Lecture 5 ctd: Recall the

<u>Dimension shifting th^m</u> Let G, H be prof. grps,

$F: \mathcal{C}_H \to \mathcal{C}_G$ an exact functor pres. induced s,

$\lambda_m: H^m(G, F-) \Longrightarrow H^m(H, -)$.

Thm 1) $\exists!$ seq. of nat. transf.

$$\lambda_i : H^i(G, F-) \Longrightarrow H^i(H, -)$$

for $i \geq m$, commuting with δ.

2) If λ_m is iso so is λ_i for $i \geq m$.

Corestriction

Let $G \supset H$, H open.

Shall construct $\mathrm{cor}: H^n(H, \overset{\overset{\mathcal{C}_G}{\in}}{A}) \longrightarrow H^n(G, A)$

Apply dim^n shifting th^m to

- $F: \mathrm{Inf}_H^G: \mathcal{C}_G \to \mathcal{C}_H$
- checked last time: pres. induced mod
- clearly exact
- $\lambda_0: A^H \longrightarrow A^G$

$$a \longmapsto \sum_{[x] \in G/H} xa$$

verified last time: this is well-defined

Hence obtain $\mathrm{Cor} = \lambda_i$.

Th^m $\mathrm{Cor} \circ \mathrm{Res} : H^i(G,A) \to H^i(G,A)$
is mult^n by $[G:H]$.

Pf Apply the uniqueness part of dim^n shifting th^m to $\mathrm{id}: \mathcal{C}_G \to \mathcal{C}_G$

$$\lambda_0 : A^G \to A^H \to A^G$$

$$a \longmapsto \sum_{[x]\in G/H} x\, a = [G:H]\, a$$

$\lambda_n =$ mult^n by $[G:H]$ satisfies all assumptions

$\lambda_n' = \mathrm{Cor} \circ \mathrm{Res}$ — " —

$\therefore \lambda_n = \lambda_n'$ □

Cor If G is finite, then $|G| \cdot H^n(G,A) = 0$ for $n>0$

Pf $H = \{1\} \subset G$

$[G:H] = |G|$

$$\begin{tikzcd} H^n(G,A) \arrow[r] \arrow[rr, bend right, "\cdot |G|"'] & \overbrace{H^n(H,A)}^{=0} \arrow[r] & H^n(G,A) \end{tikzcd}$$

□

Cor If A is rat^l then $H^n(G,A) = 0$ $\forall$ G prof.

בס״ד

Lecture 6: Free pro-p groups I

From now on: - fix prime p
- only consider pro-p groups
[i.e. order a power of p (poss. p^∞)
i.e. inverse limits of finite p-groups]

Defn Let G be a pro-p group.
Call $E \subset G$ a system of generators if
subset

- any closed subgp. of G ctg. E equals G
- If $U \subset G$ open subgp. then $E \setminus U$ is finite.

Call E minimal if no proper subset generates.

Rmk - 2^{nd} condition is strange but justified in process
- will see: every pro-p group admits a system of generators

Construction Let I be a set,
F_I = free group on I.

Let $\mathcal{U} = \{N \subset F_I \mid N$ normal s.t. 1) 2)$\}$

1) $[G:N]$ a power of p

2) $I \setminus N$ is finite.

We call $F(I) = \lim_{N \in \mathcal{U}} F_I/N$ the **free pro-p group on I**.

Rmk
- $F_I \longrightarrow F(I)$ is dense
- One may prove that this map is injective (but do not need that)

Characterisation of free pro-p groups

Thm G be a pro-p group, I a set.

$$\mathrm{Hom}_{cts}(F(I), G) \xrightarrow{\cong} \left\{ (t_i \in G)_{i \in I} \;\middle|\; \begin{array}{l} \forall U \subset G \text{ open subgp.} \\ \text{then all but finitely many} \\ t_i \in U \end{array} \right\}.$$

$$\alpha \longmapsto (\alpha(i))_{i \in I}$$

Pf $F(I) \xrightarrow{\alpha} G$. $\alpha^{-1}(U) \subset F(I)$ is an open subgp & hence $I \setminus \alpha^{-1}(U)$ is finite

$\therefore$ $\{t_i\} \setminus U$ is also finite.

i.e.,

and holds.

Conversely, suppose given $(t_i)_{i\in I}$. $\exists! \,\tilde{\alpha} : F_I \longrightarrow G$.
Since $F_I \longrightarrow F(I)$ is dense, there is at most one cts ext^n.

Let $U \subset G$ be open & normal

$$\Phi(U) = \ker\left(F_I \xrightarrow{\tilde{\alpha}} G \longrightarrow G/U \right).$$

Then $\Phi(U) \in \mathcal{U}$.

Hence obtain

$$\begin{tikzcd} F(I) \arrow[r] \arrow[rrd, bend right] & \lim_{U} F_I/\Phi(U) \arrow[r] & \lim_{U} G/U \arrow[d, equal] \\ & & G \end{tikzcd}$$

this is the desired map. □

Cor I is a min. generating set for $F(I)$.

Pf Clearly generating set (b/c $F_I \longrightarrow F(I)$ has dense image).

Fix $i \in I$.

$$F(I) \longrightarrow \mathbb{Z}/p$$
$$i \longmapsto 1$$
$$i \neq j \longmapsto 0$$

$\exists$ by th^m

$\therefore I \setminus i$ does not generate $F(I)$ □

Let $G^* \subset G$ be the smallest normal closed subgroup containing all p^{th} powers & commutators.

Then G/G^* is the largest profinite quotient group which is elementary abelian.

Ex $F(I)/F(I)^* \cong (\mathbb{Z}/p)^I$

infinite product (in general) so has non-triv. topology

Thm $\varphi : G_1 \to G_2$ mor. of pro-p groups, Then φ is surjective $\Leftrightarrow$ $G_1/G_1^* \to G_2/G_2^*$ is surj.

Rmk This is reminiscent of Nakayama's lemma.

Pf $\Rightarrow$: clear.

$\Leftarrow$: Suppose $\varphi(G_1) \neq G_2$.

Then exists a finite quotient $G_2 \to P$ s.t. $G_1 \to G_2 \to P$ is not surjective.

By a theorem about finite p-groups $\exists P' \subset P$ normal of index p s.t. the image is contained in P'.

Let $G' \subset G_2$ be the preimage of P'.

Then $G_2^* \subset G'$ $\left(P/P' \cong G_2/G' \cong \mathbb{Z}/p \right)$.

Hence $\varphi(G_1/G_1^*) \subset G'/G_2^* \neq G_2/G_2^*$

i.e. $G_1/G_2^* \longrightarrow G_2/G_1^*$ is not surj. □

Th^m Let G be a pro-p group.
Then G is free ($G \cong F(I)$ for some set I)
$\iff$ G is projective as a pro-p group

i.e. given

$$\begin{tikzcd} H \arrow[r, two heads] & H' \\ & G \arrow[u] \arrow[ul, dashed, "\exists"] \end{tikzcd}$$

surj. of pro-p gps

Pf $\Rightarrow$: Technical lemma If $G \to H$ is a surjection of profinite groups, then $\exists$ a cts. section. (proof omitted, uses Zorn's lemma) not a homom^n

$$\begin{tikzcd} H \arrow[r, two heads] & H' \arrow[l, dashed, bend right, "s"'] \\ & F(I) \arrow[u, "\varphi"'] \arrow[ul, dashed, bend left, "\varphi'"] \end{tikzcd}$$

φ' is determined by $\varphi'(i) = s\,\varphi(i)$.

[NB: if $U \subset H$ open then $\varphi'(I) \setminus U$ is finite b/c s is cts.] This is the desired lift.

$\Leftarrow$: Digression: Pontryagin duality

$\{\text{torsion } \overset{\text{discrete}}{\text{abelian}}\}$ $\qquad$ $\{\text{profinite} \qquad\}$

$$\{\ \text{groups}\ \} \xleftrightarrow[\operatorname{Hom}_{cts}(-,\, \mathbb{Q}/\mathbb{Z})]{} \{\ \text{abelian groups}\ \}$$

are inverse equivalences.

In particular

$$\left\{\begin{matrix}\text{torsion ab. gps of}\\ \text{exponent } p\end{matrix}\right\} \xleftrightarrow[\operatorname{Hom}_{cts}(-,\, \mathbb{Z}/p)]{} \left\{\begin{matrix}\text{profinite abelian}\\ \text{gps of exponent } p\end{matrix}\right\}$$

$$\|$$

$$\{\mathbb{F}_p\text{-vector spaces}\}$$

<u>Ex</u> Any $\mathbb{F}_p$-vector spaces has a basis : $\cong \bigoplus_I \mathbb{F}_p$

$\therefore$ Any profinite ab gp. of exponent p

is $\cong \prod_I \mathbb{F}_p$,

Back to G projective $\Rightarrow$ G free.

By the above, $G/G^* \cong \mathbb{F}_p^I$

$$\begin{tikzcd}
F(I) \arrow[r, two heads] & F(I)/F(I)^* \arrow[r, phantom, "\cong"] & \mathbb{F}_p^I \arrow[d, phantom, "\wr"] \\
& G \arrow[ul, dashed, "\varphi"] \arrow[r] & G/G^*
\end{tikzcd}$$

$\exists$ b/c G projective.

Since $\varphi : G/G^* \cong F(I)/F(I)^*$ is a surjection,

also $\varphi : G \longrightarrow F(I)$ is.

Since free $\Rightarrow$ projective, this surjection splits:

$$\begin{tikzcd} G \arrow[r, two heads] & F(I) \\ \exists \psi & F(I) \arrow[u, "id"'] \arrow[ul, bend left] \end{tikzcd}$$

$\varphi\psi = id \Rightarrow \psi$ injective

ψ is on elementary ab. quot. $\Rightarrow \psi$ surjective.

$\therefore \psi$ iso of profinite gps. $\square$

Ex Let $F_S(I)$ be the p-completion of F_I.

Then $Hom_{cts}(F_S(I), \underset{\substack{\uparrow \\ \text{any pro-p group}}}{G})$ = arbitrary families of elts of G indexed by I

Thus $F_S(I)$ is projective in cat. of pro-p groups.

$\therefore$ $F_S(I) \cong F(I')$ for some set I'.

NB: We will see that I' is char. as follows:

$$\mathbb{F}_p^{I} \cong \mathbb{F}_p\{I'\}$$

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Next time: G is free $\iff$ $H^2(G, \mathbb{F}_p) = 0$.