Algebraic Number Theory

Exercises 12

Dr. Tom Bachmann Winter Semester 2021/2

Exercise 1. Let $K = \mathbb{Q}(\sqrt{3}, \sqrt{-1})$.

(1) Show that $K/\mathbb{Q}$ is Galois of degree 4.
(2) Give the structure of $\mathcal{O}_K^\times$.

Exercise 2. Let A be a Dedekind domain with field of fractions K, L/K a finite separable extension of degree n and B the integral closure of A in L.

(1) Show that if B is a PID, then $C(A)$ is n-torsion.
(2) Suppose that that there exists another finite separable extension L'/K of degree n' and integral closure B' of A in L', such that $(n, n') = 1$ and B' is also a PID. Show that then A is a PID.

Exercise 3. Let $L = \mathbb{Q}(\sqrt{5}, \sqrt{-1})$.

(1) Show that $L/\mathbb{Q}$ is Galois of degree 4.
(2) Let A be a PID with field of fractions K, L/K a finite separable extension of degree n, B the integral closure of A in L. Suppose that for some family $x_1, \dots, x_n \in B$ the discriminant $D_A^B(x_1, \dots, x_n) \in A$ is square-free. Show that $(x_1, \dots, x_n)$ form an integral basis of B/A.
(3) Using (2) with $B = \mathcal{O}_L$, $A = \mathbb{Z}[\sqrt{-1}]$, $x_1 = 1$, $x_2 = (1 + \sqrt{5})/2$, find a basis of $\mathcal{O}_L$ over $\mathbb{Z}[\sqrt{-1}]$. Deduce that $\mathcal{O}_L = \mathbb{Z}[x_2, \sqrt{-1}]$.
(4) Let p be an odd prime such that -1 is not a square modulo p. For the decomposition of $p\mathcal{O}_L$ compute the corresponding e and f. Show that p does not ramify in L.

Exercise 4. Let K be a number field and I a non-zero ideal in $\mathcal{O}_K$.

(1) Show that there exists $m > 0$ such that I^m is principal.
(2) Let $n > 0$ such that I^n is principal. Show that there exists an extension L/K of degree $\leq n$ such that $I\mathcal{O}_L$ is principal.