Mathematisches Institut — LMU München — WS 2016-17

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Mathematical Quantum Mechanics

Problem Sheet 12

Hand-in deadline: 01/26/2017 before noon in the designated MQM box (1st floor, next to the library).

Ex. 1: Prove the easy direction of the HVZ theorem, i.e. (in the notation used in the lecture) $[\Sigma, \infty) \subset \sigma_{\mathrm{ess}}(H_{2,Z})$.

Ex. 2: Let $V : \mathbb{R}^3 \to \mathbb{R}$ be a bounded potential, and assume that

$$\|V\|_R^2 := \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|V(x)||V(y)|}{|x-y|^2}\mathrm{d}x\mathrm{d}y < \infty.$$

We set $H_0 := -\Delta$ and $H := -\Delta + V$. Prove the following assertions.

1. For any $E > 0$ the operator $K_E := V^{1/2}(H_0+E)^{-1}|V|^{1/2} \in \mathfrak{S}^2(L^2(\mathbb{R}^3))$ and

   $$\lim_{E\to\infty} \|K_E\|_{\mathfrak{S}^2(L^2(\mathbb{R}^3))} = 0.$$

   Here $V^{1/2} := V/|V|^{1/2}$.

2. For E sufficiently large, we have $-E \in \rho(H) \cap \rho(H_0)$ and

   $$(H+E)^{-1} - (H_0+E)^{-1} = \sum_{n=0}^{\infty}(H_0+E)^{-1}|V|^{1/2}[-K_E]^n V^{1/2}(H_0+E)^{-1}$$

3. Prove that $\sigma_{\mathrm{ess}}(-\Delta + V) = \sigma_{\mathrm{ess}}(-\Delta)$.

4. For $E \geq 0$ denote by $N_E(V)$ the total number of eigenvalues (bound states) of $H = H_0 - V$ that are less or equal to $-E$. Denote by $B_E(V)$ the total number of eigenvalues of K_E that are less or equal to -1. Prove that for $E < 0$ we have

   $$N_E(V) = B_E(V).$$

5. Prove that

   $$N_0(V) \leq \frac{1}{16\pi^2}\|V\|_R^2.$$

   In particular, if $\|V\|_R < 4\pi$, then H has no bound states.