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Tutorial 1 (MQM 1 2017/18)

Weak derivatives and Sobolev spaces

Motivation: • Usual derivative too restrictive for many purposes. E.g. $u(x) := |x|$ is not diff.'able at $x=0$, but

$$|x| = \int_0^x \operatorname{sgn}(t)\,dt \quad \text{(Fund. thm. of calculus)}$$

so $u'(x) = \operatorname{sgn}(x)$ is not a bad candidate for a derivative. However, for $u(x) = \operatorname{sgn}(x)$,

$$\operatorname{sgn}(x) \neq \int_0^x 0\,dt,$$

so $u'(x) = 0$ is <u>not</u> a good candidate.

• In higher dim. we have the integration by parts formula

$$\int_\Omega u\,\partial^\alpha \varphi\,dx = (-1)^{|\alpha|} \int_\Omega \partial^\alpha u\,\varphi\,dx$$

for all $\varphi \in C_0^\infty(\Omega)$, $u \in C^{|\alpha|}(\Omega)$. <u>This</u> is the starting point for the def. of weak derivatives.

$\boxed{\text{Def.}}$ $u \in L^1_{loc}(\Omega)$ is <u>α-times weakly diff'able</u> in Ω if $\exists\, v_\alpha \in L^1_{loc}(\Omega)$ s.t. $\int_\Omega u\,\partial^\alpha\varphi\,dx = (-1)^{|\alpha|}\int_\Omega v_\alpha \varphi\,dx$ for all $\varphi \in C_0^\infty(\Omega)$. We call v_α the weak derivative of u and write $v_\alpha = \partial^\alpha u$.

Rk. By IBP: u α-times diff'able $\Rightarrow$ u α-times weakly diff'able.

Exercise 1 $n=1$, $\Omega = (-1,1)$, $u(x) = |x|$.

Does u' exist? Compute it!
What about u''?

Exercise 2 $n=2$, $\Omega = B_1(0)$, $u_a(x) = |x|^a$, $a \in \mathbb{R}$.

For which a does the first weak der. exist?

Exercise 3 Prove uniqueness of the weak derivative. (Hint: Fundamental lemma of calculus of variations)

Exercise 4 Show: If u is α-times weakly diff. and $\partial^\alpha u$ is β-times weakly diff $\Rightarrow$ u is $\alpha+\beta$-times weakly diff.

Def. For $m \in \mathbb{N}_0$, $1 \leq p \leq \infty$

$$H^{m,p}(\Omega) := \{ f \in L^p(\Omega) : f \text{ is } m\text{-times weakly diff'able} \}$$

$$\|u\|_{H^{m,p}(\Omega)} := \begin{cases} \left(\sum_{|\alpha| \leq m} \|\partial^\alpha u\|^p_{L^p(\Omega)} \right)^{\frac{1}{p}}, & 1 \leq p < \infty \\ \max_{|\alpha| \leq m} \|\partial^\alpha u\|_{L^\infty(\Omega)}, & p = \infty \end{cases}$$

THM $H^{m,p}(\Omega)$ is a Banach space.

Rk. The notation $H^{m,p}(\Omega) \equiv W^{m,p}(\Omega)$ is also customary.

Proof: Exercise 5

The fundamental result is:

THM (Meyers - Serrin)
$C^\infty(\Omega) \cap H^{m,p}(\Omega)$ is dense in $H^{m,p}(\Omega)$ for $1 \le p < \infty$ ($p=\infty$ not included!)

Proof: See e.g. Dobrowolski.

Exercise 6: Prove: $u, v \in H^{1,2}(\Omega) \Longrightarrow uv \in H^{1,1}(\Omega)$ and $\partial(uv) = u\partial v + v\partial u$.
(Hint: Use the Meyers - Serrin thm. to approximate u, v by C^∞ functions).