MATHEMATISCHES INSTITUT
LMU MÜNCHEN
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Mathematical Quantum Mechanics

Problem Sheet 3

Hand-in deadline: 11/10/2016 before 12:00 in the designated MQM box (1st floor, next to the library).

Exercise 1: Let $\dot{T}$, T_1, T_2 be the following operators[1] on $L^2(\mathbb{R}^d)$:

$$\begin{aligned}
&\dot{T} := -\Delta, \quad \mathcal{D}(\dot{T}) := C_c^\infty(\mathbb{R}^d),\\
&T_1 := \{\text{Friedrichs extension of } \dot{T}\},\\
&T_2 := \mathcal{F}^{-1} M_{|\cdot|^2} \mathcal{F}, \quad \mathcal{D}(T_2) := \{\psi \in L^2(\mathbb{R}^d) : |\cdot|^2 \mathcal{F}\psi \in L^2(\mathbb{R}^d)\}.
\end{aligned}$$

Prove that $T_1 = T_2$.

Exercise 2: Let $0 < \alpha < d$, $0 < \beta < d$ and $0 < \alpha + \beta < d$. Prove that

$$(|\cdot|^{\alpha-d} * |\cdot|^{\beta-d})(y) := \int_{\mathbb{R}^d} |z|^{\alpha-d} |y-z|^{\beta-d} \mathrm{d}z = \frac{c_{d-\alpha-\beta} c_\alpha c_\beta}{c_{\alpha+\beta} c_{d-\alpha} c_{d-\beta}} |y|^{\alpha+\beta-d},$$

where $c_\alpha := \pi^{-\alpha/2}\Gamma(\alpha/2)$. *Hint: Use the formula*

$$c_\alpha |k|^{-\alpha} = \int_0^\infty \exp(-\pi |k|^2 \lambda) \lambda^{\alpha/2-1} \mathrm{d}\lambda.$$

Exercise 3: Compute the Fourier transform of $\mathbb{R}^3 \ni k \mapsto (|k|^2 - z)^{-1}$ for $z \in \mathbb{C} \setminus \mathbb{R}_+$.

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[1] Here, $M_{|\cdot|^2} f(p) := |p|^2 f(p)$ and $\mathcal{F}$ is the L^2-Fourier transform $(\mathcal{F}\psi)(p) = \lim_{R\to\infty} (2\pi)^{-d/2} \int_{|x|\le R} e^{-ip\cdot x} \psi(x) \mathrm{d}x$, where the limit is taken in $L^2(\mathbb{R}^d)$.