Lecture 13: Global fields

Again to simplify: only treat global fields of char. 0, i.e. number fields.

Maximal p-ext^n

Let k be a finite global field of char. 0 (i.e. $[k:\mathbb{Q}]<\infty$) & p a fixed prime.

$G = \mathrm{Gal}(\hat{k}/k)$

$G_v = \mathrm{Gal}(\widehat{k_v}/k_v)$

Since $(\hat{k})_v/k_v$ is a p-ext^n, $\exists\ (\hat{k})_v \xhookrightarrow{\varphi_v} \widehat{k_v}$.

Hence obtain $\varphi_v^*: H^*(G) \to H^*(G_v)$.

Lemma φ_v^* is indep. of the choice of embedding φ_v

__Pf__

$$\begin{tikzcd} (\hat{k})_v \arrow[r, hook, "\varphi_1"] \arrow[dr, hook, bend right, "\varphi_2"'] & \widehat{k_v} \arrow[d, dashed, "\exists \text{ autom}^n\ \gamma"] \\ & \widehat{k_v} \end{tikzcd}$$

$$\rightsquigarrow \begin{tikzcd} G & G_v \arrow[l] \\ & G_v \arrow[u, "\tilde{\gamma}"'] \arrow[ul] \end{tikzcd} \qquad \tilde{\gamma}(\tau) = \tau^\gamma.$$

Since inner autom^s act identically on coh (e.g. by dimension shifting), can conclude. $\square$

Lemma $\{\varphi_v\}$, $\{I_v\}$ is admissible in the

inertia subgp of G_v

sense of lecture 8 (rel^{ns}).

Pf I.e. (1) G_v/I_v free & (2) given $U \subset G$ open,

$\varphi_v(I_v) \subset U$ for almost all v.

(1) $G_v/I_v \cong$ (Gal. gp. of res. field ext^n) $= \begin{cases} 0 : v \text{ infinite} \\ \mathbb{Z}_p : v \text{ finite} \end{cases}$

b/c residue field is finite.

(2) Equiv. $\hat{k}^U/k$ is unramified at v for almost all v.

True b/c ext^n is finite. $\square$

In particular obtain $\varphi^*: H^2(G) \xrightarrow{(\varphi_v^*)} \bigoplus_v H^2(G_v)$

sum over all places

NB: $H^2(G_v) = 0$ if v is infinite unless $p=2$ & $k_v = \mathbb{R}$.

Th^m φ^* is injective.

Pf First suppose that $\delta(k)=1$ (i.e. $\zeta_p \in k$).

Then $\mathbb{Z}/p \hookrightarrow k^\times$ & obtain

$$
\begin{tikzcd}
H^2(G) \arrow[rr] \arrow[d, "\cong"'] & & \bigoplus_v H^2(G_v) \arrow[d, "\cong"] \\
H^2(G, \hat{k}^\times)_p \arrow[rr] \arrow[dr] & & \bigoplus_v H^2(G_v, \hat{k_v}^\times)_p \\
& \bigoplus_v H^2(\mathrm{Gal}((\hat{k})_v / k_v), \mathbb{G}_m)_p \arrow[ur] &
\end{tikzcd}
$$

inj. by Hasse local-global principle

inj. by inflation-transgression sequence, using that $H^2(\hat{k_v}/(\hat{k})_v, \mathbb{G}_m)=0$

$\leadsto$ Result follows.

Now suppose $\delta(k)=0$. Put $k' = k(\zeta_p)$

$G' = \mathrm{Gal}(\hat{k'}/k')$.

$$
\begin{tikzcd}
H^2(G) = H^2(\mathrm{Gal}(\hat{k}/k)) \arrow[rr] \arrow[d, "\mathrm{inf}"] & & \bigoplus_v H^2(G_v) \arrow[dd] \\
H^2(\mathrm{Gal}(\hat{k'}/k)) \arrow[d, "\mathrm{res}"] & & \\
H^2(G') = H^2(\mathrm{Gal}(\hat{k'}/k')) \arrow[rr] & & \bigoplus_{v'} H^2(G_{v'})
\end{tikzcd}
$$

prev. calc

last time!

$\rightsquigarrow$ result follows. $\square$

<u>Conclusion</u> The local presentations $1 \to R_v \to F_v \to G_v \to 1$ lift to

$$\begin{tikzcd} 1 \arrow[r] & R_v \arrow[r] \arrow[d, dashed] & F_v \arrow[r] \arrow[d, dashed] & G_v \arrow[r] \arrow[d, "\varphi_v"] & 1 \\ 1 \arrow[r] & R \arrow[r] & F \arrow[r] & G \arrow[r] & 1 \end{tikzcd}$$

& R is generated as a normal subgp. by the image of the R_v.

(Recall that R_v has rank $\delta(k_v)$.)

Restricted ramification

Unfortunately #gens & #relns are both infinite. ☹

Let S be any set of places of k.

<u>Def^n</u> k_S = composite of all finite p-ext^{ns} of k which are unramified outside S.

As before we see that this is a Galois ext^n of k.

Put $G_S = \mathrm{Gal}(k_S/k)$.

NB: In a p-ext^n, a place v is unramified as soon as $\quad - |\mathcal{O}_v/m_v| \equiv 1 \ (p)$

$\quad - \ v$ complex

$\quad - \ v$ real, $p \neq 2$.

Assume that such v are not in S.

Have the map $\varphi_S^* : H^2(G_S) \longrightarrow \bigoplus_{v \in S} H^2(G_v)$.

Def^n $\underset{\text{"Sha"}}{Ш}_S = \ker(\varphi_S^*)$

Def^n $V_S = \left\{ \alpha \in k^\times \,\middle|\, \begin{array}{l}\text{frac. ideal of} \\ \alpha \text{ is divisible} \\ \text{by } p\end{array} \ \& \ \begin{array}{l}\alpha \in k_v^{\times p} \\ \text{whenever } v \in S\end{array} \right\}$

NB: $k^{\times p} \subset V_S$

Def^n $Б_S = \mathrm{Hom}_{\mathrm{cts}}\left(V_S / k^{\times p}, \mathbb{Z}/p \right)$

Lemma $Б_S$ is finite.

Pf $S_1 \supset S_2 \Rightarrow V_{S_1} \subset V_{S_2}$.

$\therefore$ WMA $S = \emptyset$.

Consider $V_\emptyset \longrightarrow Cl(k)$

$$\alpha \longmapsto I^{?}(\alpha).$$

Well-defined by unique factorization of fractional ideals.

Check this induces

$$1 \to \mathcal{O}^\times / \mathcal{O}^{\times p} \longrightarrow \frac{V_\phi}{(k^\times)^p} \longrightarrow Cl(k)_p \to 0.$$

$$\therefore \dim \text{Б}_\phi = \dim \mathcal{O}^\times/\mathcal{O}^{\times p} + \dim Cl(k)_p$$

both finite by classical ANT. $\square$

<u>Thm</u> There is an injection $Ш_S \hookrightarrow \text{Б}_S$.

"<u>Pf</u>" Fairly elaborate, use all the previous techniques. pp. 110–112 of Koch's book. $\square$

In other words: can present G_S using local rel^{ns}, plus at most $\dim \text{Б}_S$ extra ones.

How many generators?

<u>Thm</u> There is an exact sequence

$$0 \to H^1(G_\phi) \to H^1(G_S) \to \bigoplus H^1(I_v)^{G_v} \to \text{Б}_\phi \to \text{Б}_S \to 0.$$

Pf $H^1(G) = \mathrm{Hom}_{ch}(G/G^*, \mathbb{Z}/p)$

$G^* = G^p[G,G]$

Note that $G_S/G_S^* = \mathrm{Gal}($max. elementary ab. ext^n with restricted ram.$)$

$\cong \mathcal{J}/U_S \mathcal{J}^p k^\times$ idele class group

Class field theory

Explicit messing about yields the following exact sequence:

$$0 \to V_S/k^{\times p} \to V_\emptyset/k^{\times p} \to U_\emptyset/U_S U_\emptyset^p \to \mathcal{J}/U_S \mathcal{J}^p k^\times \to \mathcal{J}/U_\emptyset \mathcal{J}^p k^\times \to 0.$$

Desired sequence obtained by applying $\mathrm{Hom}_{ch}(-, \mathbb{Z}/p)$. $\square$

Have seen: - $\dim B_\emptyset = \dim Cl(k)_p + \dim \mathcal{O}^\times/\mathcal{O}^{\times p}$

Easy: - $\mathcal{J}/U \mathcal{J}^p k^\times \cong Cl(k)/pCl(k)$

Since $Cl(k)$ finite: $\dim \dfrac{Cl(k)}{p\,Cl(k)} = \dim Cl(k)_p$

Dirichlet's unit thm: $\dim \mathcal{O}^\times/\mathcal{O}^{\times p} = \delta(k) + r - 1$

\# of infinite

Prev. word: $H^2(I_v)^{G_v} = \ldots$ places

From this we can read off the following:

<u>Th^m</u> $\dim H^2(G_S) = \sum_{\substack{v \in S \\ \dim(\mathcal{O}_v/m_v) = p}} [k_v : \mathbb{Q}_p] - \delta(k) - r + 1$

$+ \dim Б_S + \underbrace{\sum_{v \in S} \delta(k_v)}$

Only term which is poss. infinite

<u>Ex</u> $k = \mathbb{Q}$, $p \neq 2$. Since $\mathbb{Z}$ has unique fact^n, $V_S = k^{\times p}$ & hence $Б_S = 0$. (Any S.)

$\therefore$ only need local rel^{ns}

$\dim H^2 = \sum_{v \in S} \delta(k_v)$

NB: $= \#$ rel^{ns} !?!

<u>Rmk/ex</u> Working more carefully, in some cases can determine explicit rel^{ns}. Ex due to Koch:

$k = \mathbb{Q}$, $p \neq 2$, $S = \{p_1, p_2\}$, $p_i \equiv 1 \ (p)$,

$\not\equiv 1 \ (p^2)$

$p_1 \not\equiv x^2 \ (p_2)$.

In this case G_S is the unique non-ab. gp of order p^3 & exponent p^2.