MSP I – Math part

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- General points:
  - i) In this course: thermal equilibrium
  - ii) Statistical mechanics deals with systems having a large number of degrees of freedom
    - ↳ mathematically: take the thermodynamic limit (TDL) infinite volume, infinite number of "particles" at finite (i.e. non-zero) density
    - ↳ study of collective behaviour ⤳ thermal phases & phase transitions (liquid ↔ solid, magnetised ↔ disordered, ...)
  - iii) Programme: * ideal (i.e. non-interacting) Fermi & (all quantum) Bose gases; Bose-Einstein condensation.
    - * thermal equilibrium: general (C^*-algebraic) theory
    - * phase transitions in quantum lattice systems.
  - iv) In TDL, the Heisenberg picture of mechanics is most suited

- Recall Heisenberg picture:
  i) Central objects: the <u>observables</u>, realised as <u>operators</u> on a Hilbert space.
  ii) Dynamics: time evolution of observables.
  iii) Auxiliary objects: <u>states</u>, realised as <u>vectors</u> in a Hilbert space, which allow to compute expectation values

Mathematically:

i) Set of observables is a <u>C^*-algebra</u> $\mathcal{A}$, namely:
  * $\mathcal{A}$ is an associative algebra
  * equipped with a norm s.t. $\|AB\| \leq \|A\| \|B\|$
  * complete w.r.t. $\|\cdot\|$.
  * equipped with an involution: $*: \mathcal{A} \to \mathcal{A}$:

$$(A^*)^* = A$$
$$(A + \lambda B)^* = A^* + \bar{\lambda} B^* \quad , \lambda \in \mathbb{C}$$
$$(AB)^* = B^* A^*$$

  * C^*-property: $\|A^* A\| = \|A\|^2$

Remarks: a) Observables $A \in \mathcal{A}$ do not need to be self-adjoint.

b) Typical quantum mechanical example: If $\mathcal{H}$ is a separable Hilbert space, then $\mathcal{A} = \mathcal{L}(\mathcal{H})$, the set of bounded operators on $\mathcal{H}$ is a C^*-algebra

c) Physically, there are also unbounded observables. At least for the self-adjoint ones, one can always consider instead the corresponding unitary group (on a given Hilbert space, they are one-to-one by Stone's theorem).

d) $\mathcal{A}$ does not necessarily have an identity.

(ii) A pair $(\mathcal{A}, \tau)$ is a C^*-dynamical system if $\mathcal{A}$ is a C^*-algebra and $\mathbb{R} \ni t \mapsto \tau_t$ is a strongly continuous group of $*$-automorphisms of $\mathcal{A}$:

* $\tau_t : \mathcal{A} \to \mathcal{A}$ s.t.

$$\tau_t(A^*) = \tau_t(A)^*$$
$$\tau_t(A + \lambda B) = \tau_t(A) + \lambda \tau_t(B)$$
$$\tau_t(AB) = \tau_t(A)\,\tau_t(B)$$
$$\|\tau_t(A)\| = \|A\|$$

* $\tau_0(A) = A$; $\tau_{t+s}(A) = \tau_t(\tau_s(A))$
* For any $A \in \mathcal{A}$: $\|\tau_{t+\varepsilon}(A) - \tau_t(A)\| \to 0$ $(\varepsilon \to 0)$.

Now: τ is always generated by a <u>$*$-derivation</u>:

Let $\delta_t : \mathcal{A} \to \mathcal{A}$

$$A \mapsto \delta_t(A) = t^{-1}(\tau_t(A) - A)$$

Let $D(\delta) := \{A \in \mathcal{A} : \lim_{t \to 0} \delta_t(A) \text{ exists}\}$

and let $\delta : \mathcal{A} \to \mathcal{A}$

$$A \mapsto \delta(A) = \lim_{t \to 0} t^{-1}(\tau_t(A) - A)$$

Then δ is a closed, densely defined map s.t.

$$\mathbb{1} \in D(\delta) \text{ and } \delta(\mathbb{1}) = 0$$
$$\delta(AB) = \delta(A)B + A\delta(B)$$
$$\delta(A^*) = \delta(A)^*$$

In fact, by a thm of Hille-Yosida, there is a one-to-one correspondence between τ_t's and δ's

* Remark: In the quantum mechanical example

$$\mathcal{A} = \mathcal{L}(\mathcal{H})$$

The dynamics is generated by a self-adjoint H, namely

$$\tau_t(A) = e^{itH} A e^{-itH}$$

It is a *-automorphism because e^{itH} is unitary and a strongly continuous group because e^{itH} is so.

Corresponding *-derivation:

$$\delta(A) = \frac{d}{dt}\tau_t(A)\Big|_{t=0} = i[H, A]$$

Sometimes written as $\tau_t(A) = e^{i[H,\cdot]}(A)$.

(ii) Finally, a state over $\mathcal{A}$ is a positive, normalised linear functional over $\mathcal{A}$: $\omega : \mathcal{A} \to \mathbb{C}$

$$A \mapsto \omega(A) \in \mathbb{C}$$

s.t. $\omega(A^*A) \geq 0$ (positivity)

$$\|\omega\| := \sup_{A \in \mathcal{A}} \frac{|\omega(A)|}{\|A\|} = 1 \quad \text{(normalisation)}$$

* Remarks : a) The positivity of the quadratic form
$$\lambda \mapsto \omega((A+\lambda B)^*(A+\lambda B))$$
implies : $$\omega(A^*B) = \overline{\omega(B^*A)}$$
$$|\omega(A^*B)|^2 \leq \omega(A^*A)\,\omega(B^*B)$$
namely the Cauchy-Schwarz inequality.

b) In the QM setting, any normalised vectors $\Psi \in \mathcal{H}$ defines a state (in the above sense) by
$$\mathcal{A} \ni A \mapsto \omega_\Psi(A) = \langle \Psi, A\Psi \rangle \in \mathbb{C}$$

c) Also any density matrix $\rho \in \mathcal{L}(\mathcal{H})$ defines a state:
$$\mathcal{A} \ni A \mapsto \omega_\rho(A) = \mathrm{Tr}(\rho A)$$

d) If $\mathbb{1} \in \mathcal{A}$, the normalisation condition is equivalent to $\omega(\mathbb{1}) = 1$.

- Why is this useful/necessary?
Because the same $\mathcal{A}$ can have different, <u>inequivalent representations</u> which represent thermodynamically different situations.

* A <u>representation</u> of a C^*-algebra $\mathcal{A}$ on a Hilbert space $\mathcal{H}$ is a $*$-morphism $\pi : \mathcal{A} \to \mathcal{L}(\mathcal{H})$, namely
$$\pi(AB) = \pi(A)\pi(B) \qquad ; \qquad \pi(A+\lambda B) = \pi(A) + \lambda\pi(B)$$
$$\pi(A^*) = \pi(A)^*.$$
(if π is bijective, it is called a $*$-isomorphism).

* Two representations π_1, π_2 of $\mathcal{A}$ in $\mathcal{H}_1, \mathcal{H}_2$ are <u>equivalent</u> if there is a unitary map $U : \mathcal{H}_1 \to \mathcal{H}_2$ s.t.
$$U\pi_1(A) = \pi_2(A)\,U$$

* A representation is <u>irreducible</u> if the commutant:

$$\pi(\mathcal{A})' := \{B \in \mathcal{L}(\mathcal{H}) : [B, \pi(A)] = 0 \ \forall A \in \mathcal{A}\} = \mathbb{C} \cdot \mathbb{1}$$

* Now: given a representation π in $\mathcal{H}$ and a vector $\xi \in \mathcal{H}$, (normalized) the map

$$\omega_\xi : \mathcal{A} \to \mathbb{C}$$
$$A \mapsto \omega_\xi(A) = \langle \xi, \pi(A) \xi \rangle$$

defines a state. Reciprocally:
Given a state ω on $\mathcal{A}$, there exists a Hilbert space $\mathcal{H}_\omega$, a rep. $\pi_\omega : \mathcal{A} \to \mathcal{L}(\mathcal{H}_\omega)$ and a normalised $\Omega_\omega \in \mathcal{H}_\omega$:

$$\omega(A) = \langle \Omega_\omega, \pi_\omega(A) \Omega_\omega \rangle$$

and $\{\pi_\omega(A)\Omega_\omega : A \in \mathcal{A}\}$ is dense in $\mathcal{H}_\omega$.
Any two such <u>GNS representations</u> are equivalent.

* Physics: termodynamically different states will have inequivalent representations, while states that differ from each other only "locally" will be equivalent.
  Example: a spin-$\frac{1}{2}$ chain with the two states
  $|\cdots\uparrow\uparrow\uparrow\uparrow\cdots\rangle$ & $|\cdots\downarrow\downarrow\downarrow\downarrow\downarrow\cdots\rangle$.
  No observable could relate one to the other

• To close this chapter: topologies:
* At the level of the abstract C^*-algebra: only norm

$$A_n \to A \text{ in } \mathcal{A} \quad \text{if} \quad \|A_n - A\| \to 0$$

* In a representation $\pi(\mathcal{A}) \in \mathcal{L}(\mathcal{H})$ so there are all uniform, strong and weak topologies.
* For states:
  i) Norm: $\omega_n \to \omega$ if $\|\omega_n - \omega\| \to 0$.
  ii) Weak-* topology: $\omega_n \xrightarrow{*} \omega$ if $|\omega_n(A) - \omega(A)| \to 0$ for all $A \in \mathcal{A}$ (no uniformity in A).

  This is very much the convergence of the thermodynamic limit: ω_n is an equilibrium state in a finite volume Λ_n, with $\Lambda_n \to \mathbb{R}^3$ as $n \to \infty$.
  ω is an equilibrium state in the infinite volume (so-called KMS-state)
  $\omega_n \xrightarrow{*} \omega$ if the expectation value of any local observable A in ω_n converges.
* Remark: $\pi(\mathcal{A})$ is usually a strict *-subalgebra of $\mathcal{L}(\mathcal{H})$. There are however useful operators in $\mathcal{L}(\mathcal{H}) \setminus \pi(\mathcal{A})$ such as spectral projectors or infinite volume observables. A physically relevant algebra associated to a representation π is the closure of $\pi(\mathcal{A})$ in the weak operator topology. One has

$$\overline{\pi(\mathcal{A})}^{\text{weak}} = \left(\pi(\mathcal{A})'\right)'$$

and it is called a <u>von Neumann algebra</u>

Remark about the set of states $\mathcal{E}(\mathcal{A})$ over a C^*-algebra with an identity.

* $\mathcal{E}(\mathcal{A})$ is a convex set
* $\mathcal{E}(\mathcal{A})$ is compact in the weak-* topology

⇝ in particular: a sequence of states ω_n on $\mathcal{A}$ (e.g. thermal states in finite volume) always has weakly-* convergent subsequence:

$$\omega_{n_k}(A) \longrightarrow \overline{\omega}(A) \quad (k \to \infty)$$

As we shall see, the uniqueness or not of these limit points is one way to identify <u>phase transitions</u>

1. Ideal Gases

- Gas of non-interacting point particles. The TDL is usually obtained by considering N particles in a volume $\Lambda \subset \mathbb{R}^d$ and letting $N \to \infty$ while $\Lambda \to \mathbb{R}^d$ so that

$$N/|\Lambda| \to \rho$$

and $0 < \rho < \infty$, at fixed temperature.

- Nature is so that there are two types of indistinguishable particles: bosons and fermions
  - N bosons have a symmetric wave function:

$$\psi(x_1, \dots, x_N) = \psi(x_{\sigma(1)}, \dots, x_{\sigma(N)})$$

for all permutations $\sigma \in S_N$.

  - N fermions have an antisymmetric wave function:

$$\psi(x_1, \dots, x_N) = \mathrm{sgn}(\sigma)\, \psi(\sigma(x_1), \dots, \sigma(x_N))$$

where $\mathrm{sgn}(\sigma)$ is the signature of σ.

- In order to describe a system with an arbitrary number of particles, we use the Fock space built upon the one-particle Hilbert space $\mathcal{H}$.

Action of the symmetric group on $\mathcal{H}^N := \otimes^N \mathcal{H}$:

$$P_\sigma : \psi_1 \otimes \dots \otimes \psi_N \longmapsto \psi_{\sigma^{-1}(1)} \otimes \dots \otimes \psi_{\sigma^{-1}(N)}$$

(particles are permuted as σ, states as σ^{-1}, $P_{\sigma\sigma'} = P_\sigma P_{\sigma'}$)

With that: $\mathcal{H}^N_\pm = \{ \psi \in \mathcal{H}^N : P_\sigma \psi = \delta(\sigma) \psi \}$

where $\delta(\sigma) = \begin{cases} 1 & (+) \\ \mathrm{sgn}(\sigma) & (-) \end{cases}$

i.e. $\mathcal{H}^N_+$ is the Hilbert space of bosons;
$\mathcal{H}^N_-$ ———"——— fermions;

Let $\mathcal{H}^0_\pm := \mathbb{C}$ with unit vector Ω, the <u>vacuum</u>.

<u>Fock space</u>: $\mathcal{F}_\pm(\mathcal{H}) := \bigoplus_{N \in \mathbb{N}\cup\{0\}} \mathcal{H}^N_\pm$

contain an arbitrary but <u>finite</u> number of bosons/fermions
(i.e. finite density at finite volume but zero density in infinite volume)

$\psi \in \mathcal{F}_\pm(\mathcal{H})$ represented by $(\psi^N)_{N\in\mathbb{N}\cup\{0\}}$ with $\psi^N \in \mathcal{H}^N_\pm$

$$\|\psi\| = \sum \|\psi^N\|_{\mathcal{H}^N}$$

* Number operator:
$$(\mathcal{N}\psi)^N = N\psi^N .$$

* Second quantisation: If $A: \mathcal{H} \to \mathcal{H}$ is a linear operator:
$\Gamma(A), d\Gamma(A): \mathcal{H}^N_\pm \to \mathcal{H}^N_\pm$:

$$\Gamma(A) = A \otimes \cdots \otimes A \qquad (N \text{ times})$$

$$d\Gamma(A) = \sum_{i=1}^N 1 \otimes \cdots \otimes A \otimes \cdots 1$$
↖ i^{th} position

Remark: $\mathcal{N} = d\Gamma(1)$

Also: $\frac{d}{dt}\Gamma(e^{itA})|_{t=0} = i\,d\Gamma(A)$ namely

$$\Gamma(e^{itA}) = e^{it\,d\Gamma(A)}$$

* Annihilation operator: $\varphi \in \mathcal{H}$. Then $b(\varphi) : \mathcal{H}^N \to \mathcal{H}^{N-1}$,

$$b(\varphi)\, \psi_1 \otimes \cdots \otimes \psi_N = \sqrt{N} \langle \varphi, \psi_1 \rangle\, \psi_2 \otimes \cdots \otimes \psi_N$$

$$b(\varphi)\, \Omega = 0$$

Since $b(\varphi) : \mathcal{H}_\pm^N \to \mathcal{H}_\pm^{N-1}$ for all N, it extends to

$$b(\varphi) : \mathcal{F}_\pm(\mathcal{H}) \longrightarrow \mathcal{F}_\pm(\mathcal{H}).$$

* Creation operator: $\varphi \in \mathcal{H}$. Then $b^*(\varphi) : \mathcal{H}_\pm^{N-1} \to \mathcal{H}_\pm^N$

$$b^*(\varphi)\, \psi^{N-1} = \frac{1}{\sqrt{N}} \sum_{k=1}^{N} (\pm 1)^{k-1} P_{\pi_k} \left(\varphi \otimes \psi^{N-1} \right),$$

where $\pi_k^{-1} = (k, 1, 2, \dots, k-1, k+1, \dots, N)$

* Remarks:

i) $b^*(\varphi) = b(\varphi)^*$

ii) $\varphi \mapsto b(\varphi)$ is antilinear; $\varphi \mapsto b^*(\varphi)$ is linear.

iii) Commutation: $\mathcal{N} b(\varphi) = b(\varphi)(\mathcal{N} - 1)$

iv) If $U : \mathcal{H} \to \mathcal{H}$ is unitary, then

$$\Gamma(U)\, b^\#(\varphi)\, \Gamma(U)^* = b^\#(U\varphi)$$

v) Canonical commutation relations: on $\mathcal{F}_+(\mathcal{H})$: (Bosons)

$$[b(\varphi), b^*(\psi)] = \langle \varphi, \psi \rangle \qquad \text{(CCR)}$$

$$[b(\varphi), b(\psi)] = [b^*(\varphi), b^*(\psi)] = 0$$

Canonical anticommutation relations: on $\mathcal{F}_-(\mathcal{H})$: (fermions)

$$\{ b(\varphi), b^*(\psi) \} = \langle \varphi, \psi \rangle \qquad \text{(CAR)}$$

$$\{ b(\varphi), b(\psi) \} = \{ b^*(\varphi), b^*(\psi) \} = 0$$

where $\{A, B\} = AB + BA$.

All relations easily checked on $\mathcal{F}_\pm^{\text{fin}}(\mathcal{H}) = \{ \psi \in \mathcal{F}_\pm(\mathcal{H}) : \psi^N = 0 \ \forall N \geq N_0 \}$.

* Important: the algebraic relations CCR/CAR determine the bosonic/fermionic nature of the wavefunction:

1.1 Fermions

- The relations (CAR) indicate that the creation/annihilation operators on the fermionic Fock space are a representation of the abstract CAR algebra $\mathrm{CAR}(\mathcal{H})$, defined as the C^*-algebra generated by $\mathbb{1}$ and $a(\varphi)$, $\varphi \in \mathcal{H}$ satisfying
  * $\varphi \mapsto a(\varphi)$ is antilinear
  * $\{a(\varphi), a^*(\psi)\} = \langle \varphi, \psi \rangle \mathbb{1}$; $\{a^*(\varphi), a^*(\psi)\} = 0 = \{a(\varphi), a(\psi)\}$.
- Since $(a^*(\varphi)a(\varphi))^2 = a^*(\varphi)\{a(\varphi), a^*(\varphi)\}a(\varphi) = \|\varphi\|^2 a^*(\varphi)a(\varphi)$, the C^*-property implies that

$$\|a^*(\varphi)\| = \|\varphi\|$$

so that $\varphi \mapsto a^*(\varphi)$ is continuous (fermions).
- Remarks:
  (i) If h is a prehilbert space and $\bar{h} = \mathcal{H}$, then

$$\mathrm{CAR}(h) = \mathrm{CAR}(\mathcal{H})$$

  (ii) $\mathrm{CAR}(\mathcal{H})$ is unique up to $*$-isomorphism.
- Now: We investigate the thermal equilibrium states over $\mathrm{CAR}(\mathcal{H})$.

a) Finite volume $\Lambda \subset \mathbb{R}^d$, one-part. Hilbert space $\mathcal{H}_\Lambda$

The Gibbs state at inverse temperature $0 < \beta < \infty$

and chemical potential $\mu \in \mathbb{R}$ is given by

$$\omega_{\beta,\mu}(A) := Z_{\beta,\mu}^{-1} \operatorname{Tr}\left(e^{-\beta K_\mu} A\right)$$

where $A \in CAR(\mathcal{H}_\Lambda)$, and

$$K_\mu := d\Gamma(H) - \mu N = d\Gamma(H - \mu \mathbb{1})$$

whenever $\exp(-K_\mu) \in I_1$ (trace-class).

<u>Lemma</u>: $\exp(-\beta H) \in I_1(\mathcal{H}_\Lambda) \iff \exp(-\beta K_\mu) \in I_1(\mathcal{F}_-(\mathcal{H}_\Lambda))$ for all $\mu \in \mathbb{R}$.

Typical example: $\Lambda = [-\frac{L}{2}, \frac{L}{2}]^d$
$\mathcal{H}_\Lambda = L^2(\Lambda)$
$H = -\Delta_\Lambda$ (any self-adjoint realization) has compact resolvent, discrete spectrum and $\exp(+\Delta_\Lambda) \in I(L^2(\Lambda))$

<u>But</u>: $-\Delta$ on $\mathbb{R}^d$ has purely absolutely continuous spectrum so $\exp(\Delta)$ cannot be trace-class and the Gibbs state cannot be defined as above.

Since
$$\exp(-\beta K_\mu) = \Gamma\left(\exp\left(-\beta(H - \mu \mathbb{1})\right)\right)$$
$$= z\, \Gamma\left(\exp(-\beta H)\right)$$
where $z = e^{\beta\mu}$ is the "activity", we have

$$e^{-\beta K_\mu} b^\#(\varphi) = z b^\#(e^{-\beta H}\varphi) e^{-\beta K_\mu} \qquad (*)$$

allowing to prove:

<u>Proposition</u>: If $e^{-\beta H} \in I_1$, then

$$\omega_{\beta,\mu}\left(a^+(\varphi)a(\psi)\right) = \left\langle \psi, z e^{-\beta H}(1 + z e^{-\beta H})^{-1}\varphi \right\rangle$$

and $\omega_{\beta,\mu}\left(a^\#(\varphi)\right) = 0$.

<u>Proof</u>: By $(*)$:

$$\omega_{\beta,\mu}\left(a^+(\tilde\varphi)a(\psi)\right) = Z_{\beta,\mu}^{-1} \operatorname{Tr}\left(z b^+(e^{-\beta H}\tilde\varphi) e^{-\beta K_\mu} b(\psi)\right)$$

$$= z\,\omega_{\beta,\mu}\left(a(\psi) a^+(e^{-\beta H}\tilde\varphi)\right)$$

by cyclicity. Using the CAR, this equals

$$\omega_{\beta,\mu}\left(a^+(-z e^{-\beta H}\tilde\varphi)a(\psi)\right) + \langle \psi, z e^{-\beta H}\tilde\varphi \rangle$$

so that $\omega_{\beta,\mu}\left(a^+\left((\mathbb{1} + z e^{-\beta H})\tilde\varphi\right)a(\psi)\right) = \langle \psi, z e^{-\beta H}\tilde\varphi \rangle$

which is the first claim since $z e^{-\beta H} \geq 0$, and hence $\mathbb{1} + z e^{-\beta H}$ is invertible.

If ψ is such that $\mathcal{N}\psi = N\psi$ (fixed particle number)

Then $\langle \psi, e^{-\beta K_\mu} b(\varphi)\psi \rangle = N^{-1} \langle \psi, e^{-\beta K_\mu}(\mathcal{N}+1) b(\varphi)\psi \rangle$

$$= N^{-1} \langle \psi, (\mathcal{N}+1) e^{-\beta K_\mu} b(\varphi)\psi \rangle$$

$$= \frac{N+1}{N} \langle \psi, e^{-\beta K_\mu} b(\varphi)\psi \rangle$$

Hence the $\langle \ldots \rangle = 0$. Computing $\operatorname{Tr}(e^{-\beta K_\mu} b(\varphi))$ in a basis of such vectors proves the second claim. □

Now by a similar argument:

$$\omega_{\beta,\mu}\left(a^*(\varphi_1)\cdots a^*(\varphi_n)a(\psi_m)\cdots a(\psi_1)\right) = \delta_{n,m}\det\left(\langle \psi_i, \rho\,\varphi_j\rangle\right) \quad (**)$$

with $\rho = z e^{-\beta H}\left(\mathbb{1} + z e^{-\beta H}\right)^{-1}$

This gives a explicit formula for $\omega_{\beta,\mu}$ on all of $CAR(\mathcal{H})$ since the algebra is generated by $a(\varphi)$, $a^*(\varphi)$, $\varphi \in \mathcal{H}$.

↝ "quasi-free state" or "Gaussian state"

Remark: The crucial property of the state, expressed by (*) is

$$\omega_{\beta,\mu}(BA) = \omega_{\beta,\mu}\left(A\,\tau_{i\beta}(B)\right)$$

"KMS condition".

b) <u>Thermodynamic limit.</u> Here: $\mathcal{H} = L^2(\mathbb{R}^d)$

$$(H\psi)(x) = (-\Delta\psi)(x) = (2\pi)^{-d/2}\int |\xi|^2\,\hat{f}(\xi)\,e^{i\xi x}\,d\xi$$

with domain $\mathcal{D}(-\Delta) = H^2(\mathbb{R}^d)$

↳ well-defined dynamics: $\tau_t(a^\#(\varphi)) = a^\#(e^{itH}\varphi)$

Now: properly understood, $f(-\Delta_\Lambda)$ converges to $f(-\Delta)$ strongly for any bounded continuous function f

In particular, this applies to

$$e^{-itx} \; ; \; e^{-\beta x} \; ; \; z e^{-\beta x}(1+z e^{-\beta x})^{-1}$$

We consider finite volumes $\Lambda_L = [-\frac{L}{2}, \frac{L}{2}]^d$, $L \in \mathbb{N}$, with $\mathcal{H}_L = L^2(\Lambda_L)$, $H_L = -\Delta_L$ (for ex. with Dirichlet B.C.).

<u>Theorem</u>: Let $\omega_{\beta,\mu}^L$ be the Gibbs state associated with H_L on $CAR(\mathcal{H}_L)$. Then

$$\omega_{\beta,\mu}^L(A) \longrightarrow \omega_{\beta,\mu}(A) \qquad (L \to \infty)$$

for any $A \in CAR(\mathcal{H}_{L'})$ and any L', where $\omega_{\beta,\mu}$ is the quasi-free state on $CAR(\mathcal{H})$ with two point function

$$\omega_{\beta,\mu}(a^*(\varphi)a(\psi)) = (2\pi)^{-d/2} \int \overline{\hat{\psi}(\xi)} \frac{z e^{-\beta|\xi|^2}}{1+z e^{-\beta|\xi|^2}} \hat{\varphi}(\xi)\, d\xi$$

<u>Note</u>: in other words, the sequence of states $(\omega_{\beta,\mu}^L)_{L\in\mathbb{N}}$ converges to $\omega_{\beta,\mu}$ in the weak-$*$ topology.

<u>Proof</u>: It suffices to prove the convergence of the two-point function, see (**). But this follows by the weak convergence of

$$z e^{-\beta H_L}(1+e^{-\beta H_L})^{-1} \xrightarrow{w} z e^{-\beta H}(1+e^{-\beta H})^{-1}$$

$\square$

<u>Remark</u>: Here, it suffices to prove the convergence alone

for $\varphi, \psi \in C_c^\infty(\mathbb{R}^d)$, since $CAR(C_c^\infty(\mathbb{R}^d))$ is equal to $CAR(L^2(\mathbb{R}^d))$ as $C_c^\infty(\mathbb{R}^d)$ is dense in $L^2(\mathbb{R}^d)$.

Crucially: for given β, μ, the limit of $\omega_{\beta,\mu}^L$ exists and is unique and $\omega_{\beta,\mu}$ defines the thermal equilibrium state on the infinite volume algebra $CAR(L^2(\mathbb{R}^d))$.

Finally: Density of the free Fermi gas:

$$\rho(\beta,\mu) = \lim_{L\to\infty} L^{-d} \sum_{n\geq 1} \omega_{\beta,\mu}^L(a^*(\varphi_n)a(\varphi_n)) = (2\pi)^{d/2} \int \frac{z e^{-\beta|\xi|^2}}{1+z e^{-\beta|\xi|^2}} d\xi$$

where $(\varphi_n)_{n\in\mathbb{N}}$ is any basis of $L^2(\Lambda_L)$ (for ex: eigenbasis of $-\Delta_L$)

$\leadsto$ momentum density distribution $0 < \dfrac{z e^{-\beta|\xi|^2}}{1+z e^{-\beta|\xi|^2}} < 1$

and its limit as $\beta\to\infty$ (namely zero temperature limit)

$$\lim_{\beta\to\infty} \left(1+e^{\beta(|\xi|^2-\mu)}\right)^{-1} = \begin{cases} 1 & \text{if } |\xi|^2 < \mu \\ 0 & \text{if } |\xi|^2 > \mu \end{cases}$$

is called the Fermi sea and μ is the Fermi energy.

- $\omega_{\beta,\mu}$ having finite density cannot be represented on Fock space.

$\leadsto$ Araki–Wyss representation:

$$\mathcal{H}_\rho = \mathcal{F}_-(\mathcal{H}) \otimes \mathcal{F}_-(\mathcal{H}) \quad ; \quad \Omega_\rho = \Omega\otimes\Omega$$

$$\pi_\rho(a^\#(\varphi)) = b^\#(\sqrt{1-\rho}\,\varphi)\otimes \mathbb{1} + (-\mathbb{1})^{\mathcal{N}} \otimes b(\sqrt{\rho}\,\varphi)$$

If $\rho = \rho^2$ is a projection (at $\beta = \infty$), the two Fock space have a natural interpretation: holes & particles.

1.2. Bosons

- Unlike in the fermionic case, the bosonic $b^\#(f)$ are unbounded operators on Fock space and cannot represent a C^*-algebra.
Let $\Phi(\varphi) := \frac{1}{\sqrt{2}} (b(\varphi) + b^*(\varphi))$ has a self-adjoint extension and by Stone's theorem

$$W(t\varphi) := \exp\left(it\Phi(\varphi)\right) \qquad (t \in \mathbb{R})$$

defines a strongly continuous unitary group.
If $\mathcal{H}$ is a Hilbert space, the Weyl algebra $CCR(\mathcal{H})$ is the C^*-algebra generated by $W(\varphi)$, $\varphi \in \mathcal{H}$ s.t.

- $W(\varphi) = W(-\varphi)^*$
- $W(\varphi)W(\psi) = \exp\left(-\frac{i}{2} \operatorname{Im}\langle \varphi, \psi\rangle\right) W(\varphi + \psi)$

Remarks: i) Again: $CCR(\mathcal{H})$ is unique up to $*$-isomorph.
ii) Here: if h is a prehilbert space with $\bar{h} = \mathcal{H}$:
$$CCR(h) = CCR(\mathcal{H}) \text{ iff } h = \mathcal{H}.$$
iii) $W(\varphi)$ is unitary for all $\varphi \in \mathcal{H}$ with $W(0) = \mathbb{1}$.
iv) $\|W(\varphi) - \mathbb{1}\| = 2$ for all $\varphi \in \mathcal{H}$, $\varphi \neq 0$
(no norm continuity of $\varphi \mapsto W(\varphi)$)
v) $W(\varphi)W(\psi) = \exp\left(-i \operatorname{Im}\langle \varphi, \psi\rangle\right) W(\psi)W(\varphi)$.

vi) The map $\varphi \mapsto W(\varphi)$ is called the "Weyl quantization"

- Finite volume thermal equilibrium state: A state given by

$$\omega_{\beta,\mu}(A) = Z^{-1}_{\beta,\mu} \operatorname{Tr}_{\mathfrak{F}_s(\mathcal{H})}\left(\exp(-\beta K_\mu) A\right) \qquad (*)$$

where $K_\mu = d\Gamma(H_\Lambda) - \mu N$, on $CCR(\mathcal{H}_\Lambda)$, $\Lambda \subset \mathbb{R}^d$, whenever $\exp(-K_\mu)$ is trace class.

Lemma: $\exp(-\beta H) \in \mathcal{I}_1(\mathcal{H}_\Lambda)$ and $H - \mu > 0$

$\Leftrightarrow \exp(-\beta K_\mu) \in \mathcal{I}_1(\mathfrak{F}_s(\mathcal{H}_\Lambda))$

$\leadsto$ For bosons the Gibbs state exists only for those chemical potentials $\mu \in \mathbb{R}$ s.t. $H - \mu > 0$.

Now: although $b^{\#}(\varphi)$ do not belong to the algebra, the expression on the r.h.s of $(*)$ is finite for any polynomial in them since $b(\varphi_1) \dots b(\varphi_n) \exp(-\beta/2\, K_\mu)$ have a bounded closure which is Hilbert-Schmidt, and the map

$\varphi_1, \dots, \varphi_n, \psi_1, \dots, \psi_m \mapsto \omega_{\beta,\mu}\left(b^+(\varphi_1) \dots b^+(\varphi_n) b(\psi_m) \dots b(\psi_1)\right)$

is continuous, see exercises.

Also:
$$\exp\left(-\tfrac{\beta}{2} K_\mu\right) b^{\#}(\varphi) = b^{\#}\left(e^{-\frac{\beta}{2}(H-\mu)}\varphi\right) \exp\left(-\tfrac{\beta}{2} K_\mu\right) \qquad (\circ)$$

and we have:

Proposition: Let $0 < \beta < \infty$, $\mu \in \mathbb{R}$ and $H = H^*$ s.t.

(i) $H - \mu > 0$ and (ii) $\exp(-\beta H) \in \mathcal{I}_1(\mathcal{H}_\Lambda)$.

A bit more precisely:
the bosonic $b(\varphi)$, $b^*(\psi)$ are unbounded, so that it is a priori unclear if $\mathrm{Tr}\left(e^{-\beta K_\mu} b^*(\psi_1) \cdots b^*(\psi_n) b(\varphi_m) \cdots b(\varphi_1)\right)$ is well-defined. In fact, it is not. However, it is formally equal to

$$\mathrm{Tr}\left(e^{-\frac{\beta}{2}K_\mu} b^*(\psi_1) \cdots b^*(\psi_n) b(\varphi_m) \cdots b(\varphi_1) e^{-\frac{\beta}{2}K_\mu}\right)$$

and we have:

<u>Lemma</u>: Let $\Phi = (\varphi_1, \dots, \varphi_m)$ and $B(\Phi) := b(\varphi_m) \cdots b(\varphi_1) e^{-\frac{\beta}{2}K_\mu}$.
Then $B(\Phi)$ is closeable with bounded closure and

$$B(\Phi) \in \mathcal{I}_2(\mathcal{F}_s(\mathcal{H})).$$

<u>Proof</u>: See exercises.

It follows that for $\Psi = (\psi_1, \dots, \psi_n)$, $\Phi = (\varphi_1, \dots, \varphi_m)$,

$$B(\Psi)^* B(\Phi) \in \mathcal{I}_1(\mathcal{F}_s(\mathcal{H}))$$

and we <u>define</u>

$$\omega_{\beta,\mu}\left(b^*(\psi_1) \cdots b^*(\psi_n) b(\varphi_m) \cdots b(\varphi_1)\right) := \mathrm{Tr}\left(B(\Psi)^* B(\Phi)\right)$$

Then: $\omega_{\beta,\mu}(b^*(\varphi)b(\psi)) = \langle \psi, ze^{-\beta H}(1-ze^{-\beta H})^{-1}\varphi\rangle$

and $\omega_{\beta,\mu}(b(\varphi)) = 0$

for all $\varphi,\psi \in \mathcal{H}_\Lambda$.

Note that $1 - z\exp(-\beta H)$ is invertible since $H-\mu > 0$ implies $\exp(-\beta(H-\mu)) < \mathbb{1}$ and hence $\ker(\mathbb{1} - ze^{-\beta H}) = \{0\}$.

Proof: By ($\diamond$):

$$\omega_{\beta,\mu}(b^*(\varphi)b(\psi)) = Z_{\beta,\mu}^{-1}\,\mathrm{Tr}\left(b^*(e^{\frac{\beta}{2}(H-\mu)}\varphi)e^{-\beta(H-\mu)}b(e^{-\frac{\beta}{2}(H-\mu)}\psi)\right)$$

$$= \omega_{\beta,\mu}\left(b(e^{-\frac{\beta}{2}(H-\mu)}\psi)b^*(e^{-\frac{\beta}{2}(H-\mu)}\varphi)\right)$$

$$= \omega_{\beta,\mu}\left(b^*(\dots)b(\dots)\right) + \langle \psi, e^{-\beta(H-\mu)}\varphi\rangle$$

by the CCR. Iterating this identity n times:

$$\omega_{\beta,\mu}(b^*(\varphi)b(\psi)) = \omega_{\beta,\mu}\left(b^*(e^{-n\frac{\beta}{2}(H-\mu)}\varphi)b(e^{-n\frac{\beta}{2}(H-\mu)}\psi)\right) + \sum_{j=1}^{n}\langle \psi, e^{-j\beta(H-\mu)}\varphi\rangle$$

Since $-\beta(H-\mu) < 0$, $\|e^{-n\frac{\beta}{2}(H-\mu)}\| \to 0$ as $n \to \infty$, and so does the first term on the r.h.s by continuity, while $\sum_{j=1}^{n}(e^{-\beta(H-\mu)})^j \to e^{-\beta(H-\mu)}(\mathbb{1} - e^{-\beta(H-\mu)})^{-1}$ in norm.

The second claim follows as in the fermionic case. $\square$

- Further calculations would yield the state on the Weyl ops:

$$\omega_{\beta,\mu}(W(\varphi)) = \exp\left(-\frac{1}{4}\left\langle \varphi, \frac{1+ze^{-\beta H}}{1-ze^{-\beta H}}\varphi\right\rangle\right) \qquad (\diamond\diamond)$$

- Now: TDL: If $H-\mu > 0$ uniformly in the volume, then the limit can be taken as in the fermionic case.

$\mathcal{H}_L = L^2([-\frac{L}{2}, \frac{L}{2}]^d)$; H_L: Dirichlet Laplacian with eigenvalues $E_{\underline{n}}(L) = \frac{\pi^2}{L^2}(n_1 + \dots + n_d)$, $\underline{n} \in \mathbb{N}^d$, and ground state energy

$$E_{\underline{1}}(L) = \pi^2/L^2 \to 0^+ \quad \text{as } L \to \infty.$$

Let $\mu < 0$, so that $0 < z = \exp(\beta\mu) < 1$ and

$$H_L - \mu > -\mu > 0 .$$

<u>Theorem</u>: $0 < \beta < \infty$, $\mu < 0$, and let $\omega_{\beta,\mu}^L$ denote the Gibbs state over $CCR(\mathcal{H}_L)$ associated with H_L.

$$\lim_{L\to\infty} \omega_{\beta,\mu}^L(A) = \omega_{\beta,\mu}(A)$$

for any $A \in CCR(\mathcal{H}_L)$, where $\omega_{\beta,\mu}$ is the state over $CCR(L^2(\mathbb{R}^d))$ with

$$(\ast) \quad \omega_{\beta,\mu}(W(\varphi)) = \exp\left(-\frac{1}{4}\int |\hat{\varphi}(\xi)|^2 \frac{1+e^{-\beta(|\xi|^2-\mu)}}{1-e^{-\beta(|\xi|^2-\mu)}} d\xi\right).$$

<u>Proof</u>: Since the Weyl operators generate the whole algebra, it suffices to prove the convergence of $\omega_{\beta,\mu}^L(W(\psi))$. But the positive function $x \mapsto (1+\exp(-x))(1-\exp(-x))^{-1}$ is bounded on any interval $[C,\infty)$, $C>0$, so that

$$\left\langle \psi, \frac{\mathbb{1}+e^{-\beta(H_L-\mu)}}{\mathbb{1}-e^{-\beta(H_L-\mu)}}\varphi\right\rangle \longrightarrow \int \overline{\hat{\psi}(\xi)} \frac{1+e^{-\beta(|\xi|^2-\mu)}}{1-e^{-\beta(|\xi|^2-\mu)}} \hat{\varphi}(\xi)\, d\xi$$

for all $\psi, \varphi \in L^2(\mathbb{R}^d)$, which implies $\omega_{\beta,\mu}^L \xrightarrow{\ast} \omega_{\beta,\mu}$. $\square$

- **Note**: Similarly: $\omega_{\beta,\mu}^L(b^*(\varphi)b(\psi)) \longrightarrow \int \overline{\hat\varphi(\xi)}\,\frac{e^{-\beta(|\xi|^2-\mu)}}{1-e^{-\beta(|\xi|^2-\mu)}}\,\hat\psi(\xi)\,d\xi$.

- The finite volume density
$$\rho_L(\beta,z) = L^{-d}\sum_{\underline{n}}\omega_{\beta,\mu}^L\left(b^*(\psi_{\underline{n}})b(\psi_{\underline{n}})\right) = L^{-d}\sum_{\underline{n}}\frac{e^{-\beta(E_{\underline{n}}(L)-\mu)}}{1-e^{-\beta(E_{\underline{n}}(L)-\mu)}}$$
is a monotone increasing function of z at fixed β, L, and since $\rho_L(\beta,z)\longrightarrow\infty$ as $z\longrightarrow E_{\underline{1}}(L)^-$ its range is all of $(0,\infty)$: any density can be reached by choosing μ.

<u>But</u>: this is not true if the TDL is taken first (which is of course the correct order of limits). At $z=0$, the integrand in $(*)$ scales like
$$\frac{e^{-\beta|\xi|^2}}{1-e^{-\beta|\xi|^2}}\sim\left(\beta|\xi|^2\right)^{-1}$$
which is integrable in dimensions $d\geq 3$ at $\xi=0$.

Natural question: How can a density
$$\bar\rho > \rho_c(\beta) := \rho(\beta,1) = \int\frac{e^{-\beta|\xi|^2}}{1-e^{-\beta|\xi|^2}}\,d\xi$$
be obtained in the infinite volume limit?

<u>Short answer</u>: The excess density $\bar\rho-\rho_c(\beta)$ condensate into the ground state
$\rightsquigarrow$ <u>Bose-Einstein condensation</u> $(d\geq 3)$

- First: Periodic B.C. and a scaling limit $z(L) = 1-1/(\rho_0 L^d)$.

Number of particles in the ground state:

$$\mathcal{N}_{0,L}(\beta) := \omega^L_{\beta,\mu(L)}\left(b^*(\psi_0)\, b(\psi_0)\right) = \frac{z}{1-z} = \rho_0 L^d - 1$$

since $E_0(L) = 0$ for all L, while

$$\mathcal{N}_{\underline{n},L}(\beta) = \left(z(L)^{-1} e^{\beta E_{\underline{n}}(L)} - 1\right)^{-1} \leq \left(\beta E_{\underline{n}}(L)\right)^{-1} \leq \text{const } L^2 .$$

Hence if $d=3$, the ground state ψ_0 is the only <u>macroscopically occupied state</u> (i.e. $\mathcal{N}_{0,L}(\beta) \sim$ volume).

Precisely, this means:

Let $\underline{k} = \underline{k}(\underline{n}) = \frac{4\pi^2}{L^2}(n_1^2 + \dots + n_d^2)$. For any $\varphi \in C_c^\infty(\mathbb{R}^d)$:

$$L^{-3} \sum_{\underline{k}} \mathcal{N}_{\underline{k},L}(\beta)\varphi(\underline{k}) \longrightarrow \int \mathcal{N}_s(\beta,\xi)\varphi(\xi)\, d\xi$$

where $\qquad \mathcal{N}_s(\beta,\xi) = \bar{\mathcal{N}}(\beta,\xi) + \rho_0 \delta(\xi)$

and $\qquad \bar{\mathcal{N}}(\beta,\xi) = (2\pi)^{-3} \dfrac{e^{-\beta|\xi|^2}}{1-e^{-\beta|\xi|^2}}$

Indeed, for any $\underline{n} \neq 0$:

$$\mathcal{N}_{\underline{n},L}(\beta) - (2\pi)^3 \bar{\mathcal{N}}(\beta, k(\underline{n})) = (1 - z^{-1})\, e^{+\beta E_{\underline{n}}(L)} \mathcal{N}_{\underline{k},L}(\beta)\, \bar{\mathcal{N}}(\beta,\underline{k})(2\pi)^3$$

since $E_{\underline{n}}(L) = |k(\underline{n})|^2$. But $e^{\beta E_{\underline{n}}(L)} \mathcal{N}_{k(\underline{n}),L}(\beta) \leq$ const. L^2

so that

$$L^{-3} \sum_{\underline{n}\neq 0} \left|\mathcal{N}_{\underline{n},L}(\beta) - (2\pi)^3 \bar{\mathcal{N}}(\beta, k(\underline{n}))\right| \leq \frac{1}{\rho_0 L^3} \text{const.} L^2 \left(\left(\frac{2\pi}{L}\right)^3 \sum_{\underline{n}\neq 0} \bar{\mathcal{N}}(\beta, k(\underline{n}))\right)$$

and the second factor is bounded above by

$$\int \sqrt{r}(\beta,\xi)\,d\xi$$

which is finite in $d=3$, which concludes the argument.

In terms of densities:

$$\rho_L(\beta, z(L)) \longrightarrow \rho_s(\beta) = \rho_c(\beta) + \rho_0$$

where ρ_0 is the condensate density and $\rho_c(\beta) = \frac{1}{(2\pi)^3}\int \frac{e^{-\beta|\xi|^2}}{1-e^{-\beta|\xi|^2}}\,d\xi$.

- A more "physical" TDL: fixed density $\bar{\rho}$, with Dirichlet B.C.

Let z_L be s.t. $\rho_L(\beta, z_L) = \bar{\rho}$

and if $\bar{\rho} \leq \rho_c(\beta)$, let $\bar{z}$ be s.t. $\rho(\beta, \bar{z}) = \bar{\rho}$.

<u>Proposition</u>: Let $d \geq 3$, $0 < \bar{\rho} < \infty$ and $0 < \beta < \infty$.

i) If $\bar{\rho} \leq \rho_c(\beta)$, then $\lim_{L\to\infty} z_L = \bar{z}$

ii) If $\bar{\rho} > \rho_c(\beta)$, then $\lim_{L\to\infty} z_L = 1$. Moreover,

$$L^{-d}\,\frac{z_L e^{-\beta E_1(L)}}{1 - z_L e^{-\beta E_1(L)}} \longrightarrow \bar{\rho} - \rho_c(\beta) \qquad (L\to\infty)$$

In other words:

If the density is fixed, and larger than the critical density, then the activity must converge to 1 as in the discussion above and the excess density $\bar{\rho} - \rho_c(\beta)$ accumulates in the ground state.

We prove only the asymptotic behavior of z_L as $L\to\infty$.

Proof: (i) Since $\frac{at}{1-at}$ is convex for any $a > 0$, so is $t \mapsto f_L(\beta, t)$ and hence

$$\partial_t f_L(\beta, t_2) \leq \frac{f_L(\beta, t_1) - f_L(\beta, t_2)}{t_1 - t_2} \leq \partial_t f_L(\beta, t_1)$$

whenever $t_2 < t_1$. Furthermore,

$$\partial_t \frac{at}{1-at} = \frac{1}{t} \frac{at}{1-at} \frac{1}{1-at} \geq \frac{1}{t} \frac{at}{1-at}$$

so that $\partial_t f_L(\beta, t) \geq \frac{f_L(\beta, t)}{t}$.

Now, by Riemann approximation,

$$f_L(\beta, t) \leq f(\beta, t) \qquad \text{(Dirichlet B.C.)}.$$

so that $t_L \geq \bar{t}$ since both functions are increasing.

Altogether, with $t_1 = t_L$, $t_2 = \bar{t}$:

$$\frac{f_L(\beta, \bar{t})}{\bar{t}} \leq \frac{\bar{f} - f_L(\beta, \bar{t})}{t_L - \bar{t}}.$$

Since $f_L(\beta, \bar{t}) \longrightarrow f(\beta, \bar{t}) = \bar{f}$ pointwise,

$$0 \leq t_L - \bar{t} \leq \frac{\bar{t}\left(\bar{f} - f_L(\beta, \bar{t})\right)}{f_L(\beta, \bar{t})}$$

implies $\lim_{L \to \infty} t_L = \bar{t}$.

(ii) Assume $t_L \leq 1$. Then

$$f_c(\beta) < \bar{f} = f_L(\beta, t_L) \leq f(\beta, t_L) \leq f_c(\beta), \text{ a contradiction.}$$

Hence: $1 < z_L < e^{\beta E_1(L)} \longrightarrow 1 \quad (L \to \infty)$

so that $\lim_{L\to\infty} z_L = 1^+$. □

- Full fledged theorem: Let μ_L be such that $\rho_L(\beta, z_L) = \bar{\rho}$.
Then $\omega^L_{\beta,\mu_L} \xrightarrow{*} \omega_\beta$ as $L \to \infty$, where

i) If $\bar{\rho} \le \rho_c(\beta)$, ω_β is given by (X) on p. 20, with $z = \bar{z}$ solving $\rho(\beta, \bar{z}) = \bar{\rho}$.

ii) If $\bar{\rho} > \rho_c(\beta)$, then $\omega_\beta(W(\psi)) = \exp\left(-\frac{1}{4} q_\beta(\psi)\right)$, where

$$q_\beta(\psi) = 2^{d+1}\left(\bar{\rho} - \rho_c(\beta)\right)|\hat{\psi}(0)|^2 + \int |\hat{\psi}(\xi)|^2 \frac{1 + e^{-\beta|\xi|^2}}{1 - e^{-\beta|\xi|^2}} d\xi \qquad \text{(BEC)}$$

- Remarks. * Rescaling $\zeta := \beta^{1/2}\xi$, one sees that $\rho_c(\beta) = \text{const}\, \beta^{-d/2}$ so that at fixed density, the BEC regime (ii) is reached by lowering the temperature.
$\rightsquigarrow$ BEC is the high density / low temperature regime.

* In the BEC regime, the thermal equilibrium state is not unique. At any fixed β and activity $z = 1$, there are infinitely many thermal states, parametrized by the density $\bar{\rho} \in [\rho_c(\beta), \infty)$.
$\rightsquigarrow$ BEC phase transition.

* BEC for interacting gases: completely open problem.
For example: Does the critical temperature increase or decrease with inter.?

- GNS representation of ω_β with density $\bar{\rho} > \rho_c(\beta)$.

It has two parameters $\rho_0 = \bar{\rho} - \rho_c(\beta) > 0$ and a phase $\theta \in [-\pi, \pi)$.

Araki-Woods representation $(\mathcal{H}_0, \pi_0, \Omega_0)$:

$$\mathcal{H}_0 = \mathcal{F}_s(\mathcal{H}) \otimes \mathcal{F}_s(\mathcal{H}) \otimes L^2(S^1)$$

$$\Omega_0 = \Omega \otimes \Omega \otimes 1$$ ← constant function 1 on the circle.

$$\pi_0(W(\psi)) = W_F(\sqrt{1+\mu}\,\psi) \otimes W_F(\sqrt{\mu}\,\psi) \otimes e^{-i\alpha(\psi,\theta)}$$

where $\alpha(\psi,\theta) = (2\pi)^{3/2}\sqrt{2\rho_0}\,\left(\mathrm{Re}\,\hat{\psi}(0)\cos\theta + \mathrm{Im}\,\hat{\psi}(0)\sin\theta\right)$

and $(\mu\psi)(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{e^{-\beta|\xi|^2}}{1-e^{-\beta|\xi|^2}}\,\hat{\psi}(\xi)\, e^{i\xi x}\, d\xi$

The corresponding annihilation operator is given by

$$a_0(\psi) = b(\sqrt{1+\mu}\,\psi) \otimes \mathbb{1} \times 1 + \mathbb{1} \otimes b^*(\sqrt{\mu}\,\psi) \times 1 - (2\pi)^{3/2}\sqrt{\rho_0}\,\hat{\psi}(0)\, \mathbb{1} \otimes \mathbb{1} \otimes \exp(i\theta)$$

⇝ Here: The "ground state" component of the one-particle wave-function, $\hat{\psi}(0)$, goes into the condensate factor of $\mathcal{H}_0$, with an arbitrary phase θ.