

Lecture 13: Global fields

Again to simplify: only treat global fields of char. 0, i.e. number fields.

Maximal p-extⁿ

Let k be a finite global field of char. 0 (i.e. $[k:\mathbb{Q}] < \infty$) & p a fixed prime.

$$G = \text{Gal}(k/k)$$

$$G_v = \text{Gal}(\widehat{k}_v/k_v)$$

Since $(\widehat{k})_v/k_v$ is a p -extⁿ, $\exists (\widehat{k})_v \xrightarrow{\varphi_v} \widehat{k}_v$.

Hence obtain $\varphi_v^*: H^*(G) \rightarrow H^*(G_v)$.

Lemma φ_v^* is indep. of the choice of embedding φ_v .

$$\begin{array}{ccc} \text{PF } (k)_v & \xrightarrow{\varphi_v} & \widehat{k}_v \\ & \searrow G & \downarrow \exists \text{ autom } \gamma \\ & & \widehat{k}_v \end{array}$$

$$\begin{array}{ccc} \rightsquigarrow G & \longleftarrow & G_v \\ & \nearrow & \uparrow \tilde{\gamma} \\ & & G_v \end{array} \quad F(\tau) = \tau^\gamma$$

Since inner autom^s act identically on coh (e.g. by dimension shifting), can conclude. \square

Lemma $\{\varphi_v\}, \{\mathbb{I}_v\}$ is admissible in the sense of lecture 8 (rel⁴⁵).

↑ inertia subgp of G_v

PF I.e. (1) G_v/\mathbb{I}_v free & (2) given $U \subset G$ open,

$\varphi_v(\mathbb{I}_v) \subset U$ for almost all v .

$$(1) \quad G_v/\mathbb{I}_v \cong (\text{Gal. gp. of KB-field ext}^n) = \begin{cases} 0 & : v \text{ infinite} \\ \mathbb{Z}_p & : v \text{ finite} \end{cases}$$

↑ b/c residue field is

(2) Equiv. \hat{k}^U/k is unramified finite at v for almost all v .

True Stc extⁿ is finite. \square

In particular Stein $\varphi^* : H^2(G) \xrightarrow{(\varphi_v^*)} \bigoplus_v H^2(G_v)$

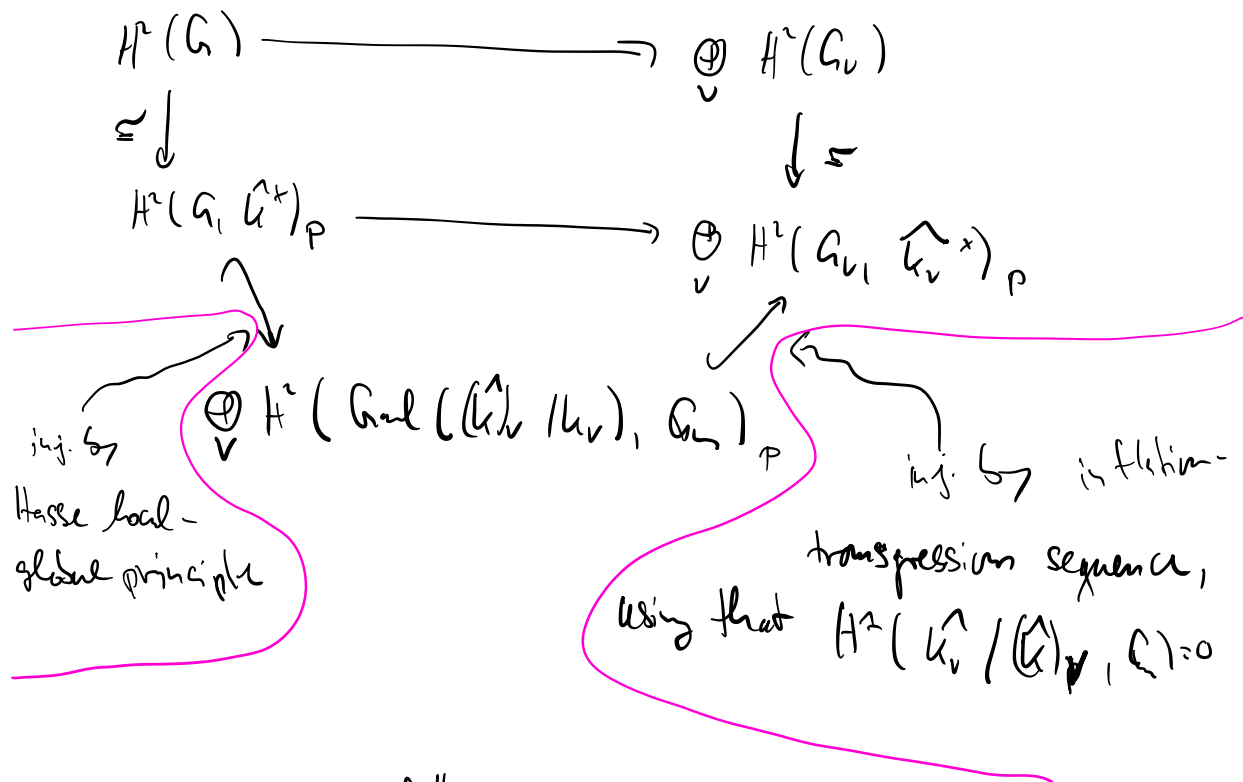
↑
Sum over all places

NB: $H^2(G_v) = 0$ if v is infinite unless $p=2$ & $k_v = \mathbb{R}$.

Thm φ^* is injective.

Pf First suppose that $\delta(k) = 1$ (i.e. $\wp_p \in k$).

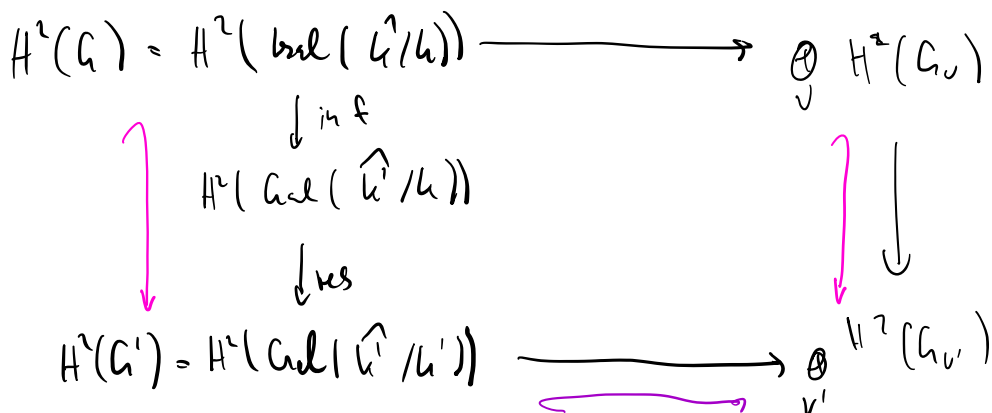
Then $\mathbb{Q}_p \hookrightarrow k^\times$ & obtain



→ Result follows.

Now suppose $\delta(k) = 0$. Put $k' = k(\wp_p)$

$$G' = \text{Gal}(\widehat{k}'/k').$$



last time!

prev. can

→ result follows. \square

Conclusion The ^{local} V presentations $1 \rightarrow R_v \rightarrow F_v \rightarrow G_v \rightarrow 1$

lift to

$$1 \rightarrow R \rightarrow F \rightarrow G \rightarrow 1$$

$\downarrow \quad \downarrow \quad \downarrow$
 $\downarrow \ell_v$

& R is generated as a normal subgroup by the images of the R_v .

(Recall that R_v has rank $\delta(K_v)$.)

Restricted ramification

Unfortunately #qs & #rels are both infinite. $\textcircled{!}$

Let S be any set of places of k .

Defⁿ k_S = composite of all finite p -ext^{ns} of k which are unramified outside S .

As before we see that this is a Galois extⁿ of k .

Put $G_S = \text{Gal}(k_S/k)$.

ps: In a p -extⁿ, a place v is unramified as soon

as $|\sigma_v/m_v| \equiv 1 \pmod{p}$

- v complex

- v real, $p \neq 2$.

Assume that such v are not in S .

Have the map $\varphi_S^*: H^2(G_S) \rightarrow \bigoplus_{v \in S} H^2(Q_v)$.

Defⁿ $\mathbb{I}_S = \ker(\varphi_S^*)$
"I_S"

Defⁿ $V_S = \left\{ \alpha \in k^+ \mid \begin{array}{l} \text{free ideal } \mathfrak{f} \\ \alpha \text{ is divisible} \\ \text{by } \mathfrak{p} \end{array} \text{ \& } \left. \begin{array}{l} \alpha \in k_v^{+p} \\ \text{whenever } v \in S \end{array} \right\}$

NB: $k^{+p} \subset V_S$

Defⁿ $\mathcal{I}_S = \text{Hom}_{\text{cts}}(V_S/k^{+p}, \mathbb{Z}/p)$

Lemma \mathcal{I}_S is finite.

Pf $S_1 \supset S_2 \Rightarrow V_{S_1} \subset V_{S_2}$.

$\therefore \text{wMA } S = \emptyset$.

Consider $V_{\emptyset} \longrightarrow \text{cl}(k)$

$$\alpha \longmapsto I^{\mathbb{P}^1}(\alpha).$$

Well-defined by unique factorization of fractional ideals.

Check this induces

$$1 \rightarrow \frac{\mathcal{O}^X}{\mathcal{O}^{X,P}} \rightarrow \frac{V_{\mathcal{O}}}{(k^X)^P} \rightarrow \text{Cl}(k)_P \rightarrow 0.$$

$$\therefore \dim \mathcal{S}_{\mathcal{O}} = \dim \frac{\mathcal{O}^X}{\mathcal{O}^{X,P}} + \dim \text{Cl}(k)_P$$

\uparrow \nearrow
 both finite by classical AVT. \square

Th There is an injection $\mathbb{I}_S \hookrightarrow \mathcal{S}_S$.

Pf Fairly elaborate, use all the previous techniques.
pp. 110-112 of Koch's book. \square

In other words: can present \mathcal{G}_S using local rel^s,
plus at most $\dim \mathcal{S}_S$ extra ones.

How many generators?

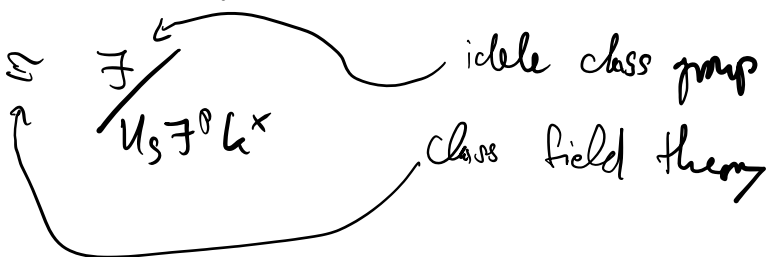
Th There is an exact sequence

$$0 \rightarrow H^1(\mathcal{G}_{\mathcal{O}}) \rightarrow H^1(\mathcal{G}_S) \rightarrow \bigoplus H^1(\mathbb{I}_v) \xrightarrow{h_v} \mathcal{S}_{\mathcal{O}} \rightarrow \mathcal{S}_S \rightarrow 0.$$

Pf $H^2(G) = \text{Hom}_{\text{cb}}(G/G^*, \mathbb{Z}/p)$

$G^* = G^{\text{f}}(G, G)$

Note that G/G^* = Gal(max. elementary ab. ext. with restricted ram.)



Explicit messiness that yields the following exact sequence:

$$0 \rightarrow V_S/k^{*p} \rightarrow V_{\mathfrak{O}}/k^{*p} \rightarrow U_{\mathfrak{O}}/U_S \mathfrak{O}^p \rightarrow \mathbb{F}/U_S \mathbb{F} \mathfrak{O}^p \rightarrow \mathbb{F}/U_{\mathfrak{O}} \mathbb{F} \mathfrak{O}^p \rightarrow 0$$

Derived sequence obtained by applying $\text{Hom}_{\text{cb}}(-, \mathbb{Z}/p)$. \square

Have seen: $\dim \bar{E}_{\mathfrak{O}} = \dim \text{Cl}(k)_p + \dim \frac{\mathcal{O}^{\times}}{\mathcal{O}^{\times p}}$

Easy: $\mathbb{F}/U_{\mathfrak{O}} \mathbb{F} \mathfrak{O}^p \cong \frac{\text{Cl}(k)}{p \text{Cl}(k)}$

Since $\text{Cl}(k)$ finite: $\dim \frac{\text{Cl}(k)}{p \text{Cl}(k)} = \dim \text{Cl}(k)_p$

Dirichlet's unit thm: $\dim \frac{\mathcal{O}^{\times}}{\mathcal{O}^{\times p}} = \sigma(k) + r - 1$

\uparrow
of primes

Prev. work: $H^2(I_U)^{G_U} = \dots$ places

From this we can read off the following:

$$\underline{\text{Th}} \quad \dim H^2(G_S) = \sum_{\substack{v \in S \\ \dim(\mathcal{O}_{X_v}) = p}} [k_v : \mathbb{Q}_p] - \delta(k) - r + 1 \\ + \dim \mathcal{I}_S + \underbrace{\sum_{v \in S} \delta(k_v)}$$

only term which is poss.
infinite

Ex $k = \mathbb{Q}$, $p \neq 2$. Since \mathbb{Z} has unique factⁿ,
 $V_S = k^{+p}$ & hence $\mathcal{I}_S = \mathcal{O}$. (Any S .)

\therefore only need local relⁿ

$$\dim H^2 = \sum_{v \in S} \delta(k_v)$$

NB: $= \# \text{rel}^{4s}$!!

Reh/ex Working more carefully, in some cases can
determine explicit relⁿ, Ex due to Koch:

$$k = \mathbb{Q}, p \neq 2, S = \{p_1, p_2\}, \begin{matrix} p_i \equiv 1 \pmod{p}, \\ \neq 1 \pmod{p^2} \end{matrix}$$

$$p_1 \not\equiv x^2 \pmod{p_2}.$$

In this case G_S is the unique non-abelian group of order p^3 & exponent p^2 .