Golden-Thompson inequality

CIRM

• Let A, B be Hermitian $n\times n$ matrices s.t. $[A,B]=0$. Then

$$e^{A+B} = \sum_{k=0}^{\infty} \frac{(A+B)^k}{k!} = \sum_{k=0}^{\infty} \sum_{l=0}^{k} \frac{1}{k!} \binom{k}{l} A^l B^{k-l}$$

$$= \sum_{k=0}^{\infty} \sum_{l=0}^{k} \frac{A^l}{l!} \frac{B^{k-l}}{(k-l)!}$$

On the other hand,

$$e^{tA} e^{tB} = \left(\sum_{k=0}^{\infty} \frac{(tA)^k}{k!} \right) \left(\sum_{l=0}^{\infty} \frac{(tB)^l}{l!} \right) = \sum_{k=0}^{\infty} C_k t^k$$

with $C_k = \sum_{l=0} \frac{A^l}{l!} \frac{B^{k-l}}{(k-l)!}$, so (taking $t=1$)

$$e^{A+B} = e^A e^B$$

• If $[A,B]\neq 0$,

$$e^{A+B} = \sum_{k=0}^{\infty} \frac{(A+B)^k}{k!} = \sum_{k=0}^{\infty} \frac{1}{k!} \Big(A^k + A^{k-1}B + A^{k-2}B^2$$

$$+ \dots + AB^{k-1} + \dots + BA^{k-1} + B^2A^{k-2} + \dots + B^k \Big)$$

$$\neq e^A e^B$$

Rather: $e^{A+B} = e^A e^B e^{-\frac{1}{2}[A,B]} \dots$ (CBH)

multiple commutators in the exponential.

• But: Still have the identity

$$\boxed{\det(e^{A+B}) = \det(e^A e^B)}$$

CIRM - Centre International de Rencontres Mathématiques
Campus de Luminy - Case 916 - F 13288 Marseille cedex 9 - T + 33 (0)4 91 83 30 00 - Fax + 33 (0)4 91 83 30 05
www.cirm-math.fr - twitter.com/@_CIRM - google + CIRM - youtube.com/CIRMchannel - www.chairejeanmorlet.com

This is because for any $n \times n$ matrix L (with eigenvalues λ_j, $j = 1, \dots, n$)

$$\det(e^L) = \det\left(e^{\operatorname{diag}(\lambda_1, \dots, \lambda_n)}\right) = \prod_j e^{\lambda_j}$$
$$= e^{\sum_j \lambda_j} = e^{\operatorname{tr} L}$$

and since commutators are traceless.
(and since det is a homomorphism)

There is another relationship between e^{A+B} and $e^A e^B$:

$$\boxed{\operatorname{tr}(e^{A+B}) \leq \operatorname{tr}(e^A e^B)} \quad \text{(Golden–Thompson)}$$

For the proof we need

Lemma (Non-commutative Hölder inequality)
Let $p \in \mathbb{N}$. Then for any $n \times n$ matrices $A_1, \dots, A_p$, we have

$$|\operatorname{tr}(A_1 \cdots A_p)| \leq \|A_1\|_p \cdots \|A_p\|_p$$

where

$$\|A\|_p^p := \operatorname{tr}\left[(A^* A)^{p/2}\right]$$

is the p-Schatten norm.

Proof. We prove this only for $p \in 2^{\mathbb{N}}$ (by induction).

- For $p = 2$: $\langle A, B \rangle_{\mathcal{J}^2} := \mathrm{tr}(B^* A)$ is a scalar product on $\mathcal{J}^2$ (the Hilbert-Schmidt class).

$$\Rightarrow |\mathrm{tr}(A_1 A_2)| = |\mathrm{tr}(A_1 (A_2^*)^*)| = |\langle A_1, A_2^* \rangle_{\mathcal{J}^2}|$$

$$\underset{\text{Schwarz}}{\overset{\text{Cauchy-}}{\leq}} \|A_1\|_{\mathcal{J}^2} \|A_2^*\|_{\mathcal{J}^2} = \|A_1\|_{\mathcal{J}^2} \|A_2\|_{\mathcal{J}^2}$$

- Assume the claim has been proved for $p \leq 2^{N-1}$. Then for $p = 2^N$

$$|\mathrm{tr}(A_1 \cdots A_p)| = |\mathrm{tr}(\underbrace{(A_1 A_2) \cdots (A_{p-1} A_p)}_{2^{N-1} = \frac{p}{2} \text{ factors}})|$$

$$\overset{\text{Ind. Hyp.}}{\leq} \|A_1 A_2\|_{\mathcal{J}^{\frac{p}{2}}} \|A_3 A_4\|_{\mathcal{J}^{\frac{p}{2}}} \cdots \|A_{p-1} A_p\|_{\mathcal{J}^{\frac{p}{2}}}$$

We expand

$$\|A_1 A_2\|_{\mathcal{J}^{\frac{p}{2}}}^{\frac{p}{2}} = \mathrm{tr}\left[(A_1 A_2)^* (A_1 A_2)\right]^{\frac{p}{2}}$$

$$= \mathrm{tr}[\underbrace{(A_1 A_2)^* (A_1 A_2) \cdots (A_1 A_2)^* (A_1 A_2)}_{\frac{p}{2} = 2^{N-1} \text{ identical factors}}]$$

$$\underset{\text{of tr}}{\overset{\text{cyclicity}}{=}} \mathrm{tr}[A_1^* A_1 A_2 A_2^* \cdots A_1^* A_1 A_2 A_2^*]$$

$$\overset{\text{Ind. Hyp.}}{\leq} \|A_1^* A_1\|_{\frac{p}{2}}^{\frac{p}{4}} \|A_2 A_2^*\|_{\frac{p}{2}}^{\frac{p}{4}} = \|A_1\|_p^{\frac{p}{2}} \|A_2\|_p^{\frac{p}{2}}$$

and similarly for $\|A_3 A_4\|_{\frac{p}{2}}^{\frac{p}{2}}$ etc.

$$\Rightarrow |\mathrm{tr}(A_1 \cdots A_p)| \leq \|A_1\|_p \|A_2\|_p \cdots \|A_{p-1}\|_p \|A_p\|_p \qquad \square$$

<u>Proof of the Golden-Thompson inequality</u>:

Apply the Lemma to $A_1 = \cdots = A_p = AB$ $(A, B \geq 0)$

$$\Rightarrow \mathrm{tr}(AB)^p \leq \|AB\|_p^p$$

By cyclicity of tr, we have for any $p \in 2^{\mathbb{N}}$

$$\mathrm{tr}(AB)^p \leq \|AB\|_p^p = \mathrm{tr}(ABBA)^{\frac{p}{2}}$$
$$= \mathrm{tr}(A^2 B^2)^{\frac{p}{2}}$$

Iterating:

$$|\mathrm{tr}(AB)^p| \leq \mathrm{tr}(A^p B^p)$$

Apply this to $e^{A/p}$, $e^{B/p}$ inst. of A, B

$$\Rightarrow \mathrm{tr}((e^{A/p} e^{B/p})^p) \leq \mathrm{tr}(e^A e^B)$$

Now let $p \to \infty$. Since $e^{A/p} = \mathbb{1} + \frac{A}{p} + \mathcal{O}(\frac{1}{p^2})$, $e^{B/p} = \mathbb{1} + \frac{B}{p} + \mathcal{O}(\frac{1}{p^2}) \Rightarrow \mathrm{tr}((e^{A/p} e^{B/p})^p)$

$$= \mathrm{tr}((\mathbb{1} + \frac{B}{p} + \frac{A}{p} + \mathcal{O}(\frac{1}{p^2})) \cdot p) = \mathrm{tr}(e^{A + B + \mathcal{O}(\frac{1}{p})})$$

By continuity of the maps tr and exp,

$$\lim_{p \to \infty} \mathrm{tr}(e^{A+B+\mathcal{O}(\frac{1}{p})}) = \mathrm{tr}(e^{A+B}) \qquad \square$$

Duhamel's Formula

Theorem (Duhamel) *Let* $\big[A_{i,j}(t)\big]_{1\le i,j\le n}$ *be a matrix–valued function of* $t \in \mathbb{R}$ *that is* C^∞ *in the sense that each matrix element* $A_{i,j}(t)$ *is* C^∞. *Then*

$$\tfrac{d}{dt} e^{A(t)} = \int_0^1 e^{sA(t)} A'(t) e^{(1-s)A(t)}\ ds$$

Proof: We first use Taylor's formula with remainder, applied separately to each matrix element, to give

$$\begin{aligned}
A(t+h) &= A(t) + A'(t)h + \int_t^{t+h} (t+h-\tau) A''(\tau)\ d\tau \\
&= A(t) + A'(t)h + h^2 \int_0^1 (1-x) A''(t+hx)\ dx \qquad \text{where } \tau = t + hx \\
&= A(t) + A'(t)h + B(t,h)h^2 \qquad \text{where } B(t,h) = \int_0^1 (1-x) A''(t+hx)\ dx
\end{aligned}$$

Observe that $B(t,h)$ is C^∞ in t and h. Define

$$E(s) = e^{sA(t+h)} e^{(1-s)A(t)}$$

Then

$$\begin{aligned}
e^{A(t+h)} - e^{A(t)} &= E(1) - E(0) = \int_0^1 E'(s)\ ds \\
&= \int_0^1 \big\{ e^{sA(t+h)} A(t+h) e^{(1-s)A(t)} - e^{sA(t+h)} A(t) e^{(1-s)A(t)} \big\}\ ds
\end{aligned}$$

In computing $E'(s)$ we used the product rule and the fact that, for any constant square matrix C, $\frac{d}{ds} e^{sC} = Ce^{sC} = e^{sC}C$. (This is easily proven by expanding the exponentials in power series.) Continuing the computation,

$$\begin{aligned}
\tfrac{1}{h}\big[e^{A(t+h)} - e^{A(t)}\big] &= \int_0^1 e^{sA(t+h)} \tfrac{1}{h}\big[A(t+h) - A(t)\big] e^{(1-s)A(t)}\ ds \\
&= \int_0^1 e^{sA(t+h)} \big[A'(t) + B(t,h)h\big] e^{(1-s)A(t)}\ ds
\end{aligned}$$

It now suffices to take the limit $h \to 0$. ■

