Lecture 2: Profinite groups

Inverse limits

Let I be a small category & $\mathcal{C}$ any category. By an I-diagram in $\mathcal{C}$ we mean a functor $D: I \longrightarrow \mathcal{C}$.

Recall that a <u>limit of D</u> means an object $c \in \mathcal{C}$ together with maps $c \longrightarrow D(i)$ $\forall i \in I$

s.t. 1) $\forall\, \alpha: i \to j \in I$, then

$$\begin{tikzcd} & D(i) \arrow[d] \\ c \arrow[ur] \arrow[r] & D(j) \end{tikzcd}$$

commutes.

2) Given $c' \in \mathcal{C}$ together with maps $c' \to D(i)$ $\forall i$ satisfying (1), then $\exists!\ f: c' \longrightarrow c$

s.t. $\forall i \in I$

$$\begin{tikzcd} & D(i) \\ c' \arrow[ur] \arrow[r, "f"'] & c \arrow[u] \end{tikzcd}$$

commutes.

<u>Ex</u> $I = \{1 \quad 2\}$

no non-identity morphisms.

A diagram $D : I \longrightarrow \mathcal{C}$ just means objects $D(1), D(2) \in \mathcal{C}$.

A limit is an object $c \in \mathcal{C}$ together with

$$\begin{tikzcd} c \arrow[r] \arrow[dr] & D(1) \\ & D(2) \end{tikzcd}$$

Cond (1) is empty.

Cond^n 2:

$$\begin{tikzcd} c' \arrow[rr] \arrow[dd] \arrow[dr, dashed, "\exists!"] & & D(1) \\ & c \arrow[ur] \arrow[dl] & \\ D(2) & & \end{tikzcd}$$

If $\mathcal{C} = \mathrm{Set}$, take $c = D(1) \times D(2)$ — cartesian product.

More generally: if limit exists call it product of $D(1)$ & $D(2)$.

<u>Ex</u> Classical projective limit

$$I = \{1 \leftarrow 2 \leftarrow 3 \leftarrow \dots\}$$

I-diag $\longleftrightarrow$ objects $c_1 \leftarrow c_2 \leftarrow c_3 \leftarrow \dots$

Inverse limit denoted $\lim\limits_{n} c_n = \lim\limits_{\overleftarrow{n}} c_n$.

<u>Ex</u> $\mathcal{C} = Ab$ $\quad c_n = \mathbb{Z}/p^n$

$$\mathbb{Z}/p \leftarrow \mathbb{Z}/p^2 \leftarrow \mathbb{Z}/p^3 \leftarrow \dots$$

$\lim_n$ " $\mathbb{Z}/p^n$ = " $\mathbb{Z}_p^\wedge$ = "p-adic integers"

Standard result: Thm Limits, if they exist, are unique up to unique iso.

I.e.: If c, c' both limits of an I-diagram D, then $c \longrightarrow c'$ is an isomorphism.

Ex In Set, $A \times B$ & $A \times B \times \{\pi\}$ are both products of A, B.

Denote "the" limit by $\lim_I D \in \mathcal{C}$.

Construction Let $\mathcal{C}$ = top. spaces.

Suppose $D : I \longrightarrow Top$ diagram.

Consider $L \subset \prod_{i \in I} D(i)$ consisting of those points $(x_i)_{i \in I}$ s.t. $\forall\, \alpha : i \to j \in I$

$$\alpha(x_i) = x_j .$$

One may prove that L is a limit of D.

The same thing works in Ab, Set, ...

Ex $\mathbb{Z}_p^\wedge = \lim \mathbb{Z}/p^n \hookrightarrow \prod \mathbb{Z}/p^n$

i.e. $\hat{\mathbb{Z}}_p \ni x = (x_1, x_2, x_3, \dots)$

$$x_i \in \mathbb{Z}/p^i \quad \& \quad \overline{x_i} \in \mathbb{Z}/p^{i-1}.$$
$$\overline{x_i} = x_{i-1}$$

Profinite groups

Def^n A topological group is called <u>profinite</u> if can be written as an inverse limit of finite (discrete) groups.

Ex Let G be any top. group.

Let I be the set of closed (normal) subgroups of G of finite index. Make I into category via inclusions.

Obtain diagram
$$D : I \longrightarrow \text{Top Grp}$$
$$H \longmapsto G/H \quad \leftarrow \text{finite + discrete}$$

$$\rightsquigarrow \hat{G} := \lim_I D \text{ is a profinite group.}$$

NB: $G \to G/H \quad \forall H \subset F$

$$\rightsquigarrow G \to \hat{G}.$$

G/K

Call $\hat{G}$ the profinite completion of G.

Ex Fix prime p. Take instead I_p = closed normal subgps of index a power of p.

$\rightsquigarrow \hat{G_p} = \lim_I D$ the p-profinite completion.

Ex Take $G = \mathbb{Z}$. I = finite index subgps

$= \{ d\mathbb{Z} \mid d > 0 \}$

$= (\mathbb{N}, \text{divisibility})$

$I_p = \{ p^n \mathbb{Z} \mid n \geq 0 \}$

$= (\mathbb{N}_0, \leq)$

$D_p : I_p \to \text{Top Gps}$ is

$$\mathbb{Z}/p \leftarrow \mathbb{Z}/p^2 \leftarrow \mathbb{Z}/p^3 \leftarrow \cdots$$

limit = p-profinite compl^n of $\mathbb{Z}$ = $\hat{\mathbb{Z}_p}$

Also $\hat{\mathbb{Z}} = \lim_{d,\ \text{divisibility}} \mathbb{Z}/d = \prod_p \hat{\mathbb{Z}_p}$.

$\underline{\text{Th}^{m}\ 1}$ Let G be a top. grp.

Then G is profinite $\iff$ G is cpct & tot. disconnected.

Pf $\Rightarrow$: G is closed inside $\prod_i G_i$.

$G = \varprojlim_I G_i$ (finite discrete); $\prod_i G_i$ cpct. by Tychonov's th^m

$\therefore$ G is cpct.

Let $e \neq x \in G$. Then $\exists\, i$ st. $G \xrightarrow{\alpha_i} G_i$.

$\alpha_i(x) \neq \alpha_i(e) = e$

Then $\alpha_i^{-1}(\{e\}) = U \subset G$, $e \in U$, is an open subgroup.

$\alpha_i^{-1}(G_i \setminus \{e\}) = V \subset G$, $x \in V$, is also open.

$\Leftarrow$: One shows that open subgroups form a basis of open nbds of e. [We believe the topologists.]

Let $U \subset G$ be an open subgroup.

Then U is closed. $\therefore$ G/U is discrete.

$G \setminus \bigcup gU$
$[g] \in G/U$
$\overset{\shortparallel}{[e]}$

Since $G \to G/U$ & G is cpct,
G/U is finite.

Hence there are only finitely many conjugates
$g^{-1}Ug$.

$\therefore \bigcap_g g^{-1}Ug$ is open & normal.

Now $G \longrightarrow \underbrace{\varprojlim_{\substack{V \subset G \\ \text{open, normal}}} G/V}_{\text{profinite gp!}}$ is cts, injective & dense.

Since G cpct & RHS Hausdorff, map is surjective.

$\leadsto$ homeom^n & hence iso of top. gps. $\square$

Rmk
- open subgp. of top. gp. is closed
- — " — — " — prof. gp is finite index

} seen is prof

- closed subgps. of prof. gps are prof.
  (still cpct. tot. disconnected)
- quotient of prof. gp. by closed, normal subgp. is prof.

Orders

<u>Def</u>

Def^n A <u>supernatural number</u> is a formal product

$$\prod_p p^{n_p}, \quad \text{where } n_p \in \mathbb{N} \cup \{0, \infty\}.$$

Product & lcm of arbitrary collections of supernatural numbers make sense.

Def^n G profinite, $H \subset G$ closed,

$$(G:H) = \operatorname{lcm} \left\{ \left(\frac{G}{U} : \frac{H}{H \cap U} \right) \;\middle|\; U \subset G \text{ open normal} \right\}$$

"index"

$|G| = (G : \{e\})$ "order of G".

Def^n G is called a pro-p-group if the order $|G|$ is a power of p.

NB: I.e. every finite quotient of G is a p-group.

Sylow theorems

Call $P \subset G$ (closed subgp) a p-Sylow subgroup if

$(G:P)$ is coprime to p.

Thm 2 - p-Sylow subgroups exist
- any two are conjugate
- any pro-p subgroup is contained in a p-Sylow

To prove this, need some preparation.

Call a category I **cofiltered** if

- given $i, i' \in I$ can find $k \rightrightarrows \begin{smallmatrix} i' \\ i \end{smallmatrix}$ (i.e. $k \to i'$, $k \to i$)
- $I \neq \emptyset$
- Given $i \overset{\beta}{\underset{\alpha}{\rightrightarrows}} i'$ $\exists$ $k \to i \rightrightarrows i'$ s.t. $k \rightrightarrows i'$ are the same maps.

Ex $\{1 \leftarrow 2 \leftarrow 3 \leftarrow \cdots\}$ is cofiltered.

Ex G a profinite gp. Then the set of open normal subgroups is also cofiltered.
(B/c if $U_1, U_2 \subset G$ open & normal, the $U_1 \cap U_2 \subset G$ is open & normal.)

$\therefore G$ is a cofiltered limit of finite groups.

Lemma If I is a cofiltered category & $D: I \longrightarrow \mathrm{Top}$ is a diagram of finite, discrete, nonempty sets, then $\lim_{I} D \neq \emptyset$.

Pf Use Tychonov's thm (limit is cpct) & finite intersection property. "$\square$"

Proof of thm 2 If K is a finite gp, let $P(K)$ be the set of p-Sylow subgroups of K.

Note: If $K \overset{\alpha}{\twoheadrightarrow} H$ & $P \subset K$ is a p-Sylow then $\alpha(P) \subset K$ is also a p-Sylow.
$\left((H : \alpha(P)) = (K : \alpha^{-1}(\alpha(P))) \mid (K : P) \text{ is coprime to } p. \right)$

$\therefore$ set $P(K) \longrightarrow P(H)$ induced map.

Consider $D : I \longrightarrow \mathrm{Top}$

$$\begin{array}{ccccc} I & & & & \\ \| & & & & \\ \text{open normal subgps of } G & H & \longmapsto & P(G/H) & G/H \\ & \cap & & \downarrow & \downarrow \\ & H' & & P(G/H') & \end{array}$$

G/H'

By the Lemma, $\lim\limits_{I} D \neq \emptyset$.

I.e.: for every $H \subset G$ open normal obtain a choice $P_H \subset G/H$ of p-Sylow st. if $H \subset H'$, then

$$\begin{aligned} G/H &\longrightarrow G/H' \\ P_H &\longmapsto P_{H'} \end{aligned}$$

Form new diagram

$$\begin{aligned} D' : I &\longrightarrow \mathrm{TopGrp} \\ H &\longmapsto P_H \subset G/H \end{aligned}$$

Take limit: $P := \lim\limits_{I} D' \hookrightarrow \lim\limits_{I} G/H = G$

Easily check: $P \hookrightarrow G$ is a p-Sylow subgp.

Other claims proved similarly. $\square$