Algebraic Number Theory

Exercises 8

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Throughout let d be a squarefree integer and $K = \mathbb{Q}(\sqrt{d})$.

Exercise 1.   (1) If $d \in \{-7, -3, -2, -1, 2, 3, 5\}$, show that
$$\left(\frac{4}{\pi}\right)^{r_2} \frac{2}{2^2} \sqrt{|d_K|} < 2.$$

(2) Conclude that $\mathcal{O}_K$ is a PID in these cases.

Exercise 2. Let $d = -5$. Show that $C(\mathcal{O}_K) = \mathbb{Z}/2$.
  [*Hint:* use ex. 2 of sheet 3 and the structure of $\mathcal{O}_K/2$.]

Exercise 3. Let $d < 0$. Show that $\mathcal{O}_K^\times$ is finite of order 2, except if $d = -1$ in which case the order is 4, or $d = -3$ in which case the order is 6.

Exercise 4. Let A be an integral domain which is of finite type as a $\mathbb{Z}$-module. Show that $A^\times$ is a finitely generated abelian group.