Algebraic Number Theory

Exercises 2

Dr. Tom Bachmann Winter Semester 2021/2

Exercise 1. Let A be an integral domain, $K = Frac(A)$, and L/K a finite Galois extension. Let B be the integral closure of A in L.

(1) Show that if $\sigma \in Gal(L/K)$ than $\sigma B \subset B$.
(2) Show that if A is integrally closed, then $B^{Gal(L/K)} = A$.

Exercise 2. Let $K = \mathbb{Q}(\alpha)$, where $\alpha^2 = -5$. Show that $\mathcal{O}_K$ is not a PID.
[*Hint:* $2 \times 3 = (1+\alpha)(1-\alpha)$]

Exercise 3. Let $A = \mathbb{Z}[X,Y]/(X^2 - Y^3)$.

(1) Show that $A \to \mathbb{Z}[T]$, $X \mapsto T^3, Y \mapsto T^2$ is a well-defined, injective ring homomorphism. (In particular A is an integral domain.) [*Hint:* you may wish to show that $Y^i X^\epsilon$ with $i \geq 0$, $\epsilon \in \{0,1\}$ form a $\mathbb{Z}$-basis of A.]
(2) Prove that $\mathbb{Z}[T]$ is the integral closure of A.

Exercise 4. Let $P(X) = X^3 - X^2 - 2X - 8 \in \mathbb{Z}[X]$, $K = \mathbb{Q}(\alpha)$ where $P(\alpha) = 0$.

(1) Show that P is irreducible.
(2) Compute that $D_{\mathbb{Q}}^{K}(1, \alpha, \alpha^2) = -4 \times 503$.
(3) Show that $\beta := \frac{\alpha+\alpha^2}{2}$ is integral over $\mathbb{Z}$.
(4) Compute $D_{\mathbb{Q}}^{K}(1, \alpha, \beta)$ and deduce that $\{1, \alpha, \beta\}$ is a $\mathbb{Z}$-basis of $\mathcal{O}_K$.