MQM Sheet 5 Solution

Ex. 1 (a)

(i) $\Rightarrow$ (ii) : • If λ is an e.v. of infinite multiplicity then any ONS on $\mathrm{Ker}(A-\lambda)$ is a Weyl seq.

• If λ is an accumulation point of $\sigma(A)$, then $\exists (t_n)_n \subset \sigma(A)$ s.t. $t_n \neq t_k$ for $n \neq k$ and $\lim_{n\to\infty} t_n = \lambda$. Choose $(\varepsilon_n)_n \subset (0,1)$ s.t. $J_n := (t_n - \varepsilon_n, t_n + \varepsilon_n)$ are pairwise disjoint. ($\Rightarrow \varepsilon_n \to 0$)
Since $t_n \in \sigma(A) \Rightarrow P^A(J_n) \neq 0 \Rightarrow \exists (\psi_n)_n \in P^A(J_n)\mathcal{H}$ w. $\|\psi_n\|=1$
Since $J_n \cap J_k = \emptyset$ for $j \neq k \Rightarrow (\psi_n)_n$ orthonormal $\Rightarrow \psi_n \xrightarrow{w} 0$ (Bessel's inequality).

$$\|(A-\lambda)\psi_n\|^2 = \|(A-\lambda)P^A(J_n)\psi_n\|^2 = \int_{J_n} d\mu_{\psi_n}(t)\,|t-\lambda|^2$$

$$\leq \int_{J_n} d\mu_{\psi_n}(t)\left(|t_n-\lambda|+\varepsilon_n\right)^2 = \left(|t_n-\lambda|+\varepsilon_n\right)^2 \xrightarrow{n\to\infty} 0 .$$

(ii) $\Rightarrow$ (i) : Let $(\psi_n)_n$ be a Weyl seq. for A at λ. We'll show that

$$\forall \varepsilon>0 : \dim\left[P^A([\lambda-\varepsilon, \lambda+\varepsilon])\right]\mathcal{H} = \infty. \qquad (*)$$

~~Assume~~ $(*)$ holds and $\lambda \notin \sigma_{ess}(A)$. Then either $\lambda \in \rho(A)$ or λ is an isolated eigenvalue of finite multiplicity

$\Rightarrow \dim P^A(\{\lambda\})\mathcal{H} < \infty$ and $P^A([\lambda-2\varepsilon,\lambda)\cup(\lambda,\lambda+2\varepsilon]) = 0$

for some $\varepsilon>0$. $\Rightarrow \dim\left[P^A[\lambda-\varepsilon,\lambda+\varepsilon]\right]\mathcal{H}$
$\leq \dim P^A(\{\lambda\}) + \dim P^A([\lambda-2\varepsilon,\lambda)\cup(\lambda,\lambda+2\varepsilon]) < \infty$

Contradiction to $(*)$. It remains to prove $(*)$.

~~Assume~~ $\circledast$ does not hold, i.e. $\exists\, \varepsilon > 0$ s.t.

$$\dim \left(P^A([\lambda-\varepsilon, \lambda+\varepsilon])\mathcal{H}\right) < \infty .$$

$$\Rightarrow \varepsilon^2 \|\psi_n\|^2 = \int_{[\lambda-\varepsilon,\lambda+\varepsilon]} d\mu_{\psi_n}(t)\, \varepsilon^2 + \int_{\mathbb{R}\setminus[\lambda-\varepsilon,\lambda+\varepsilon]} d\mu_{\psi_n}(t) \underbrace{\varepsilon^2}_{\leq (t-\lambda)^2}$$

$$\leq \underbrace{\varepsilon^2 \| \underbrace{P^A(\lambda-\varepsilon,\lambda+\varepsilon)}_{\text{compact}} \underbrace{\psi_n}_{\xrightarrow{w} 0} \|^2}_{\longrightarrow 0} + \underbrace{\|(A-\lambda)\psi_n\|^2}_{\longrightarrow 0 \quad (n\to\infty)}$$

$$\Rightarrow \liminf_{n\to\infty} \|\psi_n\| = 0 \qquad \text{contradiction.}$$ $\square$

<u>Ex. 2</u> <u>Thm 2.11 (a)</u> Let C be H-compact, $\psi \in D(H) \cap \mathcal{H}_c$.

$$\Rightarrow \lim_{T\to\infty} \frac{1}{T} \int_0^T dt\, \|C e^{-itH}\psi\|^2 = 0$$

<u>Corollary 2.3</u> $\forall \varphi \in \mathcal{H}, \psi \in \mathcal{H}_c$

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T dt\, |\langle \varphi, e^{-itH}\psi\rangle|^2 = 0$$

<u>Proof</u> that Cor. 2.3 $\Rightarrow$ Thm. 2.11 (a):

1. $$C e^{-itH}\psi = \underbrace{C(H+i)^{-1}}_{=:\tilde{C}\ \text{compact}} \underbrace{(H+i)e^{-itH}\psi}_{=:\tilde{\psi} \in \mathcal{H}_c}$$

So we can consider $\tilde{C}, \tilde{\psi}$ inst. of C, ψ. We drop the tildes again, i.e. write C, ψ, with $\psi \in \mathcal{H}_c$, C compact.

2. ~~Assume~~ $C = |\varphi\rangle\langle\varphi|$ for some $\varphi \in \mathcal{H}$

$$\|Ce^{-itH}\psi\|^2 = |\langle\varphi, e^{-itH}\psi\rangle|^2 \, \|\varphi\|^2$$

Claim follows from Cor. 2.3.

3. ~~Assume~~ $C = \sum_{j=1}^{N} |\varphi_j\rangle\langle\varphi_j|$, $\varphi_j \in \mathcal{H}$.

$$\|Ce^{-itH}\psi\|^2 \underset{\substack{\text{triangle}\\ \text{ineq.}}}{\leq} \left(\sum_{j=1}^{N} |\langle\varphi_j, e^{-itH}\psi\rangle| \, \|\varphi_j\|\right)^2$$

$$\leq N^2 \max_{1\leq j\leq N} |\langle\varphi_j, e^{-itH}\psi\rangle|^2 \|\varphi_j\|^2$$

Claim follows from Cor. 2.3.

4. C compact $\Rightarrow$ $C = \lim_{N\to\infty} F_N$, where $F_N = \sum_{j=1}^{N} |\varphi_j\rangle\langle\varphi_j|$ is finite rank. Given $\varepsilon > 0$ choose $N \in \mathbb{N}$ s.t.

$$\|C - F_N\| \leq \varepsilon/4\|\psi\|$$

Then choose $t_0 > 0$ s.t.

$$\frac{1}{T}\int_0^T dt \, \|F_N e^{-itH}\psi\|^2 \leq \varepsilon/4 \quad \forall T \geq t_0 .$$

$$\Rightarrow \frac{1}{T}\int_0^T dt \, \|Ce^{-itH}\psi\|^2 \leq \frac{1}{T}\int_0^T dt \left(\|F_N e^{-itH}\psi\| + \varepsilon/4\right)^2$$

$$\leq 2\left(\underbrace{\frac{1}{T}\int_0^T \|F_N e^{-itH}\psi\|^2}_{\leq \varepsilon/4} + \varepsilon/4\right) = \varepsilon \qquad \square$$

Ex. 3 $\mu_{\varphi,\psi} = \frac{1}{4}\{\mu_{\varphi+\psi} - \mu_{\varphi-\psi} + i\mu_{\varphi+i\psi} - i\mu_{\varphi-i\psi}\}$

(polarization). non-negative

$$\mu_{\varphi+\alpha\psi}(B) = \langle \varphi+\alpha\psi, 1_B(A)(\varphi+\alpha\psi)\rangle$$

$$= \langle\varphi, 1_B(A)\varphi\rangle + \mu_\alpha(B) \qquad \forall\, \alpha\in\{1,-1,i,-i\}$$

where all $\mu_\alpha \ll$ Lebesgue e.g.

$$|\mu_1(B)| \le |\langle\varphi, 1_B(A)\psi\rangle| + |\langle\psi, 1_B(A)\varphi\rangle| + |\langle\psi, 1_B(A)\psi\rangle|$$

$$\le 2\|\varphi\|\,\mu_\psi(B)^{\frac{1}{2}} + \mu_\psi(B) = 0 \quad \text{if} \quad |B| = 0.$$

$\Rightarrow$ Radon-Nikodym : $\exists\, h_\alpha \in L^1(\mathbb{R})$ s.t.

$$\mu_\alpha(B) = \int_B d\lambda\, h_\alpha(\lambda).$$

Since $\mu_{\varphi,\psi} = \sum_{\alpha\in\{1,-1,i,-i\}}' \alpha\cdot\mu_\alpha$ (the terms $\langle\varphi, 1_B(A)\varphi\rangle$ cancel !)

$$\Rightarrow \quad \mu_{\varphi,\psi}(B) = \int_B d\lambda\, h(\lambda) \quad \text{w. } h := \sum_\alpha{}' \alpha\cdot h_\alpha .$$

Ex. 4 $\underline{d=1}$: $-\psi'' = E\psi$, $\psi(0) = \psi(1)$

$$\Rightarrow \begin{cases} \psi_k(x) = e^{2\pi i k x} & \text{with } k\in\mathbb{Z} \\ E_k = (2\pi k)^2 \end{cases}$$

$\underline{d\ge 2}$: Separation of variables : $\psi_k = e^{2\pi i k\cdot x}$, $E_k = |2\pi k|^2$, $k\in\mathbb{Z}^d$

$$\|(-\Delta+1)\|_{\mathcal{T}^p}^p = \sum_{k\in\mathbb{Z}^d}{}' \frac{1}{(|2\pi k|^2+1)^p} < \infty \iff 2p > d \iff p > \frac{d}{2}.$$

Remarks to Ex. 4

① $\{\psi_k\}_{k\in\mathbb{Z}^d}$ is an ONB of $L^2([0,1]^d)$ (Fourier series!)
Therefore $-\Delta$ has discrete spectrum $\sigma(-\Delta)=\{E_k\}_{k\in\mathbb{Z}^d}$.
Indeed, assume that $\lambda\in\mathbb{C}\setminus\{E_k\}_{k\in\mathbb{Z}^d}$. Then for any $g\in L^2([0,1]^d)$ there ~~exists~~ $f\in D(-\Delta)$ s.t.

$$(-\Delta-\lambda)f = g$$

i.e. $\lambda\in\rho(-\Delta)$. To see this, fix λ, g and take $f_k := \frac{\langle g,\psi_k\rangle}{|2\pi k|^2-\lambda}$,

$$f := \sum_{k\in\mathbb{Z}^d} f_k\psi_k \quad (\text{convergent in } L^2 \text{ since } |f_k|\le \tfrac{1}{c}|\langle g,\psi_k\rangle|).$$

For $N\in\mathbb{N}$ set $S_N := \sum_{|k|<N} f_k\psi_k$. Then $(S_N)_N\subseteq D(-\Delta)$,

$$S_N \xrightarrow{N\to\infty} f, \quad (-\Delta-\lambda)S_N \xrightarrow{N\to\infty} g.$$

Since $-\Delta$ is closed, we have $f\in D(-\Delta)$, $(-\Delta-\lambda)f=g$.

② $(-\Delta+1)^{-1}$ is compact since

$$\Big\|(-\Delta+1)^{-1} - \overbrace{\sum_{|k|<N} \frac{1}{E_k+1}|\psi_k\rangle\langle\psi_k|}^{\text{finite rank}}\Big\|$$

$$\le \frac{1}{\inf_{|k|\ge N} E_k+1} \xrightarrow{N\to\infty} 0 \qquad (E_k = |2\pi k|^2 \xrightarrow{k\to\infty} \infty)$$

(We used that by the proof of ①: $(-\Delta+1)^{-1}g = \sum_{k\in\mathbb{Z}^d}\frac{1}{E_k+1}|\psi_k\rangle\langle\psi_k|g\rangle$ and then

$$\Big\|(-\Delta+1)^{-1}g - \sum_{|k|<N}\frac{1}{E_k+1}|\psi_k\rangle\langle\psi_k|g\rangle\Big\|^2 = \sum_{|k|\ge N}\frac{1}{(E_k+1)^2}|\langle\psi_k|g\rangle|^2$$

$$\le \sup_{|k|\ge N}\frac{1}{(E_k+1)^2}\sum_{|k|\ge N}|\langle\psi_k|g\rangle|^2 \le \sup_{|k|\ge N}\frac{1}{(E_k+1)^2}\|g\|^2 \Big)$$