Tutorial 7

Solution of exercises from last time.

Ex. 2

$K(E) := V^{\frac{1}{2}} (-\Delta - E)^{-1} V^{\frac{1}{2}}$, $E < 0$.

Assume $V \in C_c^\infty(\mathbb{R})$. Then $K(E)$ is Hilbert-Schmidt:

$$\| K(E) \|_{HS} \leq \| V^{\frac{1}{2}} (-\Delta - E)^{-1} \|_{HS} \, \| V^{\frac{1}{2}} \|_{L^\infty}$$

$$\leq 2\pi \| V^{\frac{1}{2}} \|_{L^2} \, \| (|\cdot|^2 - E)^{-1} \|_{L^2} \, \| V^{\frac{1}{2}} \|_{L^\infty} < \infty$$

In particular, $K(E)$ is compact. Since $K(E)^* = K(E)$

$$\Rightarrow \quad \| K(E) \| = r(K(E)) \quad \text{(spectral radius)}$$

In fact, since $K(E) \geq 0$

$$\Rightarrow \overbrace{\text{largest eigenvalue of } K(E)}^{=: \mu_1(E)} = \| K(E) \| .$$

By the variational principle

$$\mu_1(E) = \sup_{\|\eta\| = 1} \langle \eta , K(E) \eta \rangle$$

Note that

$$\frac{d}{dE} \langle \eta , K(E) \eta \rangle = \frac{d}{dE} \langle V^{\frac{1}{2}} \eta , (-\Delta - E)^{-1} V^{\frac{1}{2}} \eta \rangle$$

$$= \frac{d}{dE} \int | \widehat{V^{\frac{1}{2}} \eta} (\xi) |^2 \frac{1}{|\xi|^2 - E} \, d\xi$$

$$= \int | \widehat{V^{\frac{1}{2}} \eta} (\xi) |^2 \frac{1}{(|\xi|^2 - E)^2} \, d\xi \geq 0$$

$\Rightarrow \mu_1(\cdot)$ monotonically increasing.

On the one hand, we have

$$\lim_{E\to-\infty} \int |\widehat{V^{\frac12}\eta}(\xi)|^2 \frac{1}{|\xi|^2-E}\, d\xi = 0$$

On the other hand,

$$\lim_{E\to 0} \int |\widehat{V^{\frac12}\eta}(\xi)|^2 \frac{1}{|\xi|^2-E}\, d\xi = +\infty$$

if $\widehat{V^{\frac12}\eta}(0) \neq 0$, i.e. if $\int V^{\frac12}(x)\,\eta(x)\,dx \neq 0$

since $V \not\equiv 0$ and $V(x) \geq 0$ we can choose $\eta \geq 0$

s.t. $V^{\frac12}(x)\,\eta(x) \geq 0$ and and not 0 a.e.

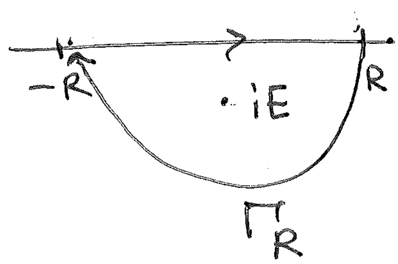
$$\Rightarrow \quad \lim_{E\to-\infty} \mu_1(E) = 0, \quad \lim_{E\to 0} \mu_1(E) = +\infty$$

If we could show that $\mu_1(\cdot)$ is cont. in $(-\infty,0)$, then $\Rightarrow \exists!\, E \in (-\infty,0)$ s.t. $\mu_1(E) = 1$. Hence E is an eigenvalue by the Birman–Schwinger principle. In fact, perturbation theory yields that $\mu_1(\cdot)$ is analytic, but we will not prove that. (This is also called the "Kato-Releich Theorem", see e.g. Reed-Simon Thm. XII.8).

Ex. 3

$$(-\Delta+E^2)^{-1}(x,y) = \int_{\mathbb{R}} e^{i(x-y)\xi}\, \underbrace{\frac{1}{\xi^2+E^2}}_{=\frac{1}{\xi-iE}\,\frac{1}{\xi+iE}}\, \frac{d\xi}{2\pi}$$

$$= \lim_{R\to\infty} \int_{\Gamma_R} e^{i(x-y)\xi} \frac{1}{(\xi-iE)(\xi+iE)} \frac{d\xi}{2\pi} \quad (\text{for } x-y \geq 0)$$



$$= \frac{2\pi i}{2\pi} \operatorname{Res}\left(\frac{e^{i(x-y)\xi}}{(\xi-iE)(\xi+iE)} ;\ \xi = iE \right)$$

$$= i\, \frac{e^{i(x-y)iE}}{2iE} = \frac{e^{-|x-y|E}}{2E}$$

Similarly for $x-y \leq 0$.

Ex. 4

$$\frac{e^{-E|x-y|}}{2E} = \frac{1}{2E} + \frac{e^{-E|x-y|}-1}{2E}$$

$$K(E) := V^{\frac{1}{2}}(-\Delta+E^2)^{-1}V^{\frac{1}{2}} = K_0(E) + K_1(E)$$

where $K_0(E)$ has kernel

$$k_0(E;x,y) = \frac{V^{\frac{1}{2}}(x)\, V^{\frac{1}{2}}(y)}{2E}$$

and $K_1(E)$ has kernel

$$k_1(E;x,y) = V^{\frac{1}{2}}(x)\, \frac{e^{-E|x-y|}-1}{2E}\, V^{\frac{1}{2}}(y)$$

Note that $|e^{-x}-1| \leq x$ for $x \geq 0$

$$|k_1(E;x,y)| \leq V^{\frac{1}{2}}(x)\,\frac{|x-y|}{2}\,V^{\frac{1}{2}}(y)$$

$$\leq \frac{1}{2}\left(|x|+|y|\right)V^{\frac{1}{2}}(x)\,V^{\frac{1}{2}}(y)$$

$$\Rightarrow \iint |k_1(E;x,y)|^2\,dx\,dy \leq \frac{1}{4}\int V(x)V(y)\,(|x|+|y|)^2\,dx\,dy$$

$$\leq \frac{1}{2}\iint V(x)V(y)\,(|x|^2+|y|^2)\,dx\,dy$$

$$= \left(\int V(x)|x|^2\,dx\right)\left(\int V(y)\,dy\right) < \infty$$

$\Rightarrow \|K_1(E)\|_{HS} \leq C(V)$ indep. of $E \in (-\infty,0)$.

On the other hand, $K_0(E)$ is rank 1, so it has just one eigenvalue, equal to $\frac{1}{2E}\int V(x)\,dx$.

$\Rightarrow$ # of singular values of $K(E)$ that are ≥ 1

$\leq 1 +$ # singular values of $K_1(E)$ that are ≥ 1

$\leq 1 + \|K_1(E)\|_{HS}^2 \leq 1 + C(V) < \infty$

Here we used Fan's inequality (a consequence of the variational principle)

$$s_{n+m+1}(K_0(E)+K_1(E)) \leq s_{n+1}(K_0(E)) + s_{m+1}(K_1(E))$$

Hence
$$s_1(K_0+K_1) \leq s_1(K_0) + s_1(K_1)$$
$$s_2(K_0+K_1) \leq \underbrace{s_2(K_0)}_{=0} + s_1(K_1)$$
$$s_3(K_0+K_1) \leq \underbrace{s_2(K_0)}_{=0} + s_2(K_1)$$
etc.