29. 11. 17

Exercise 1: Let $H = H^*$, $H \geq 0$. Prove:

$$\inf \sigma(H) = \inf_{\substack{\psi \in \mathcal{D}(H) \\ \|\psi\| = 1}} \langle \psi, H \psi \rangle$$

Exercise 2: Use the Birman–Schwinger principle to prove that $H = -\Delta + V$ on $L^2(\mathbb{R})$ (with the usual domain) has at least one negative eigenvalue, provided $V(x) \geq 0$ and $V \not\equiv 0$ (you can assume $V \in C_c^\infty(\mathbb{R})$).
Hint: $\| \sqrt{V} (H_0 - E)^{-1} \sqrt{V} \|$ is an eigenvalue of $\sqrt{V}(H_0 - E)^{-1}\sqrt{V}$ $(E < 0)$.

Exercise 3: Compute the Green function of $(-\Delta - E)^{-1}$ (in one dimension). Hint: Use contour integration.
You should find

$$(-\Delta - E)^{-1}(x,y) = \text{const.} \frac{e^{-\sqrt{-E}|x-y|}}{\sqrt{-E}}, \quad \begin{matrix} E < 0 \\ x, y \in \mathbb{R} \end{matrix}$$

Exercise 4: Prove that $H = -\Delta - V$ (on $L^2(\mathbb{R})$, $V \in C_c^\infty(\mathbb{R})$, $V \geq 0$) has only finitely many eigenvalues.
Hint: Write

$$(-\Delta - E)^{-1}(x,y) = \text{const} \left(\frac{1}{\sqrt{-E}} + \frac{e^{-\sqrt{-E}|x-y|} - 1}{\sqrt{-E}} \right)$$

and use the Birman–Schwinger principle which states that

$$\#\{E \in \sigma(H) : E \leq \underbrace{E_0}_{<0}\} = \#\{\mu \in \sigma(\sqrt{V}(H_0 - E)^{-1}\sqrt{V}) : \mu \geq 1\}.$$