Algebraic Number Theory

Exercises 9

Dr. Tom Bachmann Winter Semester 2021/2

Exercise 1. (1) Let K be a field and $A \subset B \subset K$ subrings. Show that if both A and B are dvrs with K as field of fractions, then $A = B$.

(2) Let $\mathbb{Q} \subset K$ be a number field and $A \subset K$ a dvr with field of fractions K, maximal ideal M and residue field κ. Show that $\mathcal{O}_K \subset A$. Prove that $P := M \cap \mathcal{O}_K$ is a non-zero prime ideal. [*Hint*: If $P = 0$ then $\mathcal{O}_K \to \kappa$ would be injective.] Conclude that $A = (\mathcal{O}_K)_P$.

Exercise 2. Let $\mathbb{Q} \subset K$ be a number field and $A \subset K$ a subring which is a finitely generated abelian group and whose field of fractions is K.

(1) Show that $A \subset \mathcal{O}_K$ and A is a Dedekind domain if and only if $A = \mathcal{O}_K$.
(2) Show that $A^\times$ is finitely generated.
(3) Show that $\mathcal{O}_K/A$ is finite and that if $f = |\mathcal{O}_K/A|$ then there are only finitely many maximal ideals in A containing f. Moreover show that $Q \mapsto Q \cap A$ induces a bijection between the set of maximal ideals of $\mathcal{O}_K$ not containing f and the set of maximal ideals of maximal ideals of A not containing f.

Exercise 3. (1) Show that $\mathcal{O}_K$ is a PID, where $K = \mathbb{Q}(\sqrt{-11})$.

(2) Find all integral solutions to the equation $y^3 = x^2 + 11$. [*Hint*: if (x, y) is a solution, write $x + \sqrt{-11}$ as a cube in $\mathcal{O}_K$ and use an integral basis of $\mathcal{O}_K$.]