Lecture 7: Free profinite groups II

Last time: – pro-p-group: inverse limit of finite p-groups
– free pro-p-gp on set I

$$F(I) = \varprojlim_N F_I/N \leftarrow \text{free group on } I$$

limit over $N \subset F_I$ normal index p-power gp s.t. $I \setminus N$ is finite

Thm pro-p-group G is free $\iff$ projective

Corollary Let G be a pro-p-group & $\mathbb{F}_p^I \xrightarrow{\theta} G/G^* \leftarrow$ largest elem. abelian quotient.

Then $\exists\, F(I) \xrightarrow{\varphi} G$ inducing θ.
In particular: every generating set of G/G^* lifts to G.

Pf

$$\begin{tikzcd} & F(I) \arrow[r] \arrow[dl, dashed, "\varphi"'] & F(I)/F(I)^* \cong \mathbb{F}_p^I \arrow[dl, "\theta"] \\ G \arrow[r, two heads] & G/G^* & \end{tikzcd}$$

φ exists b/c $F(I)$ is projective. $\square$

Corollary Let $E \subset G$ s.t. $\forall\, U \subset G$ open nbd. of 1, $E \setminus U$ is finite. Then E is a gen. sys.
$\iff$ image of E generates G/G^*.

Pf $\Rightarrow$: Generating sys. preserved by surjections.

$\Leftarrow$: $F(E) \xrightarrow{\text{corr. to } E} G$ is surjective by "Nakayama"
& have seen that E generates $F(E)$. $\square$

Cohomological characterisation of free pro-p-groups

$H^n(G) := H^n(G, \mathbb{F}_p)$ — triv. action
(this is the only poss. action)

Recall: $H^0(G) = \mathbb{F}_p$

$$H^1(G) = \mathrm{Hom}_{ch}(G, \mathbb{F}_p) \cong \mathrm{Hom}_{ch}(G/G^*, \mathbb{F}_p)$$

$=$ Pontryagin dual group of G/G^*.

$\rightsquigarrow$ can reformulate "Nakayama" results in terms of $H^1(G)$.

E.g. Thm $G \to G'$ mor. of pro-p-groups is surjective $\Leftrightarrow$ $H^1(G') \to H^1(G)$ is injective.

Thm pro-p-group G is free $\Leftrightarrow H^2(G) = 0$.

Proof: rest of the lecture.

Cohomological dimension

Def^n The p-cohomological dimension of a profinite group G is

$$\mathrm{cd}_p G = \sup\{n \mid H^n(G,A) \neq 0 \text{ for some } A \in \mathcal{C}_G \text{ which is } p\text{-torsion}\}$$

Prop^n G a pro-p-group. Then $\mathrm{cd}_p G \leq n$ $\Leftrightarrow H^{n+1}(G) = 0$.

Pf $\Rightarrow$: clear.

$\Leftarrow$: let A be a p-torsion discrete G-module.

Claim: $H^{n+1}(G,A) = 0$.

Pf of claim: $A = \bigcup_{\substack{N \subset G \\ \text{normal} \\ \text{open}}} \underbrace{A^N}_{G\text{-module b/c } N \text{ normal}}$; $x \in A^H$, $g \in G$ $\Rightarrow gx \in A^{H^g}$ $(ghg^{-1} \cdot gx = g\underbrace{hx}_{x \in A^H} = gx)$

$\therefore H^{n+1}(G,A) = \operatorname{colim}_N H^{n+1}(G, A^N)$.

$\therefore$ WMA that action on A is via a finite quotient of G.

If $T \subset A$ any subset, let $\langle T \rangle$ be the subspace generated by $G \cdot T$. This is a sub-G-module.

If T is finite, so is $G \cdot T$, & hence $\langle T \rangle$ has finite dimension.

$\therefore$ A is the union of its finite-dim^{l} sub-G-mod.

$\therefore$ WMA $\dim_{\mathbb{F}_p} A < \infty$.

FACT: if finite p-group P acts on finite-dim^{l} $\mathbb{F}_p$-vector space A, then there is a non-zero fixed vector.
(Orbit-stabilizer for $P \curvearrowright A \setminus 0$.)

$\therefore$ $A^G \neq 0$, have exact seq. $0 \to A^G \to A \to A' \to 0$

$\dim A' < \dim A$

Hence A is a finite ext^n of trivial rep^{ns}

$\rightsquigarrow$ exact seq. $H^{n+1}(G, A^G) \to H^{n+1}(G, A) \to H^{n+1}(G, A')$

$\therefore$ WMA $A = A^G = \mathbb{F}_p^n$

$\therefore$ WMA $A = \mathbb{F}_p$ — true by assumption.

Claim: if $H^i(G, A) = 0$ $\forall$ discrete p-torsion G-mod. A
then $H^{i+1}(G, A) = 0$ $\forall$ — " —

(This will conclude.)

SES: $0 \to A \to \underbrace{M_G A}_{\text{also } p\text{-torsion}} \to A' \to 0$

A' $\therefore$ also p-torsion

LES $\quad H^i(G, A') \to H^{i+1}(G, A) \to H^{i+1}(G, M_G A)$

$\quad\quad\quad \| \quad\quad\quad\quad\quad \| \quad\quad\quad\quad\quad \|$

$\shortparallel$ $\qquad$ $\therefore$ $\shortparallel$ $\qquad$ $\shortparallel$

$0 \qquad\qquad 0 \qquad\qquad 0 \qquad \square$

Group ext^{ns}

<u>Def^n</u> G, K profinite groups.

An <u>extension</u> of G by K is an exact sequence

$$\begin{tikzcd} 1 \arrow[r] & K \arrow[r] \arrow[dr] & E \arrow[r] \arrow[d, "\cong"] & G \arrow[r] & 1 \\ & & E' \arrow[ur] & & \end{tikzcd}$$

An iso of ext^{ns} is a commutative diagram as above.

We know that $E \to G$ admits a cts. section s.

Define a cts. action of G on K:

$$g * k = k^{s(g)}.$$

If K is abelian, this action is indep. of s.

<u>Lemma</u> For G a prof. gp

A a finite ab. gp. with cts G-action

— i.e. a finite discrete G-mod,

have a bij^n between iso classes of ext^{ns} of G by A with specified action

& $H^2(G, A)$.

Pf Given ext^n E with section $s: G \to E$,
for $g_1, g_2 \in G$ $\quad s(g_1)\cdot s(g_2) = \underbrace{\varphi(g_1,g_2)}_{\in A}\, s(g_1 g_2)$

Check that φ is a cocycle & this yields the desired iso. $\square$

Cor If G is a free pro-p-group, then
$H^2(G) = 0$. (In part. $\mathrm{cd}_p\, G \leq 1$.)

Pf
$$\begin{tikzcd} 0 \arrow[r] & \mathbb{F}_p \arrow[r] & E \arrow[r] & G \arrow[r] \arrow[dl, bend left, "\exists s"] & 0 \\ & & & G \arrow[u, equal] & \end{tikzcd}$$

s can be chosen to be a group hom^m, s.t.

G is projective. $\therefore$ $s(g_1)\cdot s(g_2) = 1 \cdot s(g_1 g_2)$
$\therefore \varphi = 1$ $\square$

Th^m Let G be a prof. group. Suppose that every ext^n of G by a finite abelian p-gp splits. Then every ext^n of G by a pro-p-group splits.

Lemma $H \subset E$ closed normal subgp. of prof. gp E,

$H' \subset H$ open. Then $\exists H'' \subset H'$ open s.t. $H'' \subset E$ is normal.

Pf Let $N = \{x \in E \mid H'^x \subset H'\}$.

$\left.\begin{array}{l} H' \text{ is a cpct. top. gp} \\ H' \subset H \text{ is open} \end{array}\right\}$ theory of top. gps. $\Longrightarrow N \subset E$ is open.

$\therefore N$ has finite index

$\therefore H'$ has only finitely many conjugates in E

Their finite intersection is open, & normal. $\square$

Pf of thm Suppose $1 \to \underset{\text{pro-}p\text{-group}}{P} \to E \to G \to 1$ is an ext^n

TPT it splits.

Let $X = \{(P', s) \mid P' \subset P$ closed, $P \subset E$ normal
s splitting of $1 \to P/P' \to E/P' \to G \to 1\}$

Order this set $(P', s) \geq (Q', t)$ if $P' \subset Q'$ &

$$\begin{tikzcd} & E/Q' \\ G \arrow[ur, "t"] \arrow[dr, "s"'] & \circlearrowleft \\ & E/P' \arrow[uu] \end{tikzcd}$$

Zorn's lemma $\leadsto$ max elt (P', s). TTP $P' = \{1\}$.

If $P' \neq \{1\}$ then P' admits a proper open subgroup

& by the lemma $\exists P'' \subsetneq P$ open normal.

Now P'/P'' is a finite p-group with an action by E.

Can find E-equiv. abelian p-group quotient of P'/P''.

∴ Increasing P'', wma that P'/P'' is elementary ab.

Now $1 \to P'/P'' \to E_s/P'' \to G \to 1$

splits, where E_s is the inverse image of

$s(G)$ under E
$\downarrow$
E/P'

Hence we obtain lifts s' of G into $E_s/P'' \subset E/P''$.

∴ $(P'', s') > (P', s)$ ⚡. $\square$

<u>Proof of the main thm</u> G pro-p-group. G free $\Leftrightarrow H^2(G)=0$.

Have seen: G free $\Rightarrow H^2(G)=0$

$H^2(G)=0 \Rightarrow H^2(G,A)=0 \quad \forall$ p-torsion A

$\Rightarrow$ any ext^n of G by an element. ab. p-gp splits

$\Rightarrow$ any ext^n of G by a pro-p-gp splits.

Now $H \twoheadrightarrow H'$ surj^n of pro-p-gps.

$\pi \qquad \uparrow$

$$\begin{tikzcd} & & H \\ K \arrow[r, hook] & G \times_{H'} H \arrow[ur] \arrow[r, two heads] & G \arrow[l, dashed, bend left] \arrow[u, bend left=60, "\alpha"'] \end{tikzcd}$$

From

pro-p-group. Ext^2 splits. Check that α is the desired lift. □

Application

Corollary If G is a free pro-p-group & $H \subset G$ is closed then H is free.

Pf $H^2(H) = H^2(G, M_G^H \mathbb{F}_p) = 0$. □

Shapiro; this lecture (free $\Leftrightarrow$ coh. dim ≤ 1)

Outlook We have seen that $\dim_{\mathbb{F}_p} H^1(G)$ is the rank of a min. gen. sys. of the pro-p-gp G.

$$R \hookrightarrow F(I) \twoheadrightarrow G.$$

I minimal

R is a free group. How many generators does

R have? i.e. what is $H^2(R)$?
I.e. how many rel^{ns} do I need?
A: $\dim H^2(G)$!
(next time)