Tutorial 1 : Infinite spin chain

Consider :

- The C^*-algebra $\mathcal{A}_{\{n\}}$ at site $n \in \mathbb{Z}$, generated by $\mathbb{1}, \sigma^x, \sigma^y, \sigma^z$ (the standard Pauli matrices). Hence $\mathcal{A}_{\{n\}} \cong M_2(\mathbb{C})$.

- For $\Lambda \subset \mathbb{Z}$ a finite subset : $\mathcal{A}_\Lambda := \bigotimes_{n \in \Lambda} \mathcal{A}_{\{n\}}$.
  The local algebra is defined by $\mathcal{A}_{loc} := \bigcup_{\substack{\Lambda \subset \mathbb{Z} \\ \text{finite}}} \mathcal{A}_\Lambda$.
- $\mathcal{A}_{\mathbb{Z}} := \overline{\mathcal{A}_\Lambda}$ (Completion in norm) is the C^*-algebra of the infinite spin chain.

Representations of $\mathcal{A}_{\mathbb{Z}}$

Let $S := \{1,-1\}^{\mathbb{Z}}$, $S^\pm := \{ s \in S : s_n = \pm 1$ for all but finitely many $n \in \mathbb{Z} \}$

$\mathcal{H}^\pm := \ell^2(S^\pm)$ (note : $S^\pm$ countable
$\Rightarrow \mathcal{H}^\pm$ separable)

ONB of $\mathcal{H}^\pm$: $\{\delta_s\}_{s \in S^\pm}$ where $\delta_s(s') = \delta_{s,s'}$.

Spin flip : $\Theta_n : S \to S$

$$(\Theta_n(s))_k := \begin{cases} -s_k & \text{if } k=n \\ s_k & \text{otherwise} \end{cases}$$

Representations:

$\pi^{\pm} : \mathcal{A}_{loc} \to \mathcal{L}(\mathcal{H}^{\pm})$ (and extended to $\mathcal{A}_{\mathbb{Z}}$ by continuity (1)).

$$\left(\pi^{\pm}(\sigma_n^x) f\right)(s) := f(\theta_n(s))$$

$$\left(\pi^{\pm}(\sigma_n^y) f\right)(s) := i s_n f(\theta_n(s))$$

$$\left(\pi^{\pm}(\sigma_n^z) f\right)(s) := s_n f(s)$$

$$\left(\pi^{\pm}(\mathbb{1}) f\right)(s) := f(s)$$

check:

- $\pi^{\pm}$ is a $*$-homomorphism, e.g.

$$\left(\pi^{\pm}(\sigma_n^y)\, \pi^{\pm}(\sigma_n^x)\right) f(s) = \pi^{\pm}(\sigma_n^y)\, f(\theta_n(s))$$

$$= i\, s_n(s) f(s) = i \left(\pi^{\pm}(\sigma_n^z) f\right)(s) = \left(\pi^{\pm}(\sigma_n^y \sigma_n^x) f\right)(s)$$

- $\pi^{\pm}$ is irreducible: There are a number of equivalent criteria for irreducibility. One of them is: any nonzero vector of $\mathcal{H}^{\pm}$ is cyclic for $\pi^{\pm}$.

  1) any basis vector δ_s is cyclic: if $s' \in S^{\pm}$, $s' \neq s$, then s' differs from s only in a finite number of sites $n \in \mathbb{Z}$, say at $n_1, \dots, n_k$. Hence,

$$\pi^{\pm}(\sigma_{n_1}^x) \cdots \cdots \pi^{\pm}(\sigma_{n_k}^x)\, \delta_s = \delta_{s'}$$

(1) Any $*$-homomorphism is automatically continuous.

Thus, $\overline{\{\pi^\pm(A)\delta_s : A \in \mathcal{A}_{\mathbb{Z}}\}}$

$$= \overline{\bigcup_{\substack{\Lambda \subset \mathbb{Z} \\ \text{finite}}} \{\pi^\pm(A)\delta_s : A \in \mathcal{A}_\Lambda\}}$$

$$\supseteq \overline{\bigcup_{k=1}^{\infty} \operatorname{span}\{\pi^\pm(\sigma^x_{n_1} \cdots \sigma^x_{n_k})\delta_s : n_1, \dots, n_k \in \mathbb{Z}\}}$$

$$= \mathcal{H}^\pm \quad (\text{since } \{\delta_s\}_{s \in S^\pm} \text{ is total in } \mathcal{H}^\pm)$$

ii) Any vector $0 \neq \psi \in \mathcal{H}^\pm$ is cyclic: write

$$\psi = \sum_{s \in S^\pm}{}' \lambda_s \delta_s$$

We can approximate ψ up to an arbitrary small error by a finite sum

$$\psi_k = \sum_{i=1}^{k}{}' \lambda_{s_i} \delta_{s_i} \qquad (\text{i.e. } \forall \varepsilon > 0\ \exists k \in \mathbb{N} \text{ s.t. } \|\psi - \psi_k\| < \varepsilon)$$

Wlog assume $\lambda_{s_1} \neq 0$. Define the projection

$$P_n^\pm := \frac{\pi^+(\mathbb{1}) \pm \pi^+(\sigma_n^z)}{2}$$

(we restrict ourselves to the π^+ representation to avoid confusion with $\pm$ signs).

The $P_n^\pm$ project onto the subspace of $\mathcal{H}^+$ consisting of vectors $s = (s_m)_{m\in\mathbb{Z}}$ s.t. $s_n = \pm 1$. Since $s_1 = (s_{1,m})_m \in \mathcal{H}^+$ has only finitely many components with $s_{1,m} = -1$, say $s_{1,m_1}, \dots, s_{1,m_N}$, we have that

$$P_{m_1}^- \cdots P_{m_N}^- \psi_k = \lambda_{s_1} \delta_{s_1}.$$

Since we assumed $\lambda_{s_1} \neq 0$ we can then recover δ_{s_1} from ψ_k by applying $\pi^+(A)$ for an $A \in \mathcal{A}_\mathbb{Z}$.

<u>Inequivalence of π^+, π^-:</u>

Consider the following observable (magnetization)

$$M_N^z := \frac{1}{2N+1} \sum_{n=-N}^{N} \sigma_n^z$$

On $\mathcal{H}^+$: $\langle \delta_{s'}, \pi^+(M_N^z)\delta_s\rangle_+ = \frac{1}{2N+1} \sum_{n=-N}^{N} s_n \underbrace{\langle \delta_{s'}, \delta_s\rangle_+}_{=\delta_{s',s}}$

$$= \frac{\delta_{s',s}}{2N+1} \sum_{n=-N}^{N} s_n \xrightarrow{N\to\infty} +\delta_{s,s'}$$ (since all but finitely many $s_n = +1$)

On $\mathcal{H}^-$: $\langle \delta_{s'}, \pi^-(M_N^z)\delta_s\rangle_- \xrightarrow{N\to\infty} -\delta_{s,s'}$

Hence, for all $\varphi, \psi \in \mathcal{H}^\pm$

$$\langle \varphi, \pi^\pm(M_N^z)\psi\rangle_\pm \xrightarrow{N\to\infty} \pm\langle\varphi,\psi\rangle_\pm$$

Assume π^+, π^- are equivalent, i.e.
$\exists\, U : \mathcal{H}^+ \to \mathcal{H}^-$ unitary s.t.

$$U \pi^+(A) U^* = \pi^-(A) \,, \quad A \in \mathcal{A}_{\mathbb{Z}} .$$

Then for any $\varphi, \psi \in \mathcal{H}^+$

$$\begin{aligned}
\langle \varphi, \psi \rangle_+ &= \lim_{N\to\infty} \langle \varphi, \pi^+(M_N^z)\psi \rangle_+ \\
&= \lim_{N\to\infty} \langle \varphi, U^* \pi^-(M_N^z) U\psi \rangle_+ \\
&= \lim_{N\to\infty} \langle U\varphi, \pi^-(M_N^z) U\psi \rangle_- \\
&= -\langle U\varphi, U\psi \rangle_- = -\langle \varphi, \psi \rangle_+ \quad ⨳
\end{aligned}$$