Algebraic Number Theory

Exercises 7

Dr. Tom Bachmann Winter Semester 2021/2

Exercise 1. Let A be a Dedekind domain and $S \subset A$ be a multiplicatively closed subset.

(1) Let $I \in F(A)$. Denote its prime decomposition by $I = P_1^{e_1} \dots P_n^{e_n}$. Determine the prime decomposition of $S^{-1}I \in F(S^{-1}A)$.

(2) Assuming that $\mathrm{Spec}\,(A) \setminus \mathrm{Spec}\,(S^{-1}A)$ has finite cardinality s, construct an exact sequence

$$0 \to A^\times \to (S^{-1}A)^\times \to \mathbb{Z}^s \to C(A) \to C(S^{-1}A) \to 0.$$

(3) Suppose that $C(A)$ is finitely generated. Show that there exists $f \in A \setminus 0$ such that $A[f^{-1}]$ is a PID.

Exercise 2. Let K be a number field.

(1) Let $I \subset K$ be a non-zero fractional ideal. Show that $I \simeq \mathbb{Z}^n$ as abelian groups, where $n = [K : \mathbb{Q}]$.

(2) Let $I \subset \mathcal{O}_K$ be an ideal. Show that $\mathcal{O}_K/I$ is finite.

Exercise 3. Let K be a number field and $x \in \mathcal{O}_K \setminus 0$. Show that

$$|N_{\mathbb{Q}}^K(x)| = |\mathcal{O}_K/(x)|.$$