Ex 1 iii) $\|a\|^2 \underset{\substack{\uparrow \\ C^* \text{property}}}{=} \|\underbrace{a^*a}_{\text{self-adj.}}\| \overset{ii)}{=} r(a^*a) \quad \forall a \in \mathcal{A}$

r depends only on the algebraic structure of $\mathcal{A}$ $\Rightarrow$ norm is unique.

Ex. 2 We'll check that $\mathcal{A}$ is a C^*-algebra. By uniqueness of the norm it then also follows[1] that $\mathcal{B}$ is not a C^*-algebra.

Only check the C^*-property:

$$\|A^*A\|_{op} = \|A\|_{op}^2 \quad \forall A \in \mathrm{Mat}_{n\times n}$$

$\geq$: $\|A\psi\|^2 = \langle A\psi, A\psi\rangle = \langle A^*A\psi, \psi\rangle \leq \|A^*A\| \, \|\psi\|^2$

$\leq$: $\langle A^*A\psi, \varphi\rangle \leq \langle A^*A\psi, \psi\rangle^{\frac{1}{2}} \langle A^*A\varphi, \varphi\rangle^{\frac{1}{2}}$

by Cauchy-Schwarz for $\langle A^*A\,\cdot, \cdot\rangle$

$$\Rightarrow \|A^*A\psi\| = \sup_{\|\varphi\|=1} |\langle A^*A\psi, \varphi\rangle|$$

$$\leq \sup_{\|\varphi\|=1} \|A\psi\| \, \|A\varphi\| \leq \|A\|^2 \|\psi\| \quad \square$$

Ex. 3 We only check that $C_b(X)$ is a Banach space that the C^*-property holds.

- $C_b(X)$ Banach space (see any standard FA textbook):

Let $(f_n)_n \subseteq C_b(X)$ be Cauchy. Then for any $x \in X$ the seq. $(f_n(x))_n \subseteq \mathbb{C}$ is Cauchy. $\longrightarrow$ $\lim_{n\to\infty} f_n(x) =: f(x)$ exists

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1. The norms are not identical except $n=1$ since $1 = \|\mathbb{1}\|_{op} \neq \sqrt{n} = \|\mathbb{1}\|_{HS}$