_{a+1}, ..., t_N) \\ &= U_1^{(\alpha_1)}(t_1, s_1; t_2, ..., t_N) U_b^{(\alpha_b)}(t_b, s_b; s_1, t_{\sigma_1(3)}, ..., t_{\sigma_1(b-1)}, t_{\sigma_1(b+1)}, ..., t_{\sigma_1(N)}) \dots \\ &\dots U_{\sigma_1(a)}^{(\alpha_{\sigma_1(a)})}(t_{\sigma_1(a)}, s_{\sigma_1(a)}; s_{\sigma_1(1)}, ..., s_{\sigma_1(a-1)}, t_{\sigma_1(a+1)}, ..., t_{\sigma_1(N)}). \end{aligned} \tag{5.27}$$

Using condition (5.21),

$$\begin{aligned} &U_1^{(\alpha_1)}(t_1, s_1; t_2, ..., t_N) U_b^{(\alpha_b)}(t_b, s_b; s_1, t_{\sigma_1(3)}, ...t_{\sigma_1(b-1)}, t_{\sigma_1(b+1)}, ..., t_{\sigma_1(N)}) \\ &= U_b^{(\alpha_b)}(t_b, s_b; t_1, ...\widehat{t_b}..., t_N) U_1^{(\alpha_1)}(t_1, s_1; s_b, t_{\sigma_1(3)}, ..., t_{\sigma_1(b-1)}, t_{\sigma_1(b+1)}, ..., t_{\sigma_1(N)}) \end{aligned} \tag{5.28}$$

Applying the induction hypothesis once more to the indices $1, 2, ...b-1, b+1, .., a$ with permutation σ_2, we see that (5.27) is equal to the right hand side of (5.25), finishing the induction.
Let $k \in \{1, ..., N\}$ and choose a permutation $\sigma \in S_N$ with $\sigma(1) = k$, then by the statement **I(N)**, formula (5.23) can be rewritten such that U_k comes first in all the summands,

$$\begin{aligned} \psi(t_1, \mathbf{x}_1 ..., t_N, \mathbf{x}_N) = \Big(&\sum_\alpha U_k^{(\alpha_k)}(t_k, s_k; t_{\sigma(2)}, ..., t_{\sigma(N)}) U_{\sigma(2)}^{(\alpha_{\sigma(2)})}(t_{\sigma(2)}, s_{\sigma(2)}; s_k, t_{\sigma(3)}, ..., t_{\sigma(N)}) \\ &\dots U_{\sigma(N)}^{(\alpha_{\sigma(N)})}(t_{\sigma(N)}, s_{\sigma(N)}; s_{\sigma(1)}, ..., s_{\sigma(N-1)}) f^\alpha \Big)(\mathbf{x}_1, ..., \mathbf{x}_N). \end{aligned} \tag{5.29}$$

The coordinate t_k only figures in the first term of each summand. Using that $U_k^{(\alpha_k)}$ are the solution operators to (5.2), this shows

$$\partial_{t_k}^n \psi(x_1, ..., x_N) = L_k(x_1, ..., x_N)\psi(x_1, ..., x_N), \tag{5.30}$$

which finishes the existence proof. □

The rather complicated structure of the above theorem and the condition (5.21) comes from the fact that we allowed the operators L_k to depend on the coordinates of all particles. If we ask they be constant in time, the structure simplifies, which we demonstrate in the following.

Corollary 5.3. *Consider a system* (5.1) *where none of the operators L_k depends on some time coordinate t_j. Assume that **(E)** holds for such a system, then we may just write $U_k^{(l)}(t) := U_k^{(l)}(t, 0; S)$ for any $S \in \mathbb{R}^{N-1}$. Provided that*

$$\left[U_j^{(l)}(t_j), U_k^{(m)}(t_k)\right] f = 0 \tag{5.31}$$

for all $j \neq k$, $f \in \mathrm{dom}(L_1) \cap \cdots \cap \mathrm{dom}(L_N)$ and all $l, m \in \{0, ..., n-1\}$, $t_j, t_k \in \mathbb{R}$, the multi-time system (5.1) *with initial values given as in* (5.22) *possesses a solution $\psi \in C^n(\mathbb{R}^N, \mathrm{dom}(L_1) \cap \cdots \cap \mathrm{dom}(L_N))$ given by*

$$\psi(t_1, \mathbf{x}_1 ..., t_N, \mathbf{x}_N) = \left(\sum_\alpha U_1^{(\alpha_1)}(t_1) U_2^{(\alpha_2)}(t_2) \dots U_N^{(\alpha_N)}(t_N) f^\alpha \right) (\mathbf{x}_1, ..., \mathbf{x}_N), \tag{5.32}$$

where the sum is over all multi-indices $\alpha = (\alpha_1, ..., \alpha_N) \in \{0, 1, ..., n-1\}^N$.

The proof is direct from the previous theorem just using the time-independence of the L_k operators.

Example: Multi-time wave equation for two particles To illustrate formula (5.32) at a simple example, consider a system of two wave equations in one spatial dimension,

$$\partial_{t_k}^2 \psi(t_1, z_1, t_2, z_2) = \partial_{z_k}^2 \psi(t_1, z_1, t_2, z_2), \qquad k = 1, 2. \tag{5.33}$$

It is well known (see e.g. [53]) that the initial value problem

$$\partial_t^2 \varphi(t, z) = \partial_z^2 \varphi(t, z), \quad \varphi(0, z) = g(z), \quad \partial_t \varphi(0, z) = h(z), \tag{5.34}$$

with $g, h \in C^2(\mathbb{R})$ has the unique solution given by D'Alembert's formula,

$$\varphi(t, z) = \frac{1}{2} \left[g(z + t) + g(z - t) + \int_{z-t}^{z+t} h(y) dy \right]. \tag{5.35}$$

We can read off from this that the solution operators in this case are given by

$$\begin{aligned} \left(U_1^{(0)}(t_1) g\right)(z_1, z_2) &= \frac{1}{2} \left(g(z_1 + t_1, z_2) + g(z_1 - t_1, z_2) \right), \\ \left(U_1^{(1)}(t_1) h\right)(z_1, z_2) &= \frac{1}{2} \int_{z-t}^{z+t} h(y, z_2) dy, \end{aligned} \tag{5.36}$$

and analogous formulas for $U_2^{(0)}$ and $U_2^{(1)}$. There is no coupling between the coordinates of the first and second particle at all, so condition (5.31) holds. Thus, Corollary 5.3 together

with the uniqueness Theorem 5.1 implies the following statement: There exists a unique solution to the system (5.33) with initial data

$$\begin{aligned} \psi(0,z_1,0,z_2) &= g(z_1,z_2) \\ \partial_{t_1}\psi(0,z_1,0,z_2) &= h_1(z_1,z_2) \\ \partial_{t_2}\psi(0,z_1,0,z_2) &= h_2(z_1,z_2) \\ \partial_{t_1}\partial_{t_2}\psi(0,z_1,0,z_2) &= j(z_1,z_2), \end{aligned} \tag{5.37}$$

where $g, h_1, h_2, j \in C^2(\mathbb{R}^2) = \mathrm{dom}(\partial_{z_1}^2) \cap \mathrm{dom}(\partial_{z_2}^2)$. We can compute this solution by eq. (5.32):

$$\begin{aligned} 4\psi(t_1,z_1,t_2,z_2) \;&=\; 4\Big(U_1^{(0)}(t_1)U_2^{(0)}(t_2)g + U_1^{(1)}(t_1)U_2^{(0)}(t_2)h_1 \\ &\qquad + U_1^{(0)}(t_1)U_2^{(1)}(t_2)h_2 + U_1^{(1)}(t_1)U_2^{(1)}(t_2)j\Big)(z_1,z_2) \\ &=\; g(z_1-t_1,z_2-t_2) + g(z_1-t_1,z_2+t_2) + g(z_1+t_1,z_2-t_2) + g(z_1+t_1,z_2+t_2) \\ &+ \int_{z_1-t_1}^{z_1+t_1} dy\; h_1(y,z_2+t_2) + h_1(y,z_2-t_2) + \int_{z_2-t_2}^{z_2+t_2} dy\; h_2(z_1-t_1,y) + h_2(z_1+t_1,y) \\ &+ \int_{z_1-t_1}^{z_1+t_1} dy \int_{z_2-t_2}^{z_2+t_2} d\tilde{y}\; j(y,\tilde{y}). \end{aligned} \tag{5.38}$$

Here we see very clearly how all combinations of solution operators figure and why the additional knowledge of the mixed derivatives, here the function j, is necessary.

5.4 Consistency condition for higher orders

Condition (5.21) from the previous section ensures that there exists a solution to the multi-time system. This sufficient condition, however, is quite indirect. One would wish for a condition that is more direct and can be assessed by looking at the operators L_k, similarly to the consistency condition (1.20) that was derived for first-order systems. We derive such a condition now, but here we can only show that it is *necessary* for the existence of solutions in the general case.

Lemma 5.4. *Let $n > 1$. Assume that for all $s_1, ..., s_N \in \mathbb{R}$ and all initial values*

$$\partial_{t_1}^{\alpha_1} \dots \partial_{t_N}^{\alpha_N} \psi(s_1, \mathbf{x}_1, ..., s_N, \mathbf{x}_N) = f^\alpha(\mathbf{x}_1, ..., \mathbf{x}_N), \quad \alpha \in \{0,1,...,n-1\}^N, \tag{5.39}$$

with $f^\alpha \in C_c^\infty(\mathbb{R}^{3N})$, there exists a solution $\psi \in C^{2n}(\mathbb{R}^n, \mathrm{dom}(L_1)) \cap \dots \cap \mathrm{dom}(L_N))$ to the multi-time system (5.1)*. Then, the following consistency conditions hold:*

$$\begin{cases} \partial_{t_j} L_k = 0 \\ [L_j, L_k] = 0 \end{cases} \quad \forall j \neq k, \tag{5.40}$$

where these equations are understood in the sense that for all $\eta \in C_c^\infty(\mathbb{R}^{3N})$,

$$\Big(\partial_{t_j} L_k(x_1, ..., x_N)\big|_{t_1=s_1,\dots,t_N=s_N}\Big)\, \eta(\mathbf{x}_1, ..., \mathbf{x}_N) = 0, \tag{5.41}$$

and

$$[L_j(s_1, \mathbf{x}_1, ..., s_N, \mathbf{x}_N), L_k(s_1, \mathbf{x}_1, ..., s_N, \mathbf{x}_N)]\, \eta(\mathbf{x}_1, ..., \mathbf{x}_N) = 0. \tag{5.42}$$

Proof. If $\psi \in C^{2n}(\mathbb{R}^n, \mathrm{dom}(L_1)) \cap \cdots \cap \mathrm{dom}(L_N))$ is a solution of (5.1), let $j \neq k$, then we can compute the derivative using the general Leibniz rule,

$$\begin{aligned} \partial_{t_j}^n \partial_{t_k}^n \psi = \partial_{t_j}^n L_k \psi &= \sum_{l=0}^{n} \binom{n}{l} \left(\partial_{t_j}^l L_k\right) \left(\partial_{t_j}^{n-l} \psi\right) \\ &= \sum_{l=1}^{n-1} \binom{n}{l} \left(\partial_{t_j}^l L_k\right) \left(\partial_{t_j}^{n-l} \psi\right) + L_k L_j \psi + (\partial_{t_j}^n L_k)\psi. \end{aligned} \tag{5.43}$$

By the commutativity of partial derivatives, we necessarily have

$$\partial_{t_j}^n \partial_{t_k}^n \psi = \partial_{t_k}^n \partial_{t_j}^n \psi, \tag{5.44}$$

which implies, inserting (5.43),

$$\begin{aligned} 0 = &\left(L_j L_k - L_k L_j + \partial_{t_k}^n L_j - \partial_{t_j}^n L_k\right) \psi \\ &+ \sum_{l=1}^{n-1} \binom{n}{l} \left(\partial_{t_k}^l L_j\right) \left(\partial_{t_k}^{n-l} \psi\right) \\ &- \sum_{l=1}^{n-1} \binom{n}{l} \left(\partial_{t_j}^l L_k\right) \left(\partial_{t_j}^{n-l} \psi\right). \end{aligned} \tag{5.45}$$

Let $s_1, ..., s_N \in \mathbb{R}$ and $\eta \in C_c^\infty(\mathbb{R}^{3N})$. We may choose initial values such that

$$\partial_{t_j}^{n-1} \psi(x_1, ..., x_N)\big|_{t_1 = s_1, ..., t_N = s_N} = \eta(\mathbf{x}_1, ..., \mathbf{x}_N), \quad f^\alpha = 0 \ \forall \alpha \neq (0, ..., 0, \underbrace{n-1}_{j-th}, 0, ..., 0). \tag{5.46}$$

Hence, (5.45) becomes

$$0 = \partial_{t_j} L_k (\partial_{t_j}^{n-1} \psi) \tag{5.47}$$

and thus condition (5.41). Since this holds for all times, L_k can actually not depend on t_j and also the higher derivatives $\partial_{t_j}^l L_k$ must vanish, so (5.45) is then

$$0 = (L_j L_k - L_k L_j)\psi, \tag{5.48}$$

which gives, applied to the general initial values that are allowed, condition (5.42). □

Remark:

(a) One might wish to prove that the consistency condition (5.40) is also sufficient for existence and uniqueness of solutions in some cases, as Petrat and Tumulka showed it for first order systems [3]. This is, however, not feasible in the great generality we allow in this chapter.

(b) Similarly to Sec. 1.5, the results in this section are proven for a multi-time wave function with domain $\mathbb{R}^{4N}$ instead of the natural domain $\mathscr{S}^{(N)}$. This makes it possible to demonstrate the structure of higher order multi-time systems and the principal way to treat such systems without distracting complications. The consistency condition (5.40) is even more restrictive than for first order equations. In the case of domain $\mathscr{S}^{(N)}$, we expect that it needs to hold only on this set, like in the model by Dirac, Fock, and Podolsky of chapter 3.

(c) The system (5.1) is not the most general linear multi-time system since, in principle, time derivatives could appear in several orders (e.g. $\partial_{t_1}^2 + \partial_{t_1}$) or differently for different particle indices, but this is an excess of generality we do not aim at here. Indeed, we have only considered multi-time systems of first order in the preceding chapters of this thesis since those are directly relevant for the description of relativistic fermions.

Reformulation to a first-order system: It is rather complicated to reduce the general system (5.1) to a multi-time system of first order since we need a vector of length n^N to do so. Therefore, we clarify the general idea only on the following simple example. Consider $n = N = 2$ and the system

$$\begin{aligned} \partial_{t_1}^2 \psi(t_1, \mathbf{x}_1, t_2, \mathbf{x}_2) &= L_1(t_1, \mathbf{x}_1, t_2, \mathbf{x}_2)\psi(t_1, \mathbf{x}_1, t_2, \mathbf{x}_2) \\ \partial_{t_2}^2 \psi(t_1, \mathbf{x}_1, t_2, \mathbf{x}_2) &= L_2(t_1, \mathbf{x}_1, t_2, \mathbf{x}_2)\psi(t_1, \mathbf{x}_1, t_2, \mathbf{x}_2). \end{aligned} \tag{5.49}$$

We shorten the equations by leaving out the argument $(t_1, \mathbf{x}_1, t_2, \mathbf{x}_2)$ from now on. We introduce a new unkown, a vector with four components,

$$\Psi := \begin{pmatrix} \psi \\ \psi_{(1)} \\ \psi_{(2)} \\ \psi_{(12)} \end{pmatrix}, \tag{5.50}$$

and recognize that (5.49) can be rewritten as

$$\partial_{t_1} \begin{pmatrix} \psi \\ \psi_{(1)} \\ \psi_{(2)} \\ \psi_{(12)} \end{pmatrix} = \begin{pmatrix} \psi_{(1)} \\ L_1\psi \\ \psi_{(12)} \\ L_1\psi_{(2)} \end{pmatrix}, \quad \partial_{t_2} \begin{pmatrix} \psi \\ \psi_{(1)} \\ \psi_{(2)} \\ \psi_{(12)} \end{pmatrix} = \begin{pmatrix} \psi_{(2)} \\ \psi_{(12)} \\ L_2\psi \\ L_2\psi_{(1)} \end{pmatrix}. \tag{5.51}$$

By introducing the vector Ψ with $n^N = 4$ entries, we are able to rewrite the system as a multi-time system of first order. However, this reformulation is only valid if some regularity conditions on ψ are assumed, since we exchanged derivatives up to third order. Scrutinizing this would lead to another possibility to prove Theorems 5.1 and 5.2.

CHAPTER
SIX

CONCLUSION

When Lieb and Seiringer introduce a single-time model to describe several relativistic Dirac particles, they provide the following remark [65, p. 185]:

> The sophisticated reader will notice that this theory is not really relativistic. To make such a theory one would have to assign a four-dimensional space-time coordinate to each electron just as we a assign a three-dimensional coordinate in non-relativistic theory. While there have been many attempts to construct such a multiple time formalism no totally satisfactory model of this kind exists.

This observation is at least as old as the works of Dirac [1, 4], Bloch [14], Tomonaga [15] and Schwinger [16]. It is a general problem of relativistic quantum mechanics that many of the suggested models can be called "not totally satisfactory", especially in case they are not mathematically well-defined. In this sense, we were able to take several steps towards more satisfactory models in the multi-time formalism. We mainly treated three different ideas for the formulation of interaction in multi-time systems with mathematical rigor in this thesis.

In the first case of multi-time Dirac equations with interaction potentials, the result is a negative one: no Poincaré invariant potential can mediate interaction in a multi-time system of the form (2.1), (2.2) since the consistency condition does not allow that. We have found that some matrix-valued potentials such as (2.12) can satisfy the consistency condition. But the message we take from our generalization of the no-go theorem by Petrat and Tumulka is that multi-time systems with interaction potentials are generally not well-posed. From a physical point of view, this is not all that surprising, since a potential $V(x_1, ..., x_N)$ governs the interaction in a non-relativistic, synchronized way. It is therefore a natural guess that relativistic interactions ought to be formulated by other means. We corroborated this guess with mathematical facts.

The second model we looked at follows Dirac, Fock, Podolsky and formulates interaction of Dirac fermions via a second-quantized scalar field φ. We already remarked that the restriction to a scalar field instead of an electromagnetic one is not decisive and was undertaken for technical simplicity. Nevertheless, for future work it could be worth doing the additional steps connected with the gauge freedom in the electromagnetic case. Theorems 3.1 and 3.2 show existence and uniqueness of solutions in a Hilbert space sense. To see that the such constructed solution also solves the multi-time equations pointwise, only a single ingredient is missing: the pointwise differentiability in time directions. The Sobolev arguments needed for this are made more difficult since the time evolutions $U_A(t, s)$ are not just of the form

$e^{-i\mathcal{H}t}$, and even $[\mathcal{H}_A(t), U_A(t,s)] \neq 0$ in general. Hence, the smoothness in time remains an interesting open question in the solution theory of multi-time systems.
The third main part of this thesis explained how interior-boundary conditions (IBCs) can be utilized in a relativistic setting. The geometric formulation of probability conservation with the help of the current form $\omega^{(n)}$ is particularly useful for relativistic considerations. The model for which we showed existence and uniqueness comes with a number of simplifications. Still, it captures the essential features we intended to demonstrate and is a rigorous multi-time IBC model where particle creation and annihilation occurs at moving sources. In future work, one could address the questions how to deal with a non-zero mass or how to increase the upper bound N on the particle number to infinity. The main question, however, is the formulation of creation and annihilation for Dirac particles in three dimensions. Here lies the real challenge for which new ideas will be necessary.
From the mathematical point of view, existence of uniqueness of solutions to systems of partial differential equations such as (1.1) and the more general (5.1) have been studied quite generally in this thesis. However, some general results in 1.5 or 5 were obtained for multi-time wave functions with domain $\mathbb{R}^{4N}$, while the natural domain of ψ is $\mathscr{S}^{(N)}$. On that domain of space-like configurations, it is mostly necessary to look at each single model separately. The way of proving existence for the model in Ch. 3 might serve as a template for similar models with cut-offs, though.
All in all, the steps towards satisfactory models for the multi-time wave function lead to a better understanding of the role of the consistency condition, of the interplay between functional analysis and pointwise evaluations, and of multi-time systems for variable particle numbers. We have not discussed multi-time integral equations as in [23], but consider them to be well worth studying since the consistency condition does not obstruct the introduction of interaction there. However, the methods for solution theory of integral equations of that kind are not that well developed and those equations require a different understanding of the role of physical equations, similarly as in Wheeler-Feynman electrodynamics.
The positive results in this thesis encourage us to take the multi-time wave function seriously as foundation of relativistic quantum mechanics. Being a map $\psi : \mathscr{S}^{(N)} \to \mathbb{C}^K$ or a sequence $(\psi^{(0)}, \psi^{(1)}, \psi^{(2)}, ...)$ of such maps for different particle numbers, it is a natural mathematical object for which many kinds of dynamics can be devised. Apart from the unsolved ultraviolet problem that appears in relativistic interactions, our results suggest that multi-time dynamics are a viable option for a manifestly Lorentz invariant formulation of quantum mechanics.

NOTATION AND CONVENTIONS

In this thesis we choose units in which Planck's constant $\hbar$, the speed of light c, and the elementary charge e is equal to one.
The chapters slightly differ in notation. However, all variations made are stated explicitly. The following list contains symbols commonly used in this work.
Note that the letter H appears in three different shapes in this thesis. Always, H denotes a Sobolev space, $\mathscr{H}$ denotes a Hilbert space and $\mathcal{H}$ a Hamiltonian operator.
We use the spacetime metric $\eta = (1, -1, ..., -1)$ for $(1+d)$-dimensional Minkowski spacetime with $d \in \mathbb{N}$ spatial dimensions. Greek indices run from 0 to d and Latin indices $a, b, ...$ only over the spatial components $1, 2, ..., d$. The Einstein summation convention is employed for Greek indices only. Particle labels are denoted also by Latin indices, $j, k, ...$ and run from 1 to the total particle number N.

List of notations

C	a generic constant that might depend on fixed parameters and might change from line to line
$\mathbb{N}$	the set of natural numbers $\{1, 2, ...\}$
$\mathbb{R}$, $\mathbb{C}$	the real and the complex numbers
i	the imaginary unit
$\Im$	imaginary part
ψ	wave function, usually multi-time wave function
N	number of particles
K	number of spin components (N-dependent, usually 4^N; or 2^N in Ch. 4)
$t, t_1, ...t_N$	time variables
$\mathbf{x}, \mathbf{x}_1, ..., \mathbf{x}_N$	spatial variables
$x_1, ..., x_N$	space-time variables, $x_k = (t_k, \mathbf{x}_k)$ for $k = 1, ..., N$.
∂_{t_k}	partial derivative $\partial_{t_k} = \frac{\partial}{\partial t_k} = \frac{\partial}{\partial x_k^0} = \partial_{k,0}$.
$< \cdot, \cdot >$	scalar product in a Hilbert space
$\| \cdot \|$	norm in a Banach space
dom	domain of an operator

$\otimes$	tensor product
$C^n(X,Y)$	functions from X to Y that are n times continuously differentiable, also $n=\infty$ possible
$C_c^\infty(X,Y)$	functions from X to Y that are smooth and have compact support
$L^p(X,Y)$	p-th Lebesgue space of measurable functions from X to Y with finite p-norm $\|f\|_p^p = \int_X \|f(x)\|^p dx$
$\widehat{\cdot}$	denotes omission in a list of elements, e.g. $t_1, \widehat{t_2}, t_3 \equiv t_1, t_3$
$\hat{\cdot}$	Fourier transform of a function
φ	single-time wave function, in Ch. 3 also scalar field (3.16)
$\mathscr{S}^{(N)}$	space-like configurations of N particles, see (1.6)
$\mathscr{S}_\delta$	space-like configurations with minimal distance δ, see (3.17)
γ^μ, γ_k^μ	Dirac's gamma matrices (see (2.4)), with k denoting the particle index the matrix acts on, compare (2.6)
$m, m_1, ..., m_N$	particle masses
$j^{\mu_1 \cdots \mu_N}$	Dirac current for N particles, defined in (1.14)
α, β	in Ch. 5: multi-indices of the form $\alpha = (\alpha_1, ..., \alpha_N) \in \mathbb{N}^N$.
γ^5	Fifth gamma matrix, see (2.5)
$\mathbb{1}$	identity matrix or identity map
$B_r(\mathbf{x})$	open ball in of radius r around $\mathbf{x}$
$\mathcal{Q}$	configuration space
$\omega^{(n)}$	current form, see (4.10)
C_b^1	set of bounded functions that are continuously differentiable with bounded derivative

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Eidesstattliche Versicherung

(Siehe Promotionsordnung vom 12.07.11, §8, Abs. 2 Pkt. 5)

Hiermit erkläre ich an Eides statt, dass die Dissertation von mir selbstständig ohne unerlaubte Beihilfe angefertigt ist.
Die Abschnitte 1.5.3, 2, 3 und 4 enthalten Ergebnisse, die aus der Zusammenarbeit mit einem Mitautor hervorgegangen sind. Verweise auf die entsprechenden Veröffentlichungen sind in den jeweiligen Abschnitten zu finden.

München, den 29.01.2019 Lukas Nickel