3. On phase transitions in quantum spin systems

- Γ : countable set (e.g. $\Gamma = \mathbb{Z}^\nu$) with metric $d(\cdot,\cdot)$

  For $\Lambda \subset \Gamma$: $|\Lambda|$ is the cardinality of Λ

$$\mathcal{F}(\Gamma) = \{\Lambda \subset \Gamma : |\Lambda| < \infty\}.$$

For any $x \in \Gamma$, let $\mathcal{H}_x$ finite dimensional (Hilbert space of a "spin" / N-level atom) and

$$\sup_{x\in\Gamma} \dim \mathcal{H}_x < \infty.$$

For $\Lambda \in \mathcal{F}(\Gamma)$: $\mathcal{H}_\Lambda := \bigotimes_{x\in\Lambda} \mathcal{H}_x$

Observables: $\Lambda \in \mathcal{F}(\Gamma)$: $\mathcal{A}_\Lambda := \mathcal{L}(\mathcal{H}_\Lambda)$ (matrices!)

Embedding:

$\Lambda_1 \subset \Lambda_2$: $\mathcal{A}_{\Lambda_1} \simeq \mathcal{A}_{\Lambda_1} \otimes \mathbb{1}_{\Lambda_2 \setminus \Lambda_1} \subset \mathcal{A}_{\Lambda_2}$

Clearly:

$A \in \mathcal{A}_X$, $B \in \mathcal{A}_Y$ and $X \cap Y = \emptyset$

$\Rightarrow$ $[A,B] = 0$.

Now: $\mathcal{A}_{loc} = \bigcup_{\Lambda\in\mathcal{F}(\Gamma)} \mathcal{A}_\Lambda$

$\mathcal{A} = \overline{\mathcal{A}_{loc}}^{\|\cdot\|}$ is a C^*-algebra with an identity.

States: Any state ω on $\mathcal{A}$ is locally normal, namely

$$\omega\restriction_{\mathcal{A}_\Lambda} = \mathrm{Tr}(\rho_\Lambda^\omega \cdot), \quad \Lambda \in \mathcal{F}(\Gamma)$$

i.e. one can think of ω as the family of density matrices $\{\rho_\Lambda^\omega : \Lambda \in \mathcal{F}(\Gamma)\}$.

- Dynamics: An interaction is a map

$$\Phi : \mathcal{F}(\Gamma) \longrightarrow \mathcal{A}$$
$$X \longmapsto \Phi(X) = \Phi(X)^* \in \mathcal{A}_X$$

the interaction between "spins" in the finite set $X \in \mathcal{F}(\Gamma)$

Hamiltonian : $H_\Lambda = \sum_{X \subset \Lambda} \Phi(X) \quad , \Lambda \in \mathcal{F}(\Gamma)$

sum of interactions within Λ.

Typical example: Two-body interaction : $\Phi(X) = 0$ whenever $|X| \neq 2$.

The finite volume dynamics

$$\tau_\Lambda^t(A) := e^{itH_\Lambda} A e^{-itH_\Lambda}$$

has a well-defined strong limit as $\Lambda \to \Gamma$ (i.e. $\exists\, \tau$, a strongly continuous, 1-parameter group of automorphisms of $\mathcal{A}$, s.t. $\|\tau_\Lambda^t(A) - \tau^t(A)\| \longrightarrow 0$ for all $t \in \mathbb{R}$, $A \in \mathcal{A}$) if Φ has sufficient decay : for example

$$\Phi(X) = 0 \quad \text{whenever } \operatorname{diam}(X) > R \quad ,$$

finite-range interaction.

- The Mermin–Wagner theorem: Absence of continuous symmetry breaking in $\nu = 1, 2$.

Here: $\mathcal{H}_x \simeq \mathcal{H}$ for all $x \in \Gamma$.

"Continuous symmetry": compact, connected Lie group G e.g. $SU(2)$.

Assume: $U : G \to \mathcal{L}(\mathcal{H})$

$$g \mapsto U_g \ : \ U_g U_g^* = U_g^* U_g = \mathbb{1}$$

is a unitary representation of G on $\mathcal{H}$, i.e.

$$U_e = \mathbb{1}, \quad U_{gg'} = U_g U_{g'}$$

Induces an action of G on $\mathcal{A}_{\{x\}}$: $A \mapsto U_g^* A U_g$, further on $\mathcal{A}_\Lambda$ by $\bigotimes_{x \in \Lambda} U_g$ (same rotation at any point) and a $*$-automorphism of $\mathcal{A}$ as the limit

$$\alpha_g(A) = \lim_{\Lambda \to \Gamma} \alpha_g^\Lambda(A) = \lim_{\Lambda \to \Gamma} \left(\bigotimes_{x\in\Lambda} U_g\right)^* A \left(\bigotimes_{x\in\Lambda} U_g\right)$$

Note: The limit exists for any $A \in \mathcal{A}_{loc}$ since $\alpha_g^\Lambda(A)$ is independent of Λ for Λ large enough, and it defines a bounded linear map (of norm 1). Hence the limit extends to the completion $\mathcal{A}$ of $\mathcal{A}_{loc}$.

Example: $G = SU(2)$

$$U_g = \exp(2\pi i\, g \cdot S)$$

where g is a point in the unit ball, and S are the three spin matrices.

- <u>Theorem</u>: Let $\mathcal{A}$ be a quantum spin system with $\Gamma = \mathbb{Z}^2$, $\{\alpha_g : g \in G\}$ the action of a compact, connected Lie group on $\mathcal{A}$ and let Φ be a G-invariant, two-body interaction: $\alpha_g(\Phi(x,y)) = \Phi(x,y)$ for all $(x,y) \in \mathbb{Z}^2 \times \mathbb{Z}^2$, $g \in G$.
  If
  $$C := \sup_{x \in \mathbb{Z}^2} \sum_{y \in \mathbb{Z}^2} \|\Phi(x,y)\| \, d(x,y)^2 < \infty, \qquad (C)$$
  then for any $0 < \beta < \infty$, all (τ^Φ, β)-KMS states are G-invariant:
  $$\omega \circ \alpha_g = \omega \qquad \text{for all } g \in G.$$

- <u>Remarks</u>. * Notation $\Phi(x,y) = \Phi(\{x,y\})$ for a two-body interaction.
  * Could be extended to N-body interaction
  * Condition (C) imposes sufficient decay on $\|\Phi(x,y)\|$ as $d(x,y) \to \infty$.
  For translation & rotation invariant Φ, the sharp decay is $\|\Phi(x,y)\| \le C \, d(x,y)^{-4}$.

- <u>Proof</u>: It suffices to check the conditions (A) and (B') of the thm on page 47.

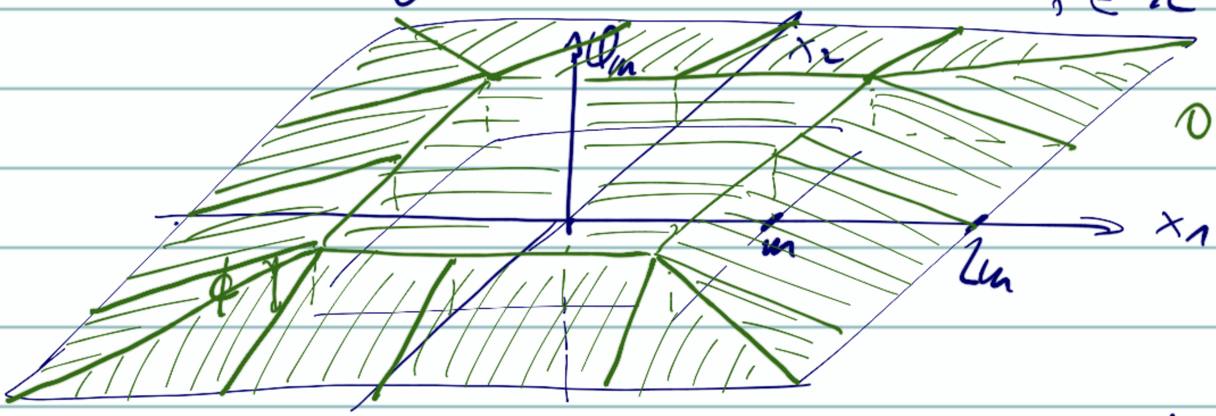
Setting: It suffices to show the claim for an arbitrary one-dimensional subgroup H of G.
$$H \simeq \mathbb{R}/\mathbb{Z} \simeq S^1$$

Let $S = S^* \in \mathcal{L}(\mathcal{H})$ be its generator:

$$U_\phi = \exp(i\phi S), \quad \phi \in [0, 2\pi).$$

1. "Smooth" approximation of α_ϕ. $m \in \mathbb{N}$, $\Lambda_m = [-m,m]^2 \cap \mathbb{Z}^2$ and $\varphi_m : \mathbb{Z}^2 \to [0, \pi)$

$$\varphi_m(x) = \begin{cases} \phi & x \in \Lambda_m \\ \phi\left(2 - \max\{|x_1|, |x_2|\}/m\right); & x = (x_1, x_2) \in \Lambda_{2m} \setminus \Lambda_m \\ 0 & x \in \mathbb{Z}^2 \setminus \Lambda_{2m}. \end{cases}$$



and let $U_\phi(m) := \bigotimes_{x \in \Lambda_{2m}} U(\varphi_m(x)) \in \mathcal{A}_{\Lambda_{2m}} \subset \mathcal{D}(\delta)$.

Let $A \in \mathcal{A}_{loc}$, $\exists\, m_0 : A \in \mathcal{A}_{\Lambda_m}$ for all $m \geq m_0$. But then $U_\phi(m)^* A U_\phi(m) = \alpha_\phi(A)$ for all $m \geq m_0$, proving that $U_\phi(m)^* \cdot U_\phi(m) \to \alpha_\phi(\cdot)$ on $\mathcal{A}_{loc}$ and by boundedness and density of $\mathcal{A}_{loc}$ also on $\mathcal{A}$.

$\Rightarrow$ Assumption (A) is ok.

Now: if $A \in \mathcal{A}_X$: $\delta(A) = i \sum_{\{x,y\} \cap \Lambda \neq \emptyset} [\Phi_{x,y}, A]$

so that

$$U_\phi(n)^* \delta(U_\phi(n)) = i \sum_{\{x,y\} \in \mathbb{Z}^2 \setminus \mathcal{N}_n} U_\phi(n)^* \Phi(x,y) U_\phi(n) - \Phi(x,y)$$

where $\mathcal{N}_n = \{\{x,y\} : x,y \in \Lambda_n \text{ or } x,y \in \mathbb{Z}^2 \setminus \Lambda_{2n}\}$

$$U_\phi^*(n) \Phi(x,y) U_\phi(n) = \alpha_\phi(\Phi(x,y)) = \Phi(x,y) \; ; \; U_\phi^*(n) \Phi(x,y) U_\phi(n) = \Phi(x,y)$$

Now:

$$\varphi_n(x) S_x + \varphi_n(y) S_y = \frac{\varphi_n(x) + \varphi_n(y)}{2}(S_x + S_y) + \frac{\varphi_n(x) - \varphi_n(y)}{2}(S_x - S_y)$$

$$=: E_n(x,y) + O_n(x,y) \in \mathcal{A}_{\{x,y\}} \quad (x \neq y)$$

and $[E_n(x,y), O_n(x,y)] = 0$.

$E_n(x,y)$ generates the <u>same</u> rotation at both x & y, so that

$$U_\phi(n)^* \Phi(x,y) U_\phi(n) = e^{iO_n(x,y)} e^{iE_n(x,y)} \Phi(x,y) e^{-iE_n(x,y)} e^{-iO_n(x,y)}$$

$$= e^{iO_n(x,y)} \Phi(x,y) e^{-iO_n(x,y)}$$

and hence

$$U_\phi(n)^* \Phi(x,y) U_\phi(n) - \Phi(x,y) = \sum_{k=1}^{\infty} \frac{i^k}{k!} [O_n(x,y), [O_n(x,y) \cdots [O_n(x,y), \Phi(\ldots)]]]$$

Denoting $U_\phi(n)^* \delta(U_\phi(n)) + U_\phi(n) \delta(U_\phi(n)^*) =: i \sum_{x,y} \Delta_n(x,y)$,

and noting that

$$U_\phi(n) \delta(U_\phi(n)^*) = U_{-\phi}(n)^* \delta(U_{-\phi}(n)) \quad ,$$

and that $O_n(x,y)$ is odd under $\phi \to -\phi$, all terms of odd order k in the series cancel out, yielding

$$\|\Delta_n(x,y)\| \leq 2 \sum_{k\geq 1} \frac{1}{(2k)!}\frac{1}{2^{2k}} |\varphi_n(x)-\varphi_n(y)|^{2k} \left\| \mathrm{ad}^{2k}_{S_x - S_y}(\Phi(x,y))\right\|$$

with $\left\| \mathrm{ad}^{2k}_{S_x-S_y}(\Phi(x,y))\right\| \leq 2^{2k}(2\|S\|)^{2k} \|\Phi(x,y)\|$

& since $|\varphi_n(x) - \varphi_n(y)| \leq |\phi| \min\left\{1, \frac{d(x,y)}{n}\right\}$:

$$|\varphi_n(x)-\varphi_n(y)|^{2k} \leq |\phi|^{2k}\left(d(x,y)/n\right)^2.$$

Altogether:

$$\sum_{x,y\in\mathbb{Z}^2 \setminus N_n} \|\Delta_n(x,y)\| \leq \frac{4e^{2\|S\||\phi|}}{n^2} \underbrace{\sum_{x\in\Lambda_{2n}}}\sum_{y\in\mathbb{Z}^2} \|\Phi(x,y)\| d(x,y)^2$$

$$\leq (2n+1)^2 \sup_{x\in\mathbb{Z}^2} \sum_{y\in\mathbb{Z}^2}$$

$$\leq 36\, C e^{2\|S\||\phi|} \quad \text{uniformly in } n.$$

Now: Since $\alpha_\pi^2 = \mathrm{id}$, $\omega \circ \alpha_\pi^2 = \omega$. By the theorem, $\omega \circ \alpha_\pi = \omega$ for all (τ, β)-KMS states. Recursively, $\omega \circ \alpha_{\pi/2^n} = \omega$ for all $n \in \mathbb{N}$.

But $\left\{\phi \in S^1 : \phi = \sum_{n=0}^{N} a_n (\pi/2^n) : a_n \in \mathbb{Z}, N \in \mathbb{N}\right\} = D$

is dense in S^1. For any $A \in \mathcal{A}$:

$$\phi \mapsto \omega\left(\alpha_\phi(A) - A\right) =: \xi_A(\phi)$$

is continuous. Since $\xi_A \restriction_D = 0$, it follows that $\xi_A \equiv 0$

$\square$

- For a proof of a phase transition for interacting quantum systems at positive temperature: Dyson-Lieb-Simon (1978) for the <u>antiferromagnetic Heisenberg model</u>

$$H_\Lambda := \sum_{|x-y|=1} \left(S_x^1 S_y^1 + S_x^2 S_y^2 + S_x^3 S_y^3\right)$$

(with an overall "+" sign, spins tend to anti-align)

What is proved?

$$\frac{1}{|\Lambda|}\sum_{x\in\Lambda} (-1)^{d(0,x)} \omega_{\beta,\Lambda}\left(S_0^3 S_x^3\right) \geq \varkappa - \frac{1}{\beta}\frac{1}{|\Lambda|}\sum_{\xi \in \Lambda^*} 1/E(\xi)$$

where $E(\xi) \sim |\xi|^2$ as $|\xi| \to 0$. As $|\Lambda| \to \infty$:

$$\geq \varkappa - \frac{1}{\beta}\underbrace{\int 1/E(\xi)\, d\xi}_{<\infty \text{ iff } \nu \geq 3}$$

Hence: If β^{-1} is small enough,

$$\lim_{\Lambda \to \mathbb{Z}^\nu} \frac{1}{|\Lambda|}\sum_{x\in\Lambda} (-1)^{d(0,x)} \omega_{\beta,\Lambda}\left(S_0^3 S_x^3\right) \geq B > 0$$

which implies Long-Range Order:

$$\lim_{|x|\to\infty} \lim_{\Lambda\to\mathbb{Z}^\nu} \inf (-1)^{d(0,x)} \omega_{\beta,\Lambda}(S_0^3 S_x^3) > 0$$

whenever $\nu \geq 3$ and β^{-1} is small enough. This shows also that the limiting states of $\omega_{\beta,\Lambda}$ as $\Lambda\to\mathbb{Z}^\nu$ cannot be invariant under rotation in $SU(2)$ $\to$ Symmetry breaking.

- Key of the proof: <u>infrared bounds</u>.

Indeed let $C_\Lambda(x) := (-1)^{d(0,x)} \omega_{\beta,\Lambda}(S_0^3 S_x^3)$:

$$\frac{1}{|\Lambda|}\sum_{x\in\Lambda} (-1)^{d(0,x)} \omega_{\beta,\Lambda}(S_0^3 S_x^3) = \widehat{C_\Lambda}(0)$$

$$= C_\Lambda(0) - \sum_{\xi\in\Lambda^*\setminus\{0\}} \widehat{C_\Lambda}(\xi)$$

meaning that one is looking for a upper bound on the amplitude of small frequency components of the correlation function.

- The ferromagnetic case is still open!