Tutorial 3 Spectral measures, spectral thm.

Exercise 1: Let $A : \mathcal{H} \supset \mathcal{D}(A) \to \mathcal{H}$ be self-adjoint. Prove that

$$\mathcal{B} \ni B \mapsto 1\!\!1_B(A) =: P(B) \in BL(\mathcal{H}) \quad (*)$$

defines a spectral measure (= projection-valued measure)

Exercise 2: With $P(B)$ given by $(*)$ prove:

$$\int_{\mathbb{R}} dP(\lambda)\, \lambda = A.$$

Exercise 3: Let $P : \mathcal{B} \to BL(\mathcal{H})$ be a spectral measure. Prove that

$$A := \int_{\mathbb{R}} dP(\lambda)\, \lambda \quad , \quad \mathcal{D}(A) := \left\{ \psi \in \mathcal{H} : \int_{\mathbb{R}} d\mu_\psi(\lambda)\, \lambda^2 < \infty \right\}$$

is self-adjoint.

Exercise 4: Let $P : \mathcal{B} \to BL(\mathcal{H})$ be a spectral measure. Prove that

$$\Pi : \left\{ \begin{array}{c} \text{bdd. meas. fcts.} \\ \mathbb{R} \to \mathbb{C} \end{array} \right\} \to BL(\mathcal{H})$$

$$f \mapsto \Pi(f) := \int_{\mathbb{R}} dP(\lambda)\, f(\lambda)$$

fulfills (some!) properties of the measurable functional calculus.