MQM Sheet 7 (1)

Ex. 1

• B is A-bounded: $B : \underbrace{(\mathcal{D}(A), \|\cdot\|_A)}_{\text{Banach space (since A closed)}} \longrightarrow \mathcal{H}$ with $\|\cdot\|_A = \|\cdot\| + \|A\cdot\|$

is compact, hence bdd.

• B has A-bound zero: If this was false, then $\exists (\psi_n)_n \subset \mathcal{D}(A)$ $\exists a > 0$ s.t.

$$\|B\psi_n\| > a\|A\psi_n\| + n\|\psi_n\|$$

Wlog. $\|B\psi_n\| = 1$ (else consider $\frac{\psi_n}{\|B\psi_n\|}$).

$\Rightarrow \|\psi_n\| < \frac{1}{n}$ and $\|A\psi_n\| < \frac{1}{a}$.

$\Rightarrow \psi_n \longrightarrow 0$ in $\mathcal{H}$ and $(A\psi_n)_n$ has a weakly convergent subseq. $A\psi_n \xrightarrow{w} \varphi$.

$\Rightarrow \forall \chi \in \mathcal{D}(A^*) : \langle \varphi, \chi \rangle = \lim_{n\to\infty} \langle A\psi_n, \chi \rangle$

$= \lim_{n\to\infty} \langle \psi_n, A^*\chi \rangle = 0$

Since A is closed $\Rightarrow \mathcal{D}(A^*)$ is dense in $\mathcal{H}$ ($\mathcal{D}(A^*)$ is dense iff A is closable) $\Rightarrow \varphi = 0$.

Hence $\psi_n \xrightarrow{w} 0$ in $(\mathcal{D}(A), \|\cdot\|_A)$[1]. Since $B : (\mathcal{D}(A), \|\cdot\|_A) \longrightarrow \mathcal{H}$ compact $\Longrightarrow B\psi_n \to 0$ in $\mathcal{H}$, Contradiction to $\|B\psi_n\| = 1$.

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1) $(\mathcal{D}(A), \|\cdot\|_A)$ is in fact a Hilbert space if we equip it with the equivalent norm $(\|\cdot\|^2 + \|A\cdot\|^2)^{1/2} = (\langle\cdot,\cdot\rangle + \langle A\cdot, A\cdot\rangle)^{1/2}$. For $\varphi \in (\mathcal{D}(A), \|\cdot\|_A)$ we have $\langle \psi_n, \varphi \rangle + \langle A\psi_n, A\varphi \rangle \longrightarrow 0$ as $n \to \infty$, hence $\psi_n \xrightarrow{w} 0$ in $(\mathcal{D}(A), \|\cdot\|_A)$.

Ex. 3 $V \in L^\infty(\mathbb{R}^n)$, $V(x) \xrightarrow{|x| \to \infty} 0 \Rightarrow \sigma_{ess}(-\Delta + V) = \sigma_{ess}(-\Delta)$

Pf By Weyl's Thm. it suffices to show that

$$V(-\Delta + 1)^{-1} \text{ is compact.}$$

This is of the form

$$A(x)B(-i\nabla) \quad \text{with} \quad A(x) = V(x), \quad B(p) = \frac{1}{|p|^2+1}$$

Since A, B are bdd. and measurable and vanish at infinity, it follows from a theorem in the lecture that $A(x)B(-i\nabla)$ is compact.

Ex. 4

i) $\exists\, a, b > 0$ s.t. $\forall \psi \in \mathcal{D}(A)$

$$\|B\psi\| \le a\|A\psi\| + b\|\psi\|$$

ii) $\exists\, a', b' > 0$ s.t. $\forall \psi \in \mathcal{D}(A)$

$$\|B\psi\|^2 \le (a')^2 \|A\psi\|^2 + (b')^2 \|\psi\|^2.$$

ii) $\Rightarrow$ i) : $\|B\psi\| \le \sqrt{(a')^2 \|A\psi\|^2 + (b')^2 \|\psi\|^2}$

$$\le a'\|A\psi\| + b'\|\psi\|$$

i) $\Rightarrow$ ii) : $\|B\psi\|^2 \le (a\|A\psi\| + b\|\psi\|)^2$

$$= a^2\|A\psi\|^2 + b^2\|\psi\|^2 + 2ab\|A\psi\|\|\psi\|$$

$$\le (a^2+\varepsilon)\|A\psi\|^2 + \left(b^2 + \tfrac{1}{\varepsilon}\right)\|\psi\|^2 \quad \forall \varepsilon > 0.$$