Algebraic Number Theory

Exercises 6

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Exercise 1. Let A be a Dedekind domain and $I, J \in F(A)$.

(1) Show that $I \subset J$ if and only if $\nu_P(I) \geq \nu_P(J)$ for all maximal ideals P.

(2) Let $I = \prod_P P^{\nu_P(I)}$, $J = \prod_P P^{\nu_P(J)}$. Compute IJ and $I \cap J$ in these terms. When is $IJ = I \cap J$?

Exercise 2. Let $K = \mathbb{Q}(\sqrt{d})$, where $d \in \mathbb{Z}$ is squarefree and $\not\equiv 1 \pmod 4$. Let p be a rational prime. Describe the decomposition of $p\mathcal{O}_K$ in $F(\mathcal{O}_K)$ in terms of $\bar{d} \in \mathbb{Z}/p$.

[*Hint:* $\mathcal{O}_K/p \simeq \mathbb{Z}/p[X]/(X^2 - d)$.]

Exercise 3. Let A be a dvr with fraction field K. Let L/K be a finite separable extension, and B the integral closure of A in L. Show that B is a PID.

[*Hint:* use exercise 4 below.]

Exercise 4. Let A be a Dedekind domain with fraction field K, $P_1, \ldots, P_n$ distinct maximal ideals, $r_1, \ldots, r_n \geq 0$ and $a_1, \ldots, a_n \in K$. Show that there exists $b \in K$ such that $\nu_{P_i}(b - a_i) \geq r_i$ for $i = 1, \ldots, n$, and $\nu_P(b) \geq 0$ for all maximal ideals P distinct from the P_i.

[*Hint:* first use the Chinese remainder theorem to treat the case where $a_i \in A$ for all i.]

Deduce that if A has only finitely many maximal ideals, then it is a PID.